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Thermodiffusion in micropolar elastic materials

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IN this paper we consider an elastic micropolar material subjected to the process of thermodiffusion. Non-linear constitutive equations for mechanical and thermodynamical quantities are derived from thermodynamic considerations for the studied model. Constitutive equations are linearized for isotropic materials and corresponding field equations are obtained.

W pracy niniejszej rozważany jest mikropolarny materiał sprężysty poddany procesowi termodyfuzji. Nieliniowe równania konstytutywne opisujące wielkości mechaniczne i termodynamiczne zostały wyprowadzone z rozważań termodynamicznych badanego modelu materiału. W przypadku materiałów izotropowych dokonano linearyzacji równań konstytutywnych i otrzymano odpowiednie równania pola.

В настоящей работе обсуждается микрополярный упругий материал подвергнутый процессу термодиффузии. Нелинейные определяющие уравнения, описывающие механические и термодинамические величины, выведены из термодинамических рассуждений исследуемой модели материала. В случае изотропных материалов проведена линеаризация уравнения поля.

1. Introduction

THE problem of thermodiffusion in elastic solids of microstructure was studied in papers [1 and 2]. In these papers it was assumed that there is non-homogeneous distribution of temperature and of chemical potential inside the macroelement. According an influence of microtemperature and of chemical micropotential appeared on the state of the body.

In the continuum theory of micropolar materials every material point is phenomenologically equivalent to a rigid body. Considering thermodiffusion in such materials we assume the temperature and the chemical potential to be homogeneous inside the macroelement. Thus from the point of view of this theory, the distribution of temperature, as well as of chemical potential, is determined at every moment by only one function of position.

The mechanical model of micropolar materials was introduced by SUHUBI and ERINGEN [3]. The theory was developed further by ERINGEN in papers [4, 5 and 6], but did not extend beyond the domain of linearity. A similar theory, based on independent rotations of material points, was suggested by AERO and KUVSHINSKII [7 and 8]. Later KAFADAR and ERINGEN formulated the non-linear theory of micropolar elastic materials [9]. In the paper [10] the micropolar elastic material is considered as an elastic Cosserat continuum.

The process of thermodiffusion results from the non-uniformity of the temperature distribution in the body. Assuming that no chemical reactions occur during the process of thermodiffusion, the law of conservation of mass is taken to be valid. In the case of a classical elastic material, the problem of thermodiffusion was treated by PODSTRIGAČ [11] and PODSTRIGACZ and PAVLINA [12 and 13], and further by NOWACKI [14 and 15]. The linear theory for coupled mechanical and thermodiffusional effects for elastic materials of grade two was derived by NAERLOVIĆ-VELJKOVIĆ [16].

2. Kinematics

The motion of a simple micropolar continuum is determined by the equations:

$$x^\varkappa = x^\varkappa(X^K, t), \quad \chi^\varkappa_{.K} = \chi^\varkappa_{.K}(X^L, t), \tag{2.1}$$

where $\chi^\varkappa_K$ stands for the orthogonal tensor expressing independent rotations of material points of the body.

Interpreting the material as a simple Cosserat continuum, we obtain the following equations of motion [10]:

$$x^\varkappa = x^\varkappa(X^K, t), \quad d^\varkappa_{.(\alpha)} = d^\varkappa_{.(\alpha)}(X^K, D^K_{.(\alpha)}, t), \tag{2.2}$$

where $d^\varkappa_{.(\alpha)}$ and $D^K_{.(\alpha)}$ are triads of directors in the deformed and underformed configutations respectively, and where

$$d^\varkappa_{.(\alpha)} = \chi^\varkappa_{.K} D^K_{.(\alpha)}. \tag{2.3}$$

Since the motion of directors represents a rigid motion, we can write:

$$g_{\varkappa l} d^\varkappa_{.(\alpha)} d^l_{.(\beta)} = G_{KL} D^K_{.(\alpha)} D^L_{.(\beta)} = C_{\alpha\beta} = \text{const}, \tag{2.4}$$

wherefrom, using (2.3),

$$g_{\varkappa l} \chi^\varkappa_{.K} \chi^l_{.L} = G_{KL}, \quad G^{KL} \chi_{\varkappa K} \chi_{lL} = g_{\varkappa l}. \tag{2.5}$$

However, there must be:

$$|\chi^\varkappa_{.K}| = +\sqrt{\frac{G}{g}}, \quad G = |G_{KL}|, \quad g = |g_{\varkappa l}|, \tag{2.6}$$

when rotation is described.

The velocity of a point of a macroelement can be expressed in the form:

$$v'^\varkappa = v^\varkappa + \nu^\varkappa_{.l} d^l, \tag{2.7}$$

where $v^\varkappa$ is the velocity of the mass center of the macroelement, d^l is the position vector of the point of the macroelement originating at the macroelement mass center and

$$\nu_{\varkappa l} = \dot{d}_{\varkappa(\alpha)} d^{(\alpha)}_{.l} = \dot{\chi}_{\varkappa K} \chi^K_{.l} \tag{2.8}$$

is the giration tensor. From $(2.5)_2$, after differentiation with respect to time, we get:

$$\nu_{\varkappa l} = \dot{\chi}_{\varkappa K} \chi^K_{.l} = -\dot{\chi}_{lK} \chi^K_{.\varkappa} = -\nu_{l\varkappa}. \tag{2.9}$$

We see that $\nu_{\varkappa l}$ is a skew-symmetric tensor, so that it has three mutually independent coordinates. The tensors $\chi_{\varkappa K}$ and $\chi_{K\varkappa}$ are mutually reciprocal, as well as the triads $d^\varkappa_{.(\alpha)}$ and $d^{(\alpha)}_{.\varkappa}$.

3. The equation of energy balance and equations of motion

The global form of the energy balance equation is given by

$$(3.1)\qquad \int_v \varrho(\dot{v}^\varkappa v_\varkappa + I^{\alpha\beta}\ddot{d}^{\varkappa}_{.(\alpha)}\dot{d}_{\varkappa(\beta)})dv + \int_v \varrho\dot{u}dv = \oint_s (T^i v_i + H^{i(\alpha)}\dot{d}_{i(\alpha)})ds$$
$$+\int_v \varrho(f^i v_i + l^{i(\alpha)}\dot{d}_{i(\alpha)})dv + \oint_s q^\varkappa ds_\varkappa + \int_v \varrho h dv,$$

or, after an identical transformation,

$$(3.2)\qquad \int_s \varrho(\dot{v}^\varkappa v_\varkappa + \Gamma^{ij}\nu_{ij})dv + \int_v \varrho\dot{u}dv = \oint_s (T^i ds - t^{i\varkappa}ds_\varkappa)v_i$$
$$+\oint_s (H^{ij}ds - m^{ij\varkappa}ds_\varkappa)\nu_{ij} + (qds - q^\varkappa ds_\varkappa) + \int_v \varrho(f^i v_i + l^{ij}\nu_{ij})dv$$
$$+\int_v (t^{i\varkappa}_{.,\varkappa}v_i + t^{i\varkappa}v_{i,\varkappa} + m^{ij\varkappa}_{,\varkappa}\nu_{ij} + q^\varkappa_{,\varkappa} + \varrho h)dv,$$

where

$$(3.3)\qquad \Gamma^{ij} = -\Gamma^{ji} = I^{\alpha\beta}\ddot{d}^{[i}_{.(\alpha)}d^{j]}_{.(\beta)} = I^{KL}\ddot{\chi}^{[i}_{.K}\chi^{j]}_{.L}$$

stands for the inertial spin and where

$$(3.4)\qquad \varrho I^{\alpha\beta}dv = D^{(\alpha)}_{.K}D^{(\beta)}_{.L}\int_{dv}\varrho_0' D^K D^L dv', \qquad I^{\alpha\beta} = I^{KL}D^{(\alpha)}_{.K}D^{(\beta)}_{.L} = i^{\varkappa l}d^{(\alpha)}_{.\varkappa}d^{(\beta)}_{.l},$$

are the director coefficients of inertia.

In (3.2) the tensor $H^{ij} = -H^{ji}$ represents *the surface couple*, the tensor $l^{ij} = -l^{ji} = l^{i(\alpha)}d^{j}_{.(\alpha)}$ is *the body couple*, $m^{ij\varkappa} = -m^{ji\varkappa}$ is *the couple stress tensor*, u — *the internal energy density*, T^i — *the stress vector*, f^i — *the body force density*, q — *the heat influx*, $q^\varkappa$ — *the heat flux vector* and h — *the heat supply density*.

Requiring the invariance of Eq. (3.2) with respect to superposed rigid body motions and taking into account the boundary conditions

$$(3.5)\qquad T^i = t^{ij}n_j, \quad H^{ij} = m^{ij\varkappa}n_\varkappa, \quad q = q^\varkappa n_\varkappa,$$

we get the equations of motion and the equation for internal energy balance in the following form:

$$(3.6)\qquad \begin{aligned} \varrho\dot{v}^i &= t^{ij}_{,j} + \varrho f^i,\\ \Gamma^{ij} &= t^{[ij]} + m^{ij\varkappa}_{.,\varkappa} + \varrho l^{ij}\\ \varrho\dot{u} &= t^{ij}v_{i,j} - t^{[ij]}\nu_{ij} + m^{ji\varkappa}\nu_{ij,\varkappa} + q^\varkappa_{,\varkappa} + \varrho h. \end{aligned}$$

4. The dissipation function and the thermodynamic forces

The volume concentration of diffused mass at a point of the body is determined by the function $c(x^\varkappa, t)$. We assume that there are no body sources of mass production and, denoting by $J^\varkappa$ the flux vector of the diffused mass, we obtain the local balance equation of diffused mass in the form:

$$(4.1)\qquad \dot{c} = \frac{1}{\varrho}J^\varkappa_{,\varkappa}.$$

In the case of a mechanically reversible model of behaviour, the entropy balance equation may be presented by

$$\varrho\theta\dot{\eta} = \varrho h + q^{\varkappa}_{,\varkappa} - \varrho M\dot{c}, \tag{4.2}$$

or, taking into account (4.1),

$$\varrho\theta\dot{\eta} = \varrho h + q^{\varkappa}_{,\varkappa} - MJ^{\varkappa}_{,\varkappa}, \tag{4.3}$$

where θ and M are absolute temperature and chemical potential at a point of the body.

Further, we express Eq. (4.3) as

$$\varrho\dot{\eta} = \frac{\varrho h}{\theta} + \left(\frac{q^{\varkappa}}{\theta}\right)_{,\varkappa} - \left(\frac{M}{\theta}J^{\varkappa}\right)_{,\varkappa} + \frac{q^{\varkappa}\theta_{,\varkappa}}{\theta^2} + \left(\frac{M}{\theta}\right)_{,\varkappa} J^{\varkappa}. \tag{4.4}$$

The first three terms on the right-hand side represent the reversible part of entropy production

$$\varrho\dot{\eta}_{(r)} = \frac{\varrho h}{\theta} + \left(\frac{q^{\varkappa}}{\theta}\right)_{,\varkappa} - \left(\frac{M}{\theta}J^{\varkappa}\right)_{,\varkappa}. \tag{4.5}$$

The entropy production due to the existence of irreversible processes in the body is connected with heat transfer and diffusion:

$$\varrho\sigma = \frac{q^{\varkappa}\theta_{,\varkappa}}{\theta^2} + \left(\frac{M}{\theta}\right)_{,\varkappa} J^{\varkappa}. \tag{4.6}$$

Hence we find the following expression for the dissipation function:

$$\varrho\Phi = \frac{q^{\varkappa}\theta_{,\varkappa}}{\theta} + \theta\left(\frac{M}{\theta}\right)_{,\varkappa} J^{\varkappa} \tag{4.7}$$

and, according to the second law of thermodynamics, the following inequalities take place:

$$\sigma \geqslant 0, \quad \Phi \geqslant 0. \tag{4.8}$$

The dissipation function (4.7) can be presented in the form:

$$\varrho\Phi = Q_{(a)}\dot{q}^{(a)}, \tag{4.9}$$

where $Q_{(a)}$ are irreversible thermodynamic forces and $\dot{q}^{(a)}$ — corresponding generalized velocities. In our case the thermodynamic forces are

$$Q_{(a)} = \left\{\frac{\theta_{,\varkappa}}{\theta}, \theta\left(\frac{M}{\theta}\right)_{,\varkappa}\right\}, \tag{4.10}$$

and hence the corresponding generalized velocities become

$$\dot{q}^{(a)} = \{q^{\varkappa}, J^{\varkappa}\}. \tag{4.11}$$

Using Onsager's constitutive equations [17, 18], we obtain linear relations between generalized velocities and irreversible thermodynamic forces:

$$\dot{q}^{(a)} = L^{(ab)}Q_{(b)}, \tag{4.12}$$

where $L^{(ab)} = L^{(ba)}$ are Onsager's phenomenological coefficients.

From (4.12), using (4.10) and (4.11), we find:

$$q_\varkappa = L^{11}\frac{\theta_{,\varkappa}}{\theta} + L^{12}\theta\left(\frac{M}{\theta}\right)_{,\varkappa}, \tag{4.13}$$
$$J_\varkappa = L^{21}\frac{\theta_{,\varkappa}}{\theta} + L^{22}\theta\left(\frac{M}{\theta}\right)_{,\varkappa},$$

with $L^{12} = L^{21}$ and

$$L = \begin{vmatrix} L^{11} & L^{12} \\ L^{21} & L^{22} \end{vmatrix} > 0. \tag{4.14}$$

The non-linear relations between irreversible thermodynamic forces and generalized velocities can be obtained by using principle of the least irreversible force which was suggested by ZIEGLER [19]. In that way we get the following expression for irreversible thermodynamic forces:

$$Q_{(a)} = \left(\frac{\partial\Phi}{\partial\dot{q}^{(b)}}\dot{q}^{(b)}\right)^{-1}\varrho\Phi\frac{\partial\Phi}{\partial\dot{q}^{(a)}}, \tag{4.15}$$

where the dissipation function depends on the following arguments:

$$\Phi = \Phi\left[\frac{\theta_{,\varkappa}}{\theta}, \theta\left(\frac{M}{\theta}\right)_{,\varkappa}\right], \tag{4.16}$$

which now we consider as generalized velocities. From (4.15), taking into account (4.16), we determine the non-linear constitutive equations for thermodynamic forces:

$$q^\varkappa = \lambda\frac{\partial\Phi}{\partial\left(\dfrac{\theta_{,\varkappa}}{\theta}\right)}, \quad J^\varkappa = \lambda\frac{\partial\Phi}{\partial\left[\theta\left(\dfrac{M}{\theta}\right)_{,\varkappa}\right]}, \tag{4.17}$$

where

$$\lambda = \varrho\Phi\left\{\frac{\partial\Phi}{\partial\left(\dfrac{\theta_{,\varkappa}}{\theta}\right)}\frac{\theta_{,\varkappa}}{\theta} + \frac{\partial\Phi}{\partial\left[\theta\left(\dfrac{M}{\theta}\right)_{,\varkappa}\right]}\theta\left(\frac{M}{\theta}\right)_{,\varkappa}\right\}^{-1}. \tag{4.18}$$

5. Free energy and constitutive equations

Introducing the free energy density at a point of the body

$$\psi = u - \theta\eta, \tag{5.1}$$

we get

$$\varrho\dot{\psi} = \varrho\dot{u} - \varrho\theta\dot{\eta} - \varrho\eta\dot{\theta}. \tag{5.2}$$

Making use of $(3.6)_3$ and (4.2), we present (5.2) in the form

$$\varrho\dot{\psi} = t^{ij}v_{i,j} - t^{[ij]}\nu_{ij} + m^{ij\varkappa}\nu_{ij,\varkappa} - \varrho\eta\dot{\theta} + \varrho M\dot{c}. \tag{5.3}$$

Next, we rewrite this equation in the following form:

$$\varrho\dot{\psi} = t^{ij}X^K_{;j}\dot{x}_{i;K} - (t^{[ij]}d^{(\alpha)}_{.j} - m^{ij\varkappa}d^{(\alpha)}_{.j;K}X^K_{;\varkappa})\dot{d}_{i(\alpha)} + m^{ij\varkappa}d^{(\alpha)}_{.j}X^K_{;\varkappa}\dot{d}_{i(\alpha);K} - \varrho\eta\dot{\theta} + \varrho M\dot{c}. \tag{5.4}$$

and we conclude that the free energy depends on the arguments below:

$$\psi = \psi(x^{\varkappa}_{;K}, d^{\varkappa}_{.(\alpha)}, d^{\varkappa}_{.(\alpha);K}, \theta, c). \tag{5.5}$$

After differentiation (5.5) with respect to the time and comparing with (5.4), we obtain the non-linear constitutive equations for anisotropic micropolar elastic materials in the coupled process of mechanical and thermodiffusional effects:

$$\begin{aligned} t^{ij} &= \varrho g^{il} \frac{\partial\psi}{\partial x^l_{;K}} x^j_{;K}, \\ t^{[ij]} &= -\varrho \frac{\partial\psi}{\partial d^l_{.(\alpha)}} g^{l[i} d^{j]}_{.(\alpha)} - \varrho \frac{\partial\psi}{\partial d^l_{.(\alpha);K}} g^{l[i} d^{j]}_{.(\alpha);K}, \\ m^{ij\varkappa} &= \varrho \frac{\partial\psi}{\partial d^l_{.(\alpha);K}} g^{l[i} d^{j]}_{.(\alpha)} x^{\varkappa}_{;K}, \quad \eta = -\frac{\partial\psi}{\partial\theta}, \quad M = \frac{\partial\psi}{\partial c}. \end{aligned} \tag{5.6}$$

However, equations $(5.6)_1$ and $(5.6)_2$ must be in accordance, i.e.,

$$\left(g^{il} \frac{\partial\psi}{\partial x^l_{;K}} x^j_{;K} + g^{il} \frac{\partial\psi}{\partial d^l_{.(\alpha)}} d^j_{.(\alpha)} + g^{il} \frac{\partial\psi}{\partial d^l_{.(\alpha);K}} d^j_{.(\alpha);K}\right)_{[ij]} = 0. \tag{5.7}$$

The last expression represents *the condition of objectivity* of the free energy density (5.5) and the constitutive equations (5.6).

If the condition (5.7) is satisfied, we find Eq. $(5.6)_2$ to be superfluous. Namely, in that case $(5.6)_2$ is included in $(5.6)_1$ as its skew-symmetric part.

Putting $d^{\varkappa}_{.(\alpha)} = \chi^{\varkappa}_{.K} D^K_{.(\alpha)}$ and $d^{\varkappa}_{.(\alpha);K} = \chi^{\varkappa}_{.L;K} D^L_{.(\alpha)}$, we present the free energy density in the following form:

$$\psi = \psi(x^{\varkappa}_{;K}, \chi^{\varkappa}_{.L}, \chi^{\varkappa}_{.L;K}, \theta, c). \tag{5.8}$$

The set of equations (5.6) is now replaced by

$$\begin{aligned} t^{ij} &= \varrho g^{il} \frac{\partial\psi}{\partial x^l_{;K}} x^j_{;K}, \\ m^{ij\varkappa} &= \varrho \frac{\partial\psi}{\partial \chi^l_{.L;K}} g^{l[i} \chi^{j]}_{.L} x^{\varkappa}_{;K}, \\ \eta &= -\frac{\partial\psi}{\partial\theta}, \quad M = \frac{\partial\psi}{\partial c}. \end{aligned} \tag{5.9}$$

The condition of objectivity (5.7) then becomes:

$$\left(g^{il} \frac{\partial\psi}{\partial x^l_{;K}} x^j_{;K} + g^{il} \frac{\partial\psi}{\partial \chi^l_{.L}} \chi^j_{.L} + \frac{\partial\psi}{\partial \chi^l_{.L;K}} \chi^j_{.L;K}\right)_{[ij]} = 0. \tag{5.10}$$

The free energy density (5.8) is a function of 23 independent variables $x^{\varkappa}_{;K}$, $\chi^{\varkappa}_{.L}$, $\chi^{\varkappa}_{.L;K}$, θ and c. Since (5.10) represents a system of three linear partial differential equations, it admits $23-3 = 20$ independent integrals. We choose following integrals [10]:

$$\Sigma_{KL} = \chi_{K\varkappa} x^{\varkappa}_{;L}, \quad K_{KLM} = \chi_{K\varkappa} x^{\varkappa}_{.L;M}; \quad \theta, \ c, \tag{5.11}$$

which can be rewritten in the form:

$$\Sigma_{KL} = \chi_{\varkappa K} x^{\varkappa}_{;L}, \quad K_{KLM} = \chi_{\varkappa K} \chi^{\varkappa}_{.L;M} = -\chi_{\varkappa L} \chi^{\varkappa}_{.K;M}; \theta, c, \tag{5.12}$$

having in mind that $\chi^{\varkappa}_{.K}$ is an orthogonal tensor. Thus, the general solution for free energy density is given by

$$\psi = \psi(\Sigma_{KL}, K_{KLM}, \theta, c). \tag{5.13}$$

Substituting now (5.13) into (5.9) and using (5.12), we obtain non-linear constitutive equations for anisotropic materials:

$$\begin{aligned} t^{ij} &= \varrho \frac{\partial \psi}{\partial \Sigma_{KL}} \chi^{i}_{.K} x^{j}_{;L}, \\ m^{ij\varkappa} &= \varrho \frac{\partial \psi}{\partial K_{KLM}} \chi^{i}_{.K} \chi^{j}_{.L} x^{\varkappa}_{;M}, \\ \eta &= -\frac{\partial \psi}{\partial \theta}, \quad M = \frac{\partial \psi}{\partial c}, \end{aligned} \tag{5.14}$$

which are form-invariant with respect to the superposed rigid body motion.

Introducing the following measure of deformation

$$\varepsilon_{KL} = \Sigma_{KL} - G_{KL} = \chi_{\varkappa K} x^{\varkappa}_{;L} - G_{KL}, \tag{5.15}$$

the constitutive equation $(5.14)_1$ reduces to

$$t^{ij} = \varrho \frac{\partial \psi}{\partial \varepsilon_{KL}} \chi^{i}_{.K} x^{j}_{;L}. \tag{5.16}$$

If we introduce the director displacement vectors $\varphi^{\varkappa}_{.(\alpha)}$:

$$d^{\varkappa}_{.(\alpha)} = D^{\varkappa}_{.(\alpha)} + \varphi^{\varkappa}_{.(\alpha)}, \quad D^{K}_{.(\alpha)} = d^{K}_{.(\alpha)} - \varphi^{K}_{.(\alpha)},$$

after multiplying by $D^{(\alpha)}_{.K}$ and $d^{(\alpha)}_{.\varkappa}$ respectively, we obtain

$$\begin{aligned} \chi^{\varkappa}_{.K} &= g^{\varkappa}_{K} + \varphi^{\varkappa}_{.K} \quad (\varphi^{\varkappa}_{.K} = \varphi^{\varkappa}_{.(\alpha)} D^{(\alpha)}_{.K}), \\ \chi^{K}_{.\varkappa} &= g^{K}_{\varkappa} - \varphi^{K}_{.\varkappa} \quad (\varphi^{K}_{.\varkappa} = \varphi^{K}_{.(\alpha)} d^{(\alpha)}_{.\varkappa}), \end{aligned} \tag{5.17}$$

where $\varphi^{\varkappa}_{.K}$ and $\varphi^{K}_{.\varkappa}$ are *micro-displacement gradients*. Making use of (5.17), we get from $(2.5)_1$

$$\varphi_{KL} + \varphi_{LK} + \varphi_{MK} \varphi^{M}_{.L} = 0 \quad (\varphi_{KL} = g^{\varkappa}_{K} \varphi_{\varkappa L}). \tag{5.18}$$

We see that in the linear approximation (5.18) leads to

$$\varphi_{KL} + \varphi_{LK} = 0. \tag{5.19}$$

Since $\chi_{\varkappa K} = \chi_{K\varkappa}$ $(\chi^{T} = \chi^{-1})$, using (5.17) we also obtain

$$\begin{aligned} \varphi_{\varkappa K} &= -\varphi_{K\varkappa}, \\ \varphi_{KL} &= -\varphi_{L\varkappa} g^{\varkappa}_{K} = -\varphi_{l\varkappa} g^{l}_{L} g^{\varkappa}_{K}, \\ \varphi_{\varkappa l} &= -\varphi_{lK} g^{K}_{\varkappa} = -\varphi_{LK} g^{L}_{l} g^{K}_{\varkappa}. \end{aligned} \tag{5.20}$$

If we introduce displacement gradients by

$$x^{\varkappa}_{;K} = g^{\varkappa}_{K} + u^{\varkappa}_{;K}, \quad X^{K}_{;\varkappa} = g^{K}_{\varkappa} - u^{K}_{;\varkappa}$$

and after substituting (5.17) and (5.18) into $(5.12)_2$ and (5.15), we may express the deformation tensors ε_{KL} and K_{KLM} as follows

$$\begin{aligned} \varepsilon_{KL} &= u_{K,L}+\varphi_{LK}+\varphi_{MK}u^{M}_{,L} = u_{K,L}-\varphi_{KL}+(u_{M,L}-\varphi_{ML})\varphi^{M}_{.K}, \\ K_{KLM} &= \varphi_{KL,M}+\varphi_{SK}\varphi^{S}_{.L,M} = -\varphi_{LK,M}-\varphi_{SL}\varphi^{S}_{.K,M}. \end{aligned} \tag{5.21}$$

In the linear theory these tensors become

$$\varepsilon_{KL} = u_{K,L}+\varphi_{LK} = u_{K,L}-\varphi_{KL}, \quad K_{KLM} = \varphi_{KL,M} = -\varphi_{LK,M}. \tag{5.22}$$

In the case of isotropic materials, we can introduce the following spatial tensors:

$$\varepsilon_{\varkappa l} = g_{\varkappa l}-\chi_{\varkappa l}X^{K}_{;l}, \quad \varkappa_{\varkappa lm} = G^{KL}\chi_{\varkappa K;m}\,\chi_{lL} = -\varkappa_{l\varkappa m}, \tag{5.23}$$

so that, instead (5.13), we find

$$\psi = \psi(\varepsilon_{\varkappa l}, \varkappa_{\varkappa lm}, \theta, c). \tag{5.24}$$

Substituting (5.24) into (5.9) and making use of (5.23), we get the non-linear constitutive equations for isotropic materials in the form:

$$\begin{aligned} t^{ij} &= \varrho\frac{\partial\psi}{\partial\varepsilon_{ij}}-\varrho\frac{\partial\psi}{\partial\varepsilon_{\varkappa j}}\varepsilon^{\cdot i}_{\varkappa}-\varrho\frac{\partial\psi}{\partial\varkappa_{\varkappa lj}}\varkappa^{\cdot\cdot i}_{\varkappa l}, \\ m^{ij\varkappa} &= \partial\frac{\psi\partial}{\partial\varkappa_{ij\varkappa}}, \quad \eta = -\frac{\partial\psi}{\partial\theta}, \quad M = \frac{\partial\psi}{\partial c}. \end{aligned} \tag{5.25}$$

The condition of objectivity (5.10) now reads:

$$\left(\frac{\partial\psi}{\partial\varepsilon_{i\varkappa}}\varepsilon^{\cdot j}_{\varkappa}-\frac{\partial\psi}{\partial\varepsilon_{\varkappa j}}\varepsilon^{\cdot i}_{\varkappa}+2\frac{\partial\psi}{\partial\varkappa_{ilm}}\varkappa^{j}_{.lm}-\frac{\partial\psi}{\partial\varkappa_{\varkappa lj}}\varkappa^{\cdot\cdot i}_{\varkappa l}\right)_{[ij]} = 0. \tag{5.26}$$

From $(2.5)_2$, using (5.17) and $(5.20)_3$, we obtain

$$\varphi_{\varkappa l}+\varphi_{l\varkappa}-\varphi_{m\varkappa}\varphi^{m}_{.l} = 0. \tag{5.27}$$

Neglecting the non-linear part of (5.27), we see that in the linear theory

$$\varphi_{\varkappa l}+\varphi_{l\varkappa} = 0. \tag{5.28}$$

Making use of (5.23), (5.17), $(5.20)_2$ and (5.27), we can express the deformation tensors $\varepsilon_{\varkappa l}$ and $\varkappa_{\varkappa lm}$ in the form

$$\begin{aligned} \varepsilon_{\varkappa l} &= u_{\varkappa,l}+\varphi_{l\varkappa}-\varphi_{m\varkappa}u^{m}_{,l} = u_{\varkappa,l}-\varphi_{\varkappa l}-(u_{m,l}-\varphi_{ml})\varphi^{m}_{.\varkappa}, \\ \varkappa_{\varkappa lm} &= \varphi_{\varkappa l,m}-\varphi_{r\varkappa}\varphi^{r}_{.l,m} = -\varphi_{l\varkappa,m}+\varphi_{rl}\varphi^{r}_{.\varkappa,m}. \end{aligned} \tag{5.29}$$

In the linear theory these tensors become:

$$\varepsilon_{\varkappa l} = u_{\varkappa,l}+\varphi_{l\varkappa} = u_{\varkappa,l}-\varphi_{\varkappa l}, \quad \varkappa_{\varkappa lm} = \varphi_{\varkappa l,m} = -\varphi_{l\varkappa,m}. \tag{5.30}$$

Disregarding the non-linear terms in (5.25), we obtain the following constitutive equations for the linear theory:

$$t^{ij} = \varrho\frac{\partial\psi}{\partial\varepsilon_{ij}}, \quad m^{ij} = \varrho\frac{\partial\psi}{\partial\varkappa_{ij}}, \quad \eta = -\frac{\partial\psi}{\partial\theta}, \quad M = \frac{\partial\psi}{\partial c}, \tag{5.31}$$

where

$$\varkappa_{ij} = \frac{1}{2}\varepsilon_{i\varkappa l}\varkappa^{\varkappa l}_{..j}, \quad m^{ij} = \frac{1}{2}\varepsilon^{i\varkappa l}m^{\cdot\cdot j}_{\varkappa l}, \tag{5.32}$$

and

$$\varepsilon_{ij} = u_{i,j} - \varphi_{ij} = u_{i,j} - \varepsilon_{ij\varkappa}\varphi^{\varkappa}, \quad \varkappa_{ij} = \frac{1}{2}\varepsilon_{i\varkappa l}\varphi^{\varkappa l}{}_{,j} = \varphi_{i,j}. \tag{5.33}$$

The free energy density is a function of the form $\psi = \psi(\varepsilon_{ij}, \varkappa_{ij}, \theta, c)$. However, if we put $\theta = \theta_0 + T$ and $c = c_0 + C$, where T and C represent increases of temperature and concentration with respect to some reference values θ_0 and c_0, then we have $\psi = \psi(\varepsilon_{ij}, \varkappa_{ij}, T, C)$.

Supposing $\psi = 0$, $t^{ij} = 0$, $m^{ij} = 0$, $\eta = 0$ and $M = 0$ for $\varepsilon_{ij} = 0$, $\varkappa_{ij} = 0$, $\theta = \theta_0$ and $c = c_0$, then in the case of infinitesimal deformations and for small temperature and concentration changes, i.e., for

$$\left|\frac{T}{\theta_0}\right| \ll 1, \quad \left|\frac{C}{c_0}\right| \ll 1,$$

the free energy density is a quadratic polynomial of the form

$$\varrho\psi = \frac{1}{2}A^{ij\varkappa l}\varepsilon_{ij}\varepsilon_{\varkappa l} + \frac{1}{2}B^{ij\varkappa l}\varkappa_{ij}\varkappa_{\varkappa l} + C^{ij}\varepsilon_{ij}T + D^{ij}\varepsilon_{ij}C + \frac{1}{2}mT^2 + \frac{1}{2}nC^2 + pTC, \tag{5.34}$$

where

$$\begin{aligned} A^{ij\varkappa l} &= \nu_1 g^{ij}g^{\varkappa l} + \nu_2 g^{i\varkappa}g^{jl} + \nu_3 g^{il}g^{j\varkappa}, \\ B^{ij\varkappa l} &= \nu_4 g^{ij}g^{\varkappa l} + \nu_5 g^{i\varkappa}g^{jl} + \nu_6 g^{il}g^{j\varkappa}, \\ C^{ij} &= \nu_7 g^{ij}, \quad D^{ij} = \nu_8 g^{ij}, \end{aligned} \tag{5.35}$$

are isotropic tensors and $m, n, p, \nu_1, \ldots, \nu_8$ are material constants.

Using now (5.34) and (5.35), and putting $\varrho \approx \varrho_0(1 - u^{\varkappa}_{,\varkappa})$, we obtain from (5.31) the linear constitutive equations in the form:

$$\begin{aligned} t^{ij} &= (\nu_1\varepsilon_I + \nu_7 T + \nu_8 C)g^{ij} + \nu_2\varepsilon^{ij} + \nu_3\varepsilon^{ji}, \\ m^{ij} &= \nu_4\varkappa_I g^{ij} + \nu_5\varkappa^{ij} + \nu_6\varkappa^{ji}, \\ \varrho_0\eta &= -mT - pC - \nu_7\varepsilon_I, \\ \varrho_0 M &= pT + nC + \nu_8\varepsilon_I, \end{aligned} \tag{5.36}$$

where $\varepsilon_I = u^{\varkappa}_{,\varkappa} = \operatorname{div}\mathbf{u}$ and $\varkappa_I = \varphi^{\varkappa}_{,\varkappa} = \operatorname{div}\boldsymbol{\varphi}$ are first invariants of the tensors ε_{ij} and $\varkappa_{ij}$ respectively.

Finally, if we put

$$\varepsilon_{ij} = u_{i,j} - \varphi_{ij} = e_{ij} + r_{ij} - \varphi_{ij} = e_{ij} + \varepsilon_{ij\varkappa}(r^{\varkappa} - \varphi^{\varkappa}), \tag{5.37}$$

where

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad r_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}), \quad r_{ij} = \varepsilon_{ij\varkappa}r^{\varkappa}, \quad \varphi_{ij} = \varepsilon_{ij\varkappa}\varphi^{\varkappa}, \tag{5.38}$$

we get the following equivalent form of the constitutive equations:

$$\begin{aligned} t^{ij} &= (\lambda e_I + \tau_1 T + \tau_2 C)g^{ij} + 2\mu e^{ij} + \nu\varepsilon^{ij\varkappa}(r_{\varkappa} - \varphi_{\varkappa}), \\ m^{ij} &= \nu_1\varphi^{\varkappa}_{,\varkappa}g^{ij} + \nu_2\varphi^{i,j} + \nu_3\varphi^{j,i}, \\ \varrho_0\eta &= -mT - pC - \tau_1 e_I, \\ \varrho_0 M &= pT + nC + \tau_2 e_I, \end{aligned} \tag{5.39}$$

where λ and μ are classical Lamé's constants. The total number of material constants, including ϱ_0, is 12.

6. The field equations

Substituting into (4.13) the values $\theta = \theta_0 + T$ and $c = c_0 + C$, we find following linear constitutive equations for the heat flux vector and for the flux vector of the diffused mass:

$$q_\varkappa = \frac{L^{11}}{\theta_0} T_{,\varkappa} + L^{12} M_{,\varkappa}, \quad J_\varkappa = \frac{L^{21}}{\theta_0} T_{,\varkappa} + L^{22} M_{,\varkappa}. \tag{6.1}$$

Taking into account the constitutive equation $(5.39)_4$, and using (5.38), we obtain the generalized forms of Fourier's law and, for Fick's law for isotropic materials in linear theory,

$$\begin{aligned} q_\varkappa &= \left(\frac{L^{11}}{\theta_0} + \frac{L^{12}}{\varrho_0} p\right) T_{,\varkappa} + \frac{L^{12}}{\varrho_0} n C_{,\varkappa} + \frac{L^{12}}{\varrho_0} \tau_2 \Delta u_\varkappa, \\ J_\varkappa &= \left(\frac{L^{21}}{\theta_0} + \frac{L^{22}}{\varrho_0} p\right) T_{,\varkappa} + \frac{L^{22}}{\varrho_0} n C_{,\varkappa} + \frac{L^{22}}{\varrho_0} \tau_2 \Delta u_\varkappa. \end{aligned} \tag{6.2}$$

Introducing new constants

$$\alpha = \frac{L^{11}}{\theta_0} + \frac{L^{12}}{\varrho_0} p, \quad \beta = \frac{L^{12}}{\varrho_0} n, \quad \gamma = \frac{L^{12}}{\varrho_0} \tau_2, \tag{6.3}$$

we may present equations (6.2) in the form

$$\begin{aligned} q_\varkappa &= \alpha T_{,\varkappa} + \beta C_{,\varkappa} + \gamma \Delta u_\varkappa, \\ J_\varkappa &= \frac{L^{22}}{L^{12}} \left[\left(\alpha - \frac{L}{\theta_0 L^{22}}\right) T_{,\varkappa} + \beta C_{,\varkappa} + \gamma \Delta u_\varkappa\right], \quad L = \begin{vmatrix} L^{11} & L^{12} \\ L^{21} & L^{22} \end{vmatrix}. \end{aligned} \tag{6.4}$$

Making use of the constitutive equations $(5.39)_{1,2}$, after substituting into $(3.6)_{1,2}$, we obtain the following equations of motion:

$$\begin{aligned} &\left(\lambda + \mu - \frac{\nu}{2}\right) u^\varkappa_{,\varkappa i} + \left(\mu + \frac{\nu}{2}\right) \Delta u_i + \nu \varepsilon_{ij\varkappa} \varphi^{j,\varkappa} + \tau_1 T_{,i} + \tau_2 C_{,i} + \varrho f_i = \varrho_0 \ddot{u}_i, \\ &\nu_2 \varepsilon^{ij\varkappa} \Delta \varphi_\varkappa + (\nu_1 + \nu_2) \varepsilon^{ij\varkappa} \varphi^l_{,l\varkappa} - \nu \varepsilon^{ij\varkappa} \varphi_\varkappa + \nu u^{[ij]} + \varrho l^{ij} = \varrho_0 I \varepsilon^{ij\varkappa} \ddot{\varphi}_\varkappa. \end{aligned} \tag{6.5}$$

Starting from the entropy balance equation (4.3), and using Eq. $(5.39)_3$ and (6.4), we find the differential equation of the temperature field in the form

$$\alpha \Delta T + \beta \Delta C + \gamma \Delta u^\varkappa_{,\varkappa} + m\theta_0 \dot{T} + p\theta_0 \dot{C} + \tau_1 \theta_0 \dot{u}^\varkappa_{,\varkappa} + \varrho h = 0. \tag{6.6}$$

Similarly, starting from the equation of balance of diffused mass (4.1), and using $(6.4)_2$, we obtain the differential equation for the field of concentration in the form

$$\left(\alpha - \frac{L}{\theta_0 L^{22}}\right) \Delta T + \beta \Delta C + \gamma \Delta u^\varkappa_{,\varkappa} - \frac{L^{12}}{L^{22}} \varrho_0 \dot{C} = 0. \tag{6.7}$$

The six equations of motion (6.5) together with the two field equations (6.6) and (6.7) represent the complete system of differential equations for the linear theory of termodiffusion in isotropic micropolar elastic materials.

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