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On the initial value problem in non-linear thermoelasticity

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The initial value problem for the dynamic equations of non-linear thermoelasticity is solved in the Sobolev space. This problem is reduced to the initial value problem for the wave equation and to the initial value problem for the non-linear system of the heat equation and the wave equation. Then, using the principle of contraction mapping a solution to the problem under consideration is found.

Problem początkowy dla równań dynamiki nieliniowej termosprężystości został rozwiązany w przestrzeni Sobolewa. Problem został sprowadzony do zagadnienia początkowego dla nieliniowego równania przewodnictwa ciepła i równania falowego. Wykorzystując następnie zasadę odwzorowania zwężającego znaleziono rozwiązanie rozważanego problemu.

Начальная задача для уравнений динамики нелинейной термоупругости решена в пространстве Соболева. Задача сведена к начальной задаче для нелинейного уравнения теплопроводности и для волнового уравнения. Затем используя принцип отображения сжатия найдено решение рассматриваемой проблемы.

We consider Shalov's basic equations of continuum mechanics (see [7], p. 919, Eqs. (30), (31)) in the following form:

$$\nabla_k \sigma^{kl} - \varrho \partial_t x^k \nabla_k \partial_t x^l - \varrho \partial_t^2 x^l = F^l, \quad l = 1, 2, 3, \tag{1}$$

$$c_{\varepsilon_{ij}} \partial_t T - \hat{\nabla}_k \wedge \frac{\partial T}{\partial \xi^k} + \left(q^{kl} + T \frac{\partial p^{kl}}{\partial T} \right) e_{kl} = \varrho Q^e, \tag{2}$$

where σ^{kl} is the symmetric stress tensor, ϱ — the mass density, T — the local absolute temperature, x^l — the function of motion, which determines the spatial position occupied by the material point at time t (Euler's coordinates), $\nabla_k = \partial/\partial x^k \pm \Gamma^l_{kl}$ — covariant derivative, ξ^k — Lagrangian coordinates of the material point, $\hat{\nabla}_k = \partial/\partial \xi^k \pm \hat{\Gamma}^l_{kl}$ — covariant derivative with respect to the Lagrangian coordinates, F^l and Q^e are the body force and the intensity of heat sources respectively, p^{kl} is the part of the stress tensor, which is independent of the velocity e_{kl} of the strain tensor ε_{ij} (cf., [7], p. 915, formula (14)), and $q^{kl} = \sigma^{kl} - p^{kl}$, $c_{\varepsilon_{ij}}$ — the specific heat at constant deformation, $\wedge$ — the coefficient of heat conduction.

For the sake of simplicity we assume (cf., [7], p. 920) that

$$x^l = \xi^l + u^l(\xi, t), \quad l = 1, 2, 3,$$

where u^l is the displacement vector field of the medium.

Now, the Eqs. (1), (2) can be written as

$$\hat{\nabla}_k \sigma^{kl} - \varrho \partial_t^2 u^l = F^l, \quad l = 1, 2, 3, \tag{3}$$

$$c_{\varepsilon_{ij}} \partial_t T - \hat{\nabla}_k \wedge \frac{\partial T}{\partial \xi^k} + \left(q^{kl} + T \frac{\partial p^{kl}}{\partial T} \right) e_{kl} = \varrho Q^e. \tag{4}$$

In the case of a homogeneous, isotropic, thermoelastic medium where the familiar relation of Duhamel-Neuman (cf., [8], formula (2.25), p. 320) is used in the form

$$\sigma^{kl} = (\lambda\hat{\nabla}_j u^j - \gamma T)g^{kl} + 2\mu\varepsilon^{kl}, \tag{5}$$

where λ, μ are the two Lamé constants of the medium, $\gamma = (3\lambda+2\mu)\cdot\alpha_t$, α_t is the linear coefficient of thermal expansion, g^{kl} — the metric tensor and $\varepsilon^{kl} = (g^{ki}\hat{\nabla}_i u^l + g^{li}\hat{\nabla}_i u^k)/2$, from Eqs. (3), (4) under assumption $q^{kl}_{\Delta} = 0$ we obtain

$$\lambda g^{kl}\frac{\partial}{\partial\xi^k}\left(g^{-1/2}\frac{\partial}{\partial\xi^m}(g^{1/2}u^m)\right) - \gamma g^{kl}\frac{\partial T}{\partial\xi^k} + \mu(g^{km}\hat{\nabla}_k\hat{\nabla}_m u^l + g^{kl}\hat{\nabla}_m\hat{\nabla}_k u^m) - \varrho\partial_t^2 u^l = F^l, \quad l = 1, 2, 3, \tag{6}$$

$$c_{\varepsilon_{ij}}\partial_t T - \Lambda g^{-1/2}\frac{\partial}{\partial\xi^l}\left(g^{1/2}g^{lm}\frac{\partial T}{\partial\xi^m}\right) + \gamma T g^{lm}\partial_t(\hat{\nabla}_l u_m + \hat{\nabla}_m u_l)/2 = \varrho Q^e, \tag{7}$$

where $g = \det(g_{kl})$ and g_{kl} are covariant components of the metric tensor g^{mn}.

Now, we assume that the coordinates ξ^l are rectangular and we set $\xi = x$. In these coordinates the Eqs. (6), (7) have the following form

$$\mu\Delta u + (\lambda+\mu)\operatorname{grad}\operatorname{div} u - \gamma\operatorname{grad} T - \varrho\partial_t^2 u = F, \tag{8}$$

$$\varkappa^{-1}\partial_t T - \Delta T + \eta T\operatorname{div}\partial_t u = \frac{\varrho}{\lambda}Q^e, \tag{9}$$

where $\varkappa = \Lambda/c_{\varepsilon_{ij}}$, $\eta = \gamma/\Lambda$.

These last equations were given by W. NOWACKI in [5]. From Shalov's concepts of continuum mechanics [7] it follows that the natural functional spaces in which one finds the solution of initial-boundary value problems is (cf. [7], p. 918, definition 3) the family of Sobolev's spaces H^s ($= B_{2,k}$ in the notation of [3], Chapter II, where $k_s(\xi)$ is the temperate weight function defined by $k_s(\xi) = (1+|\xi|^2)^{s/2}$).

For the Eqs. (8), (9) we consider the initial value problem in the half-space-time R_4^+ (cf. [6], p. 993) with the initial conditions

$$u(x, +0) = u^\circ(x), \quad (\partial_t u)(x, +0) = u^1(x), \quad T(x, +0) = T_0(x), \tag{10}$$

where u^0, u^1 are the given vector fields of classes(¹) $H^s(R_3, R_3)$ $H^{s-1}(R_3, R_3)$ respectively for $s > \frac{3}{2} + r$, r some positive integer $\geqslant 4$, and T is the given scalar function of class $H^s(R_3, R_1)$.

For the sake of simplicity we assume that the body forces F and the intensity Q^e vanish in R_4^+.

Under the foregoing assumption we seek the solution u, T of the initial value problem for the Eqs. (8), (9) with the conditions (10) in the class(²) $C([0, \vartheta], H^s)$.

(¹) We denote by $H^s(D, R_m)$ the space of maps from D to R_m of class H^s.

(²) We denote by $C(I, E)$ the space of continuous functions defined on the interval $I \subset R_1$ taking values in the Banach space E. The elements of $C(I, E)$ are called the continuous curves in E. Here $E = H^s$ means that $E = H^s(R_3, R_3)$ or $E = H^s(R_3, R_1)$.

Using Helmholtz's decomposition $u = v - \operatorname{grad}\phi$ we reduce (cf. [6], p. 994, formulae (4), (5), (6)) the initial value problem for the Eqs. (8), (9) with the conditions (10) under the assumptions $F = 0 = Q^e$ to the following initial value problems:

$$(11) \qquad L_a v = 0, \quad v(x, +0) = v^0(x), \quad (\partial_t v)(x, +0) = v^1(x),$$

$$(12) \qquad \varkappa^{-1}\partial_t T - \Delta T = \eta T \Delta \partial_t \phi, \quad T(x, +0) = T_0(x),$$

$$(13) \qquad L_b \phi = \frac{\gamma}{\varrho} T, \quad \phi(x, +0) = \varphi_0(x); \quad (\partial_t \phi)(x, +0) = \varphi_1(x).$$

Here L_j for $j = a, b$, the propagation speeds a, b of shear and compressional waves respectively, and the initial data φ_k, v^k for $k = 0, 1$ are defined by

$$(14) \qquad \begin{gathered} L_j = \partial_t^2 - j^2 \Delta, \quad a = (\mu/\varrho)^{1/2}, \quad b = \big((\lambda + 2\mu)/\varrho\big)^{1/2}, \\ \varphi_k(x) = -(4\pi)^{-1} \int_{R_3} |x-y|^{-1} (\operatorname{div} u^k)(y)\, dy, \quad v_k = u^k - \operatorname{grad}\varphi_k. \end{gathered}$$

Remark 1. The initial data φ_k for $k = 0, 1$ belong to the classes $H^{s+1}(\mathrm{R}_3, \mathrm{R}_1)$, $H^s(\mathrm{R}_3, \mathrm{R}_1)$ respectively. This follows from the assumptions on u^k and some integral representation of φ_k for $k = 0, 1$ (see [6], p. 994, formula (7) and [4], p. 31). The initial data v^k for $k = 0, 1$ belong to the classes $H^s(\mathrm{R}_3, \mathrm{R}_3)$, $H^{s-1}(\mathrm{R}_3, \mathrm{R}_3)$ respectively. This follows immediately from Helmholtz's decomposition and the regularity of u^0, u^1 and φ_0, φ_1.

The initial value problem (11) is the classical initial value problem for the wave equation (cf. [9], pp. 161–163, 168–190) and its solution takes the explicit form

$$(15) \qquad v = G_a *_3 v^1 + (\partial_t G_a) *_3 v^0.$$

Here $G_a(x, t) = (4\pi a^2 t)^{-1} H(t)\, \delta(at - |x|)$ is the fundamental solution for the wave equation $L_a v = 0$, H denotes Heaviside's function, δ is the one-dimensional Dirac delta distribution and $*_3$ denotes the three-dimensional convolution.

In order to solve the initial value problem for the Eqs. (12), (13) we may assume $b = 1$ without loss of generality, and then we reduce this problem to the following equivalent problem

$$(16) \qquad \partial_t U = A^j \partial_j U + \theta, \quad U(x, +0) = U^0(x),$$

$$(17) \qquad \varkappa^{-1}\partial_t T - \Delta T - (\eta \Delta U_4) T = 0, \quad T(x, +0) = T_0(l),$$

where (16) is the symmetric hyperbolic first order system (see [2], pp. 588–589) with the vector functions

$$U = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}, \quad \theta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\gamma}{\varrho} T \end{bmatrix}$$

and the given initial data

$$U^0 = \begin{bmatrix} \partial_1\varphi_0 \\ \partial_2\varphi_0 \\ \partial_3\varphi_0 \\ \varphi_1 \end{bmatrix}$$

here $\partial_j = \partial x_j$.

Let $\|\ \|_s$ denote the H^s-norm (cf. p. 674) for function U^0 defined on R_3 taking values in R_4, let X be the set of continuous curves (see, p. 674, the footnote(²)) $\Omega:[0, \vartheta] \to H^s(R_3, R_4)$ such that $\Omega(0) = U^0 \in H(R_3, R_4)$ and $\|\Omega(t) - U^0\|_s \leqslant M$ for $0 \leqslant t \leqslant \vartheta$. Thus, X is a complete metric space and we define S

$$X \ni \Omega \to (S\Omega)(t) = U^0 + \int_0^t A^j\partial_j(S\Omega)(s)\,ds + \int_0^t \begin{bmatrix} 0 \\ 0 \\ 0 \\ \left(\frac{\gamma}{\varrho} G_{\Delta\Omega_4} *_3 T_0\right)(s) \end{bmatrix} ds, \tag{18}$$

where the integration is done as a curve in $H^{s-1}(R_3, R_4)$ and $G_{\Delta\Omega_4}$ denotes the fundamental solution of the generalized heat equation $\varkappa^{-1}\partial_t T - \Delta T - (\eta\Delta\Omega_4)T = 0$.

R e m a r k 2. If a continuous curve $V:[0, \vartheta] \to L^q(R_N)$ for $q > N/2$ is given, then, fundamental solution G_V of the generalized heat equation $\partial_t T - \Delta T - VT = 0$ has the form

$$G_V(x, y, t) = \Gamma(x-y, t)\omega(x, y, t). \tag{i}$$

Here, Γ is the fundamental solution of the heat equation (cf. [6], p. 995 formula (12) for $\varkappa = 1$), belongs to $L^\infty(R_N) \otimes L^\infty(R_N) \otimes L^\infty(0, \vartheta)$ and satisfies the following integral equation:

$$\omega(x, y, t) - (\Gamma(x-y, t))^{-1} \int_0^t \int_{R_N} \Gamma(x-z, t-s)V(z, s)\Gamma(z-y, s)\omega(z, y, s)\,dz\,ds = 1. \tag{ii}$$

For this fundamental solution the following estimate holds:

$$\|G_{V_1}(x, \ldots, t) - G_{V_2}(x, \ldots, t)\|_{L^1} = C\|V_1(t) - V_2(t)\|_{L^q}. \tag{iii}$$

The proof of this remark is easy and quite the same as for the corresponding statements in Lemmas 1.1 and 1.2 of [1].

Using Young's inequality and the fundamental property (iii) for $V_1 = \Delta\Omega_4$, $V_2 = 0$, $q = 2$ we obtain

$$\|G_{\Delta\Omega_4} *_3 T)(t)\|_s \leqslant C\|\Delta\Omega_4(t)\|_{L^2}\|T_0\|_s. \tag{19}$$

Now, from the linear theory of first order symmetric hyperbolic systems it follows that there is such a unique map $S:X \to X$, namely for $\Omega \in X$ the unique solution W of the system

$$\partial_t W = A^j\partial_j W + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\gamma}{\varrho} G_{\Delta\Omega_4} *_3 T_0 \end{bmatrix}, \quad W(0) = U_0 \tag{20}$$

is exactly $S\Omega = W$ and belongs to X if ϑ is sufficiently small. In fact, for $\Omega \in X$ and $T_0 \in H^s(R_3, R_1)$ from the energy estimate (see [2], p. 647–650) of the solution W and the inequality (19) we conclude that S maps X into X if ϑ is sufficiently small.

Let Y be the completion of X with respect to the norm $\|\ \|_{s-1}$. Now, we note that by virtue of the energy estimate (see[2], p. 650, formula (12a)) and inequality [iii] the map $S: X \to X$ if ϑ is sufficiently small, is a contraction mapping in the H^{s-1}-topology, i.e., for Ω, $\Omega \in X$ and sufficiently small

$$\|(S\Omega)(t) - (S\tilde{\Omega})(t)\|_{s-1} \leqslant p\|\Omega(t) - \tilde{\Omega}(t)\|_{s-1} \tag{21}$$

with $p < 1$.

Thus S extends to concentration mapping on the complete metric space Y, therefore, by the contraction mapping principle S has a unique fixed point U in Y, i.e. $SU = U$, a solution in $C([0, \vartheta], H^{s-1})$ of the initial value problem for the Eqs. (16), (17) when $T = G_{\Delta U_4} *_3 T_0$. By standard technique (differentiation with respect to $x = (x_1, x_2, x_3)$ the Eqs. (16)) it can be easily seen that the fixed point U is in fact in $C[(0, \vartheta], H^s)$. From Helmholtz's decomposition, formula (15) and the existence of a fixed point of map S it is clear that the vector field u and the scalar function $T = G_{\mathrm{div}\,\partial_t u} *_3 T_0$ satisfy the Eqs. (8), (9) and the initial conditions (10). Then we deduce the following

Theorem 1. Let u°, u^1 be vector fields of classes $H^s(R_3, R_3)$ $H^{s-1}(R_3, R_3)$ respectively and let T_0 be a scalar function of class $H^s(R_3, R_1)$ for $s > \frac{3}{2} + r$, r some positive integer $\geqslant 4$. Assume that $F = 0 = Q^e$. Then there exist $\vartheta > 0$ and unique solution u, T of the initial value problem for the Eqs. (8), (9) with condition (10) in $C([0, \vartheta], H^s)$.

Remark 3. From our proof of Theorem 1 it follows immediately that: 1) the solution u, T depends continuously on u^0, u^1 T_0 in the H^s-topology, 2) if $r = \infty$, then u, T are C^∞-smooth.

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