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Stress singularities at vertices of composite plates with smooth or rough interfaces

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THE asymptotic behaviour of the stress and displacement field in a multi-wedge corner of a composite body consisting of a number of homogeneous and isotropic wedges with rough interfaces in linear elasticity was studied. The order of the stress singularity at the singular vertex was examined and its dependence on the mechanical properties of the wedges and their particular geometric configuration was studied by a simple method based on complex variables. According to this method the two complex stress functions of the Mushkelishvili formulation of the plane-stress problem are expressed as sums of powers of complex exponents and their conjugates. Introducing these forms of the stress functions into the appropriate boundary conditions for the particular type of interface considered, the order of singularity was determined. The characteristic equations for the evaluation of the order of singularity in a bi-wedge with a rough interface were established when the bi-wedge was loaded with a prescribed stresses along the remaining boundaries. Numerical results for the special case of wedges of different angles adhering along a half-space were given.

W ramach liniowej teorii sprężystości rozważono asymptotyczne zachowanie się pol naprężeń i odkształceń w wieloklinowym narożu ciała kompozytowego, składającego się z dużej liczby jednorodnych i izotopowych klinów z szorstkimi powierzchniami podziału. Zbadano rząd osobliwości naprężeń w wierzchołku osobliwym i przeanalizowano w prosty sposób za pomocą funkcji zmiennej zespolonej wpływ tej osobliwości na własności mechaniczne klinów i ich poszczególne konfiguracje geometryczne. Zgodnie z tą metodą dwie funkcje zmiennej zespolonej występujące w sformułowanym przez Muscheliszwilego rozwiązaniu zagadnienia płaskiego stanu naprężenia są wyrażone jako sumy potęg zespolonych wykładników i ich sprzężeń. Wprowadzając tak określone funkcje naprężeń do odpowiednich warunków brzegowych dla poszczególnych typów powierzchni podziału określono rząd osobliwości. Równania charakterystyczne dla wyznaczenia rzędu osobliwości w „dwuklinie" z szorstką powierzchnią podziału określono dla przypadku, gdy dwuklin ten jest obciążony przez naprężenia dane na pozostałych brzegach. Podano wyniki numeryczne dla szczególnych przypadków klinów o różnych kątach rozwarcia przylegających do półprzestrzeni.

В рамках линейной теории упругости рассмотрено асимптотическое поведение полей напряжений и деформаций в многогранной вершине композитного тела, состоящего из большого количества однородных и изотропных граней с шероховатыми поверхностями раздела. Исследован порядок особенности напряжений в особой вершине и анализируется простым образом, при помощи функций комплексного переменного, влияние этой особенности на механические свойства грани и их отдельные геометрические конфигурации. Согласно этому методу выступающие две функции комплексного переменного, в сформулированной Мусхелишвили задаче плоского напряженного состояния, выражаются как суммы степеней комплексных показателей и их сопряжений. Вводя так определенные функции напряжений в соответствующие граничные условия, для отдельных типов поверхности раздела, определен порядок особенности. Характеристические уравнения, для определения порядка особенности в биграни с шероховатой поверхностью раздела, определены для случая, когда эта бигрань нагружена напряжением заданным на остальных краях. Приведены численные результаты для отдельных случаев граней смежных под разными углами с полупространством.

1. Introduction

DUE to the great complexity of the problem of the stress and displacement distribution in a composite body consisting of a number of dissimilar wedges, a small number of contributions has been appeared. The most important region in the multiwedge is the close vicinity of the vertex of the wedges where their interfaces coalesce. This is due to the fact that a stress singularity is engendered at the vertex. Thus, the problem is reduced to that of finding the order of singularity and its dependence on the mechanical properties of the dissimilar bodies and the geometry of the wedges at the particular corner considered. Stress singularities have been investigated in many crack problems as well as in the homogeneous wedge under various loading conditions [1, 2].

A systematic study of the problem of a bimaterial wedge has been done by BOGY in a series of recent publications [3, 4]. Bogy applied the Mellin transform to reduce the problem of the bimaterial wedge to a simpler one in the transform plane. However, due to the great complexity of the solution even in the transform plane, Bogy restricted his solutions at the close vicinity of the corner, where the two interfaces coalesce. Only in the case of the problem of two edge-bonded quarter planes with a concentrated normal load applied at the boundary [4] he gave a more systematic study by calculating the stresses at the bonded interface at some distance from the singular vertex.

RAO [5] presented recently a general procedure for the determination of singularities at corners at the intersection of two or more interfaces in domains governed by harmonic and biharmonic types of equations. Several types of boundary conditions at the interfaces were considered and special cases of the bimaterial wedge were analysed.

DUNDURS [6] made a significant contribution to the multiwedge problem by pointing out that for the case of a bimaterial wedge under plane-stress or plane-strain conditions a reduced dependence of the stress field exists on only two combinations of the four elastic constants of the materials constituting the biwedge.

THEOCARIS [7] applied the complex variable technique [1] for the study of the region very close to a singular vertex in a multiwedge. This method was applied to the case of a transverse semi-infinite crack in an infinite plate, as well as to the problem of a homogeneous wedge under various boundary conditions. The results obtained coincided with the existing solutions.

The remarkable feature of the results obtained by any of the above theories for the multiwedge is that they hold regardless of the topological intricacies of the composite body. Furthermore, they can be adapted to the mechanical conditions which are imposed to the interfaces between phases, be it full adhesion or frictionless slip. Partial separation between phases in the form of cracks running along the interface may be considered by these solutions. Furthermore, if there is some friction between partly separated interfaces, the boundary conditions between phases change and, therefore, this problem necessitates special attention. Since in most of the engineering problems the adherence between phases is in general not complete, the problem of partially rough interfaces is of great importance. Indeed, we can say that this is the case in almost all practical applications where two or more bodies are in contact creating a number of interfaces. Thus, in the macrostructural analysis some friction always exists between any two surfaces in contact. Furthermore, the role of friction becomes very important in the microstructural study of the bodies in contact, where the smoothest possible surface consists in reality of a large number of zig-zag parts. Thus, a great number of microsurfaces are always in contact, while gaps always exist along large distances in any interface and these are distributed in a very complicated manner. The totality of these microcontacts and microgaps for each interface of two materials in contact creates some overall friction, which depends on the distribution of these microsurfaces and which modulates the appropriate conditions at the interface. Therefore, it is reasonable to accept for all engineering applications that the contact problem of two bodies with some friction is the most realistic.

In the present paper the above mentioned cases in composite plates were analysed. The boundary conditions at a smooth or rough interface were investigated and the order of singularity at a multiwedge corner of a composite body as a function of the mechanical properties and the geometry of the wedges as well as of the conditions at the interface was found. The special case of the bimaterial wedge was analysed in detail.

2. Boundary conditions for a multiwedge with bonded or rough interfaces

Consider the general case of a composite body consisting of n dissimilar homogeneous, isotropic and elastic wedges with their interfaces coalescing at a point, and which is under plane-strain or generalized plane-stress conditions. A system of Cartesian coordinates with origin at the vertex of the wedges is related to the body (Fig. 1). Each of the wedges denoted by S_j ($j = 1, 2, \ldots, n$) is included between the interfaces E_j and $E_{(j+1)}$ ($E_{(n+1)} \equiv \equiv E_1$), where each tangent at the interface E_j subtends an angle ϑ_j with the x-axis. The two successive wedges $S_{(k-1)}$ and S_k may be considered either as perfectly bonded along their interface E_k, or smooth and free to slip relatively to each other. A third and more realistic situation between these two extreme cases is when friction is developed between the two bodies along their interface E_k.

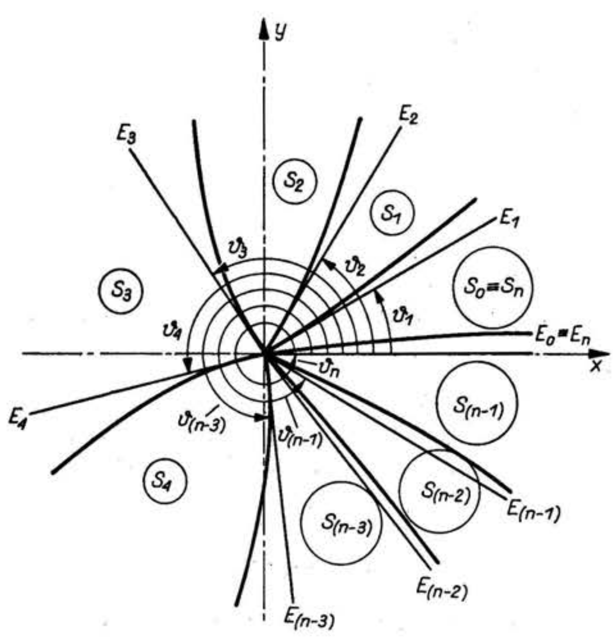

FIG. 1. Geometry of a multiwedge plate.

According to the Muskhelishvili formulation for the plane stress problem in linear elasticity where Airy's stress function is expressed in terms of two complex functions $\varphi(z)$ and $\psi(z)$, the resultant force along a generic curve E between the positions $s = 0$ and $s = s_A$ is given by [8]:

$$i\int_0^{s_A} [T_x(t)+iT_y(t)]ds = \varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)}, \tag{2.1}$$

where $T_x(t)$ and $T_y(t)$ are the components of the specified tractions along this curve. In this relation it is assumed that the direction of increasing arc-coordinates t is chosen so that the wedge always lies to the left of curve E when one follows the direction of increasing arc-coordinates.

When the displacements are specified on the boundary of the wedge the sole boundary condition is given by:

$$u\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)} = 2G[f(t)+ih(t)], \tag{2.2}$$

where G is the shear modulus, $u = (3-4\nu)$ for plane strain and $u = (3-\nu)/(1+\nu)$ for plane stress, ν is the Poisson ratio and $f(t)$ and $h(t)$ are the two components of the specified displacements.

When the two dissimilar wedges $S_{(k-1)}$ and S_k are perfectly bonded along their interface E_k, it is natural to require that the stresses and displacements at the interface E_k are continuous. Using the subscripts $(k-1)$ and k for the wedges $S_{(k-1)}$ and S_k we have for the perfectly bonded faces of the wedges $S_{(k-1)}$ and S_k:

$$(u_x+iu_y)_{(k-1)} = (u_x+iu_y)_k, \tag{2.3}$$

$$(T_x+iT_y)_{(k-1)} = -(T_x+iT_y)_k. \tag{2.4}$$

Since the positive direction on the curve E_k corresponding to wedge $S_{(k-1)}$ is reversed for the same curve corresponding to wedge S_k, an adjustment is necessary in relation (2.4) by putting a negative sign in this relation. Combining relations (2.3) and (2.4) with equations (2.1) and (2.2), we obtain the boundary conditions along the bonded interface E_k as follows:

$$\varphi_{(k-1)}(t_k)+t_k\overline{\varphi'_{(k-1)}(t_k)}+\overline{\psi_{(k-1)}(t_k)} = \varphi_k(t_k)+t_k\overline{\varphi'_k(t_k)}+\overline{\psi_k(t_k)}, \tag{2.5}$$

$$\Gamma_k[u_{(k-1)}\varphi_{(k-1)}(t_k)-t_k\overline{\varphi'_{(k-1)}(t_k)}-\overline{\psi_{(k-1)}(t_k)}] = u_k\varphi_k(t_k)-t_k\overline{\varphi'_k(t_k)}-\overline{\psi_k(t_k)}, \tag{2.6}$$

where $\Gamma_k = G_k/G_{(k-1)}$ is the ratio of the shear moduli of the materials of the two wedges and primes denote differentiation with respect to z.

When some friction is developed between the wedges $S_{(k-1)}$ and S_k along their interface E_k, the appropriate boundary conditions are expressed by:

$$(u_\theta)_{(k-1)} = (u_\theta)_k, \tag{2.7}$$

$$(\sigma_\theta)_{(k-1)} = -(\sigma_\theta)_k, \tag{2.8}$$

$$(\tau_{r\theta})_{(k-1)} = (\tau_{r\theta})_k = q(\sigma_\theta)_{(k-1)} = q(\sigma_\theta)_k, \tag{2.9}$$

where q is the friction coefficient between the two rough surfaces.

We replace Eqs. (2.8) and (2.9) by the following expressions:

$$(\sigma_\vartheta + i\tau_{r\vartheta})_{(k-1)} = -(\sigma_\vartheta + i\tau_{r\vartheta})_k, \tag{2.10}$$

$$(\tau_{r\vartheta})_k = q(\sigma_\vartheta)_k, \tag{2.11}$$

so that the boundary conditions at the interface E_k with friction are expressed by Eqs. (2.7), (2.10) and (2.11), respectively.

The components of stresses and displacements in polar coordinates are expressed by [8]:

$$\sigma_\vartheta + \sigma_r = 2[\varphi'(z) + \overline{\varphi'(z)}], \tag{2.12}$$

$$\sigma_\vartheta - \sigma_r + 2i\tau_{r\vartheta} = 2[\bar{z}\varphi''(z) + \psi'(z)]e^{2i\vartheta}, \tag{2.13}$$

$$2G(u_r + iu_\vartheta) = [u\varphi(z) - z\varphi'(z) - \overline{\psi(z)}]e^{-i\vartheta}. \tag{2.14}$$

If we substitute the expressions for stresses σ_ϑ, $\tau_{r\vartheta}$ and displacement u_ϑ obtained from the above relations into Eqs. (2.7), (2.10) and (2.11), we obtain the following boundary conditions for a rough interface E_k:

$$\begin{aligned} \Gamma_k \operatorname{Im}\big[e^{-i\vartheta_k}[u_{(k-1)}\varphi_{(k-1)}(t_k) - t_k\overline{\varphi'_{(k-1)}(t_k)} - \overline{\psi_{(k-1)}(t_k)}]\big] \\ = \operatorname{Im}\big[e^{-i\vartheta_k}[u_k\varphi_k(t_k) - z\overline{\varphi'_k(t_k)} - \overline{\psi_k(t_k)}]\big], \end{aligned} \tag{2.15}$$

$$\begin{aligned} \varphi'_{(k-1)}(t_k) + \overline{\varphi'_{(k-1)}(t_k)} + e^{2i\vartheta_k}[\bar{t}_k\varphi''_{(k-1)}(t_k) + \psi'_{(k-1)}(t_k)] = \varphi'_k(t_k) \\ + \overline{\varphi'_k(t_k)} + e^{2i\vartheta_k}[\bar{t}\varphi''_k(t_k) + \psi'_k(t_k)], \end{aligned} \tag{2.16}$$

$$\operatorname{Im}\big[e^{2i\vartheta_k}[t_k\varphi''_k(t_k) + \psi'_k(t_k)]\big] = q\operatorname{Re}\big[2\varphi'_k(t_k) + e^{2i\vartheta_k}[t\varphi''_k(t_k) + \psi'_k(t_k)]\big], \tag{2.17}$$

where Re and Im represent the real and imaginary parts of the corresponding functions. The boundary conditions for the special case of a smooth interface, where the two wedges are free to mutually slip may be deduced from the above equations by putting $q = 0$.

We assume now for the complex stress-functions $\varphi(z)$ and $\psi(z)$ in the Muskhelishvili formulation for the plane-stress problem at the close vicinity of the vertex of the multiwedge the forms:

$$\varphi_k = a_{1k}z^\lambda + a_{2k}z^{\bar{\lambda}}, \quad \psi_k = b_{1k}z^\lambda + b_{2k}z^{\bar{\lambda}}. \tag{2.18}$$

Furthermore, these expressions for φ_k and ψ_k must satisfy the appropriate boundary conditions for the particular type of mechanical conditions prevailing at the considered interface E_k.

3. The order of singularity at the vertex of the multiwedge

3.1. The general equations

From the above established boundary conditions and the particular form of the complex stress functions $\varphi(z)$ and $\psi(z)$, for which only singular terms are considered, the type of singularity can be determined, and its dependence on the mechanical properties of the dissimilar materials of the composite multiwedge, as well as of the particular vertex geometric configuration considered.

If we introduce relations (2.18) into Eqs. (2.15), (2.16), (2.17), put $t_k = re^{i\vartheta_k}$ and eliminate the coefficients of r^λ and $r^{\bar{\lambda}}$, we obtain the following conditions for the case of a rough interface with a friction coefficient equal to q:

$$(3.1)\quad \begin{aligned} &\Gamma_k[(u_{(k-1)}a_{1(k-1)}e^{i(\lambda-1)\vartheta_k}-\lambda\bar{a}_{2(k-1)}e^{i(1-\lambda)\vartheta_k}-\bar{b}_{2(k-1)}e^{-i(1+\lambda)\vartheta_k})-(u_{(k-1)}\bar{a}_{2(k-1)}\times\\ &\times e^{i(1-\lambda)\vartheta_k}-\lambda a_{1(k-1)}e^{i(\lambda-1)\vartheta_k}-b_{1(k-1)}e^{i(1+\lambda)\vartheta_k})] = (u_k a_{1k}e^{i(\lambda-1)\vartheta_k}-\lambda\bar{a}_{2k}e^{i(1-\lambda)\vartheta_k}\\ &-\bar{b}_{2k}e^{-i(1+\lambda)\vartheta_k})-(u_k\bar{a}_{2k}e^{i(1-\lambda)\vartheta_k}-\lambda a_{1k}e^{i(\lambda-1)\vartheta_k}-b_{1k}e^{i(\lambda+1)\vartheta_k}), \end{aligned}$$

$$(3.2)\quad \begin{aligned} &\lambda a_{1(k-1)}e^{i(\lambda-1)\vartheta_k}+\bar{a}_{2(k-1)}e^{i(1-\lambda)\vartheta_k}+b_{1(k+1)}e^{i(1+\lambda)\vartheta_k} = \lambda a_{1k}e^{i(\lambda-1)\vartheta_k}\\ &+\bar{a}_{2k}e^{i(1-\lambda)\vartheta_k}+b_{1k}e^{i(1+\lambda)\vartheta_k}, \end{aligned}$$

$$(3.3)\quad \begin{aligned} &\bar{\lambda}a_{2(k-1)}e^{i(\bar{\lambda}-1)\vartheta_k}+\bar{a}_{1(k-1)}e^{i(1-\bar{\lambda})\vartheta_k}+b_{2(k-1)}e^{i(\bar{\lambda}+1)\vartheta_k} = \bar{\lambda}a_{2k}e^{i(\bar{\lambda}-1)\vartheta_k}\\ &+\bar{a}_{1k}e^{i(1-\bar{\lambda})\vartheta_k}+b_{2k}e^{i(\bar{\lambda}+1)\vartheta_k}, \end{aligned}$$

$$(3.4)\quad \begin{aligned} &(1-iq)\lambda(\lambda-1)e^{i(\lambda-1)\vartheta_k}a_{1k}+(1-iq)\lambda e^{i(\lambda+1)\vartheta_k}b_{1k}-(1+iq)\lambda(\lambda-1)e^{i(1-\lambda)\vartheta_k}\bar{a}_{2k}\\ &-(1+iq)\lambda e^{-i(1+\lambda)\vartheta_k}\bar{b}_{2k} = 2iq\lambda e^{i(\lambda-1)\vartheta_k}a_{1k}+2iq\lambda e^{i(1-\lambda)\vartheta_k}\bar{a}_{2k}. \end{aligned}$$

Applying equations (3.1), (3.2), (3.3), (3.4) to the case of an interface with friction in the n interfaces of a composite body consisting of n dissimilar materials, we obtain $4n$ equations homogeneous for the unknown coefficients a_{1k}, $\bar{a}_{2k}$, b_{1k}, $\bar{b}_{2k}$. For the non-trivial solution of the system of $4n$-equations the determinant of the coefficients of the unknowns must be equal to zero. Thus, we obtain an equation from which the value of λ can be determined.

3.2. The bimaterial wedge

Consider now the special case of a composite body consisting of two dissimilar wedges with a rough interface E_1 and subjected to any type of loading on the faces E_2 and E_3 (Fig. 2). The $0x$-axis of the Cartesian coordinate system is taken to coincide with the

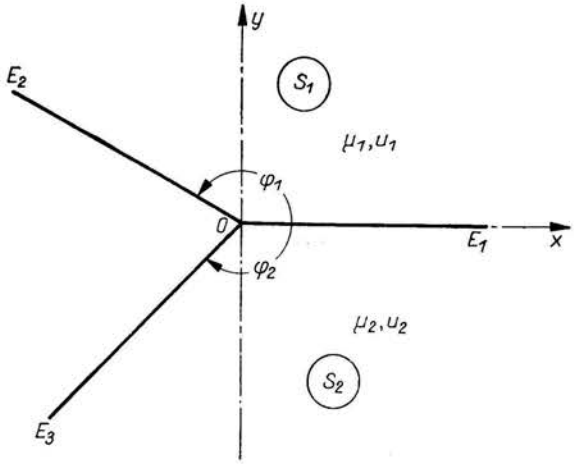

FIG 2. Geometry of a bimaterial wedge.

interface E_1, while the tangents to E_2 and E_3 at the origin 0 make angles equal to φ_1 and φ_2 with E_1 respectively.

By putting $\vartheta_k = 0$ into relations (3.3), (3.2), (3.4), (3.1) we obtain the following four equations respectively, which represent the boundary conditions at E_1 under the form of functions $\varphi(z)$ and $\psi(z)$ given by Eqs. (2.18):

$$a_{11}+\lambda\bar{a}_{21}+\bar{b}_{21} = a_{12}+\lambda\bar{a}_{22}+\bar{b}_{22}, \tag{3.5}$$

$$\bar{a}_{21}+\lambda a_{11}+b_{11} = \bar{a}_{22}+\lambda a_{12}+b_{12}, \tag{3.6}$$

$$[\lambda(1-iq)-(1+iq)]a_{12}-[\lambda(1+iq)-(1-iq)]\bar{a}_{22}+(1-iq)b_{12}-(1+iq)\bar{b}_{22} = 0, \tag{3.7}$$

$$\Gamma[(u_1+\lambda)a_{11}-(u_1+\lambda)\bar{a}_{21}-\bar{b}_{21}+b_{11}] = (u_2+\lambda)a_{12}-(u_2+\lambda)\bar{a}_{22}-\bar{b}_{22}+b_{12}. \tag{3.8}$$

In order to obtain the boundary conditions at the faces E_1 and E_2, we introduce into relations (3.2) and (3.3), which represent the continuity of stresses at the interface, the particular values for the angles of the faces of wedges, that is $\vartheta_k = \varphi_1$ for E_1 and $\vartheta_k = \varphi_2$ for E_2 and equate the right-hand sides of these relations to zero, since only one material exists at either face E_1 and E_2. Thus, we obtain:

$$a_{11}e^{2i\lambda\varphi_1}+\lambda\bar{a}_{21}e^{2i\varphi_1}+\bar{b}_{21} = 0, \tag{3.9}$$

$$\bar{a}_{21}e^{-2i\lambda\varphi_1}+\lambda a_{11}e^{-2i\varphi_1}+b_{11} = 0, \tag{3.10}$$

$$a_{12}e^{2i\lambda\varphi_2}+\lambda\bar{a}_{22}e^{2i\varphi_2}+\bar{b}_{22} = 0, \tag{3.11}$$

$$\bar{a}_{22}e^{-2i\lambda\varphi_2}+\lambda a_{12}e^{-2i\varphi_2}+b_{12} = 0. \tag{3.12}$$

Relations (3.5) to (3.12) constitute a system of eight homogeneous equations for the unknowns $a_{11}, a_{12}, \bar{a}_{21}, \bar{a}_{22}, b_{11}, b_{12}, \bar{b}_{21}, \bar{b}_{22}$. By eliminating the unknowns $b_{11}, b_{12}, \bar{b}_{21}, \bar{b}_{22}$ for a non-trivial solution of the derived system for the unknowns $a_{11}, a_{12}, \bar{a}_{21}, \bar{a}_{22}$, we must equate to zero the determinant of the coefficients of the unknowns and we obtain:

$$\begin{vmatrix} (1-e^{2i\lambda\varphi_1}) & \lambda(1-e^{2i\varphi_1}) & (1-e^{2i\lambda\varphi_2}) & \lambda(1-e^{2i\varphi_2}) \\ \lambda(1-e^{-2i\varphi_1}) & (1-e^{-2i\lambda\varphi_1}) & \lambda(1-e^{-2i\varphi_2}) & (1-e^{-2i\lambda\varphi_2}) \\ 0 & 0 & \begin{matrix}(1+iq)(1-e^{2i\lambda\varphi_2})-\\ -\lambda(1-iq)(1-e^{-2i\varphi_2})\end{matrix} & \begin{matrix}\lambda(1+iq)(1-e^{2i\varphi_2})-\\ -(1-iq)(1-e^{-2i\lambda\varphi_2})\end{matrix} \\ \begin{matrix}\Gamma[(u_1+\lambda)-\\ -(\lambda e^{-2i\varphi_1}-\\ -e^{2i\lambda\varphi_1})]\end{matrix} & \begin{matrix}-\Gamma[(u_1+\\ +\lambda)-(\lambda e^{2i\varphi_1}-\\ -e^{-2i\lambda\varphi_1})]\end{matrix} & \begin{matrix}-[(u_2+\lambda)-\\ -(\lambda e^{-2i\varphi_2}-e^{2i\lambda\varphi_2})]\end{matrix} & \begin{matrix}-[(u_2+\lambda)-\\ -(\lambda e^{2i\varphi_2}-e^{-2i\lambda\varphi_2})]\end{matrix} \end{vmatrix} = 0. \tag{3.13}$$

Relation (3.13) gives the characteristic equation for the determination of λ in a bi-material wedge in terms of the mechanical properties of the two wedges for the special configuration considered and for the case of the first fundamental problem when a rough interface exists in the bi-wedge. For the special case of a smooth interface we must put $q = 0$ into relation (3.13).

By applying some simple properties of the determinants it is possible to put equation (3.13) in an equivalent real form as follows:

$$
\tag{3.14}
\begin{vmatrix}
\lambda A(\varphi_1,\lambda)+B(\varphi_1,\lambda) & C(\varphi_1,\lambda) & \lambda A(\varphi_2,\lambda)+B(\varphi_2,\lambda) & C(\varphi_2,\lambda) \\
-D(\varphi_1,\lambda) & -\lambda A(\varphi_1,\lambda)+B(\varphi_1,\lambda) & -D(\varphi_2,\lambda) & -\lambda A(\varphi_2,\lambda)+B(\varphi_2,\lambda) \\
\Gamma D(\varphi_1,\lambda) & \Gamma[\lambda A(\varphi_1,\lambda)+E(\varphi_1 u_1,\lambda)] & q[\lambda A(\varphi_2,\lambda)+B(\varphi_2,\lambda)] & qC(\varphi_2,\lambda)+(u_2+1) \\
-\Gamma D(\varphi_1,\lambda) & -\Gamma[\lambda A(\varphi_1,\lambda)+E(\varphi_1 u_1,\lambda)] & q[\lambda A(\varphi_2,\lambda)+B(\varphi_2,\lambda)]-2D(\varphi_2,\lambda) & -[2\lambda A(\varphi_2,\lambda)-B(\varphi_2,\lambda)-qC(\varphi_2,\lambda)+E(\varphi_2 u_2,\lambda)]
\end{vmatrix} = 0,
$$

where:

$$A(\varphi,\lambda) = 1-\cos 2\varphi, \qquad D(\varphi,\lambda) = \sin 2\lambda\varphi+\lambda\sin 2\varphi,$$

$$B(\varphi,\lambda) = 1-\cos 2\lambda\varphi, \qquad E(\varphi,u,\lambda) = u+\cos 2\lambda\varphi,$$

$$C(\varphi,\lambda) = \sin 2\lambda\varphi-\lambda\sin 2\varphi.$$

3.2.1. *Special cases of bi-wedges*. As a special case of the bi-wedge considered above we study the situation where a wedge of any angle φ adheres along a semi-infinite plane.

By putting $\varphi_1 = \varphi$ and $\varphi_2 = -\pi$ in relations (3.13) or (3.14), we obtain the following equation for the determination of λ:

$$
\tag{3.15}
\begin{aligned}
&(u_1+1)\sin(2\lambda\pi)[\lambda^2\sin^2\varphi-\sin^2\lambda\varphi]-\Gamma(u_1+1)\sin^2(\lambda\pi)[\sin(2\lambda\varphi)+\lambda\sin(2\varphi)]\\
&\quad+q\sin^2(\lambda\pi)[2\Gamma(u_1-1)\sin^2(\lambda\varphi)+2\lambda[\lambda(u_2+2\Gamma-1)+\Gamma(u_1+1)]\sin^2\varphi] = 0.
\end{aligned}
$$

Relation (3.15) can be written in the form:

$$\tag{3.16} A\alpha+B\beta+C = 0$$

with

$$
\tag{3.17}
\begin{aligned}
A &= 2G(\varphi,\lambda)\cos(\lambda\pi)-K(\varphi,\lambda)\sin(\lambda\pi)+qJ(\varphi,\lambda),\\
B &= 4qG(\varphi,\lambda)\sin(\lambda\pi),\\
C &= -2G(\varphi,\lambda)\cos(\lambda\pi)-K(\varphi,\lambda)\sin(\lambda\pi)+qJ(\varphi,\lambda),
\end{aligned}
$$

where

$$
\tag{3.18}
\begin{aligned}
G(\varphi,\lambda) &= \sin^2(\lambda\varphi)-\lambda^2\sin^2\varphi,\\
K(\varphi,\lambda) &= \sin(2\lambda\varphi)+\lambda\sin(2\varphi),\\
J(\varphi,\lambda) &= 2\lambda(\lambda+1)\sin(\lambda\pi)\cos^2\varphi,
\end{aligned}
$$

if we introduce the composite material parameter α and β given by:

$$\tag{3.19} \alpha = \frac{(\Gamma u_1-u_2)+(\Gamma-1)}{(\Gamma u_1+1)+(\Gamma+u_2)}, \quad \beta = \frac{(\Gamma u_1-u_2)-(\Gamma-1)}{(\Gamma u_1+1)+(\Gamma+u_2)}.$$

Relations (3.19) have been introduced by DUNDURS [6]. Referring all the physically interesting material combinations ($0 \leqslant \nu_{1,2} \leqslant 0.5$, $0 < \Gamma < \infty$) to the $(\alpha-\beta)$-plane, a parallelogram is obtained which is symmetric with respect to the origin with sides given by [4]:

$$\alpha = \pm 1, \quad \pm\alpha \mp 4\beta + 1 = 0. \tag{3.20}$$

From relations (3.16)–(3.18) we conclude that the singularity boundary, that is the locus of points for which $\lambda = 1$, is a straight line parallel to the β-axis intersecting the α-axis at a point with α_b equal to

$$\alpha_b = -\frac{-(1+\pi q)\sin\varphi + (\pi+\varphi)\cos\varphi}{(1-\pi q)\sin\varphi + (\pi-\varphi)\cos\varphi}. \tag{3.21}$$

The variation of the position of the singularity boundary with angle φ of the wedge for several values of the friction coefficient q is presented in Fig. 3.

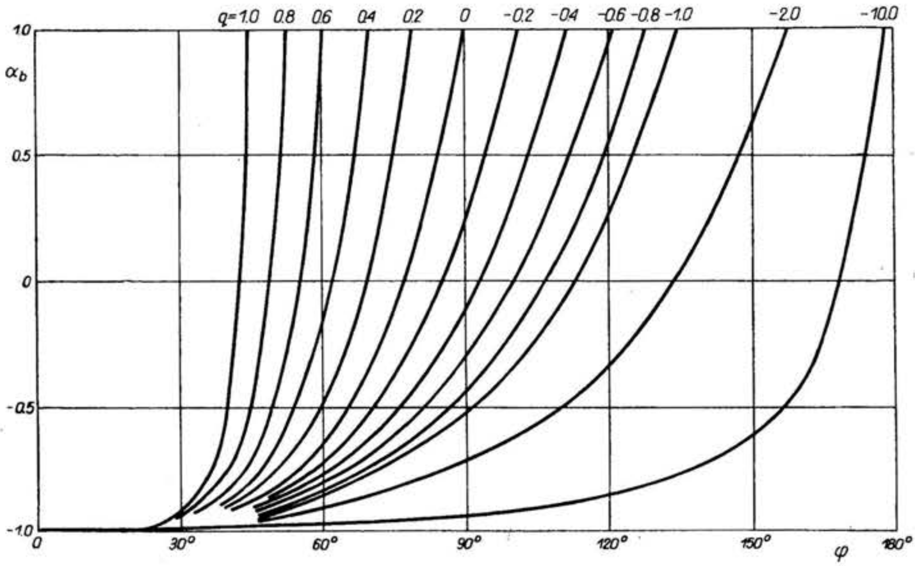

FIG. 3. Variation of the position of the singularity boundary with the angle φ of the wedge adhering along a half-plane for the following values of friction coefficient $q = 1.0, 0.8, 0.6, 0.4, 0.2, 0, -0.2, -0.4, -0.6, -0.8, -1.0, -2.0, -10.0$.

As a numerical example we consider the variation of the order of singularity λ, as the angle φ of the wedges varies for a particular value of friction coefficient q equal to $q = 1$. The contour lines of λ for $\varphi = 30°, 60°, 90°$ and $\varphi = 120°, 150°, 180°$ are presented in Figs. 4 and 5, respectively. Comparing these figures we conclude that λ decreases, i.e., the singularity becomes more severe, as the angle φ increases. We can also observe that the contour lines for λ rotate counterclockwise as φ increases becoming parallel to the α-axis for $\varphi = \pi$.

By putting $q = 0$ into Eq. (3.15) we obtain for plane-strain conditions $(u_i = 3-4\nu_i)$ the relation:

$$(\tan\lambda\pi)(\sin 2\lambda\varphi + \lambda\sin 2\varphi) = -\frac{G_1}{G_2}\frac{1-\nu_2}{1-\nu_1}[(1-\cos 2\lambda\varphi) - \lambda^2(1-\cos 2\varphi)], \tag{3.22}$$

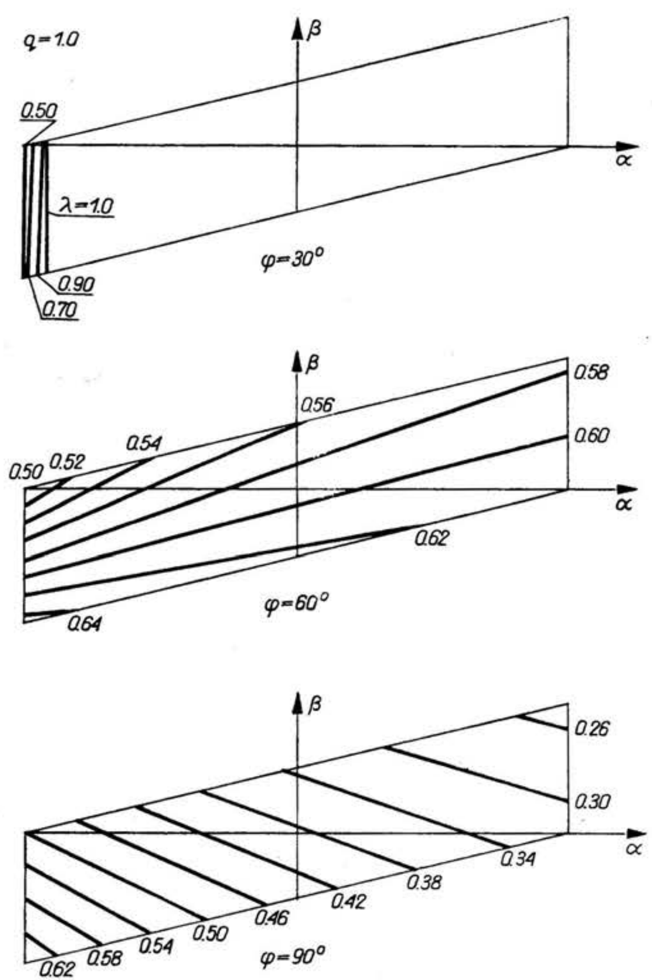

FIG. 4. Contour lines of λ in the $\alpha-\beta$ parallelogram for the following values of the angle φ of the wedge adhering along the half-plane $\varphi = 30°, 60°, 90°$. Friction coefficient $q = 1.0$.

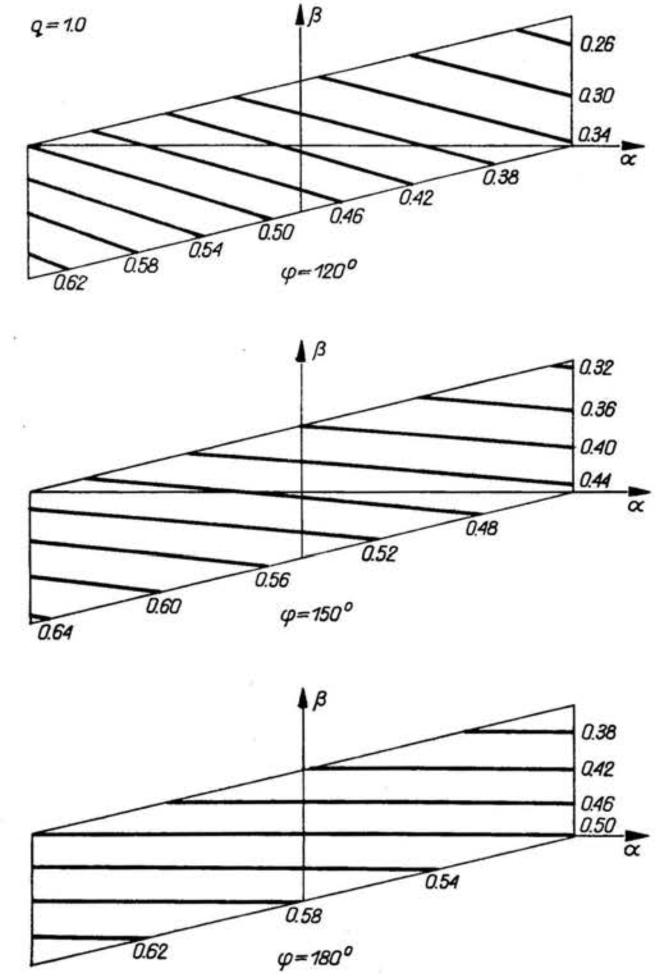

FIG. 5. Contour lines of λ in the $\alpha-\beta$ parallelogram for the following values of the angle φ of the wedge adhering along the half-plane $\varphi = 120°, 150°, 180°$, Friction coefficient $q = 1.0$.

while for plane-stress conditions $\left(u_i = (3-\nu_i)/(1+\nu_i)\right)$, we have:

$$(3.23)\qquad (\tan\lambda\pi)(\sin 2\lambda\varphi+\lambda\sin 2\varphi) = -\frac{E_1}{E_2}\left[(1-\cos 2\lambda\varphi)-\lambda^2(1-\cos 2\varphi)\right].$$

Relation (3.23.) is similar to Eq. (22) given by RAO [5]. However, Rao from Eq. (22) of [5] concluded that the order of singularity is independent of Poisson's ratios for both cases of plane-stress or plane-strain conditions. According to our results the order of singularity λ is independent of Poisson's ratios only for plane-stress conditions [Eq. (3.23)], while for plane-strain conditions [Eq. (3.22)] this quantity is a function of ν_1 and ν_2 of the two materials.

From relations (3.16)–(3.18) we obtain for $\varphi = 180°$ for $q = 0$ or for $q \neq 0$ but for indentical materials ($\alpha = \beta = 0$) $\lambda = 0.5$ that is, the order of singularity at the end of contact of any two materials which can slip along their common interface, or for identical materials with smooth, rough or bonded contact is always equal to $\lambda = 0.50$.

4. Conclusions

A realistic model for the study of the order of singularity at the end of partial contact of an interface of two dissimilar wedges was proposed and analysed. Thus, adherence of any degree varying from the extreme case of a smooth to the other extreme case of a completely bonded interface with different amounts of friction between the faces in contact can be taken into account. The boundary conditions at the interface corresponding to the above cases were considered and analysed. The most important region at the close vicinity of a multiwedge corner of a composite body consisting of many homogeneous, isotropic and elastic wedges was analysed and the asymptotic behaviour of the stress and displacement fields was investigated for the conditions on the interfaces mentioned above by applying the method of the complex stress function. The two complex stress functions of the Muskhelishvili formulation of the plane-stress problem are expressed as sums of powers of complex exponents and their conjugates and, by implying to these expressions for the stress functions to satisfy the particular boundary conditions for each case, the characteristic equations for the determination of the order of singularity were obtained. Thus, the dependence of the order of singularity on the mechanical properties and the particular configuration of a composite body consisting of n dissimilar wedges has been found.

The special case of the bi-material wedge was considered and the characteristic equation derived for the determination of the order of stress singularity λ when the two wedges adhere along their common interface, while stresses are prescribed along the two other boundaries of the bi-wedge. The particular case of a wedge adhering along a half-plane was analysed. Numerical results for λ were given for the angles $\varphi = 30°, 60°, 90°, 120°$ $150°$ and $180°$ for one value of the friction coefficient q equal to $q = 1$, in Dundurs' parallelograms for all the physically interesting material combinations of the wedge and the halfplane.

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