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On the pulsatile flow of micropolar fluid

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THE comparison of physical arguments supporting the models of Stokesian fluid and Eringen micropolar fluid permits us to suggest that the latter should better describe a flow of suspensions. The unsteady motion is discussed in the case of axi-symmetrical flow with a pulsatile pressure gradient. The solutions to this problem are derived in the dimensionless variables. Thus the influence of material constants is naturally classified with the use of similarity numbers. The results give a promising interpretation of experimental data for the pulsatile blood flow.

Porównanie założeń fizycznych, leżących u podstaw modeli płynu stokesowskiego i mikropolarnego płynu Eringena, pozwala sugerować, że ten ostatni powinien lepiej opisywać przepływy zawiesin. Rozpatrzony jest ruch nieustalony na przykładzie osiowo-symetrycznego przepływu przy pulsującym gradiencie ciśnienia. Rozwiązania takiego zagadnienia są wyprowadzone w zmiennych bezwymiarowych. Uzyskuje się klasyfikację wpływu stałych materiałowych w naturalnej zależności od liczb podobieństwa. Rezultaty umożliwiają obiecującą interpretację danych doświadczalnych dotyczących pulsacyjnego przepływu krwi.

Сравнение физических предположений, находящихся у основ моделей жидкости Стокса и микрополярной жидкости Эрингена, позволяет предполагать, что эта последняя модель должна лучше описывать течения взвесей. Рассматривается неустановившееся движение на примере осе-симметричного течения при пульсирующем градиенте давления. Решения такой задачи выведены в безразмерных переменных. Получается классификация влияния материальных постоянных в естественной зависимости от чисел подобия. Результаты дают возможность многообещающей интерпретации экспериментальных данных, касающихся пульсирующего течения крови.

Notations

- $\mathbf{g}$ the vector of body force (per unit volume),
- $\mathbf{G}$ the vector of body momentum (per unit volume),
- L characteristic linear magnitude,
- P pressure,
- t time,
- P_a pulsatile pressure gradient
- P_c constant pressure gradient,
- U characteristic linear velocity,
- w microrotational velocity component,
- v velocity component in the axial direction,
- $\mathbf{v}$ linear velocity vector,
- $\mathbf{w}$ microrotational velocity vector,
- x_k spatial coordinate;

the similarity numbers

- Rem micropolar Reynolds number,
- Rint Reynolds number of interaction between vorticity and microrotation,
- Rw Reynolds number of microrotation,
- Ni number of microinertia,
- Re Newtonian Reynolds number;

the material constants

ϱ density,
μ viscosity (plasma viscosity),
$\varkappa, \beta$ additional coefficients of viscosity,
J rotational microinertia;

Greek letters

γ the root of characteristic equation,
ε skew-symmetric tensor,
ω the frequency of pulsatile pressure.

THE linear theory of polar fluid is a generalization of Stokesian fluid theory which takes into account some effects of fluid local structure. The model of Eringen micropolar fluid is introduced by making an assumption that fluid consists of material points, i.e., fluid particles, the motion of which is given by the velocity field **v**, and additionally, by the microrotation field **w**. The microrotation field represents local average rotational velocity of the fluid particles, and is independent of the vorticity.

Comparing physical arguments supporting the mathematical models of Stokesian and Eringen fluids one can expect that media of complex structures, e.g., biological liquids or other suspensions, will be described more accurately by the latter. In the case of suspensions it is evident that a conglomerate of dispersed phase can exhibit slip — a movement relevant to the neighbourhood in which it is embedded. The more so in unsteady state, the particle angular velocity may not coincide with the regional angular velocity which is equal to the half of the vorticity. The motion of blood is a particular case of fluid suspension flow, indicating considerable disparity with the Newtonian character of flow. One can expect the axi-symmetrical flow of micropolar fluid in tubes with a pulsatile pressure gradient to be a better model of blood flow.

The equations of motion of Eringen micropolar incompressible fluid are given here in the dimensionless forms. Two characteristic quantities are selected for the purpose. They are: L or R — characteristic linear magnitude and U — linear velocity. Following introduction of dimensionless variables (dimensional variables are marked with asterisks):

$$x_i = \frac{x_i^*}{L}, \quad u_i = \frac{u_i^*}{U}, \quad g_i = \frac{g_i^*}{U^2/L}, \quad P = \frac{P^*}{\varrho U^2}, \tag{1}$$
$$t = \frac{t^*}{L/U}, \quad w_i = \frac{w_i^*}{U/L},$$

the equations of motion take the form:

$$\frac{du_i}{dt} = g_i - \frac{\partial P}{\partial x_i} + \frac{1}{\text{Rem}} \frac{\partial^2 u_i}{\partial x_k \partial x_k} + \frac{1}{\text{Rint}} \varepsilon_{ijk} \frac{\partial w_k}{\partial x_j}, \tag{2}$$

$$\frac{1}{\text{Ni}} \frac{dw_i}{dt} = G_i + \frac{1}{\text{Rw}} \frac{\partial^2 w_i}{\partial x_k \partial x_k} + \frac{1}{\text{Rint}} \left(\varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} - 2w_i \right). \tag{3}$$

In similarity numbers occurring in the above-given formulae the material constants are combined with the quantities characteristic of the flow:

$$\text{(4)} \qquad \mathrm{Rem} = \frac{\varrho L U}{\mu + \varkappa/2}, \quad \mathrm{Rint} = \frac{\varrho L U}{\varkappa}, \qquad \mathrm{Rw} = \frac{\varrho L^3 U}{\beta}, \quad \mathrm{Ni} = \frac{L^2}{J}.$$

It is advisable to begin investigation of the unsteady motion with a simple case of a start-up problem of Couette plane flow. The conclusions derived from the solution of this problem which we have presented in [8] are as follows:

changes dependent on time (that is approaching the steady flow) are of roughly exponential character and depend on the ratio of constants Ni and Rint

$$\text{(5)} \qquad \frac{\mathrm{Ni}}{\mathrm{Rint}} = \frac{L}{\varrho U}\frac{\varkappa}{J};$$

the reciprocal of effective Reynolds number, defined as a quotient of shear stress and shear rate, representing the effective dimensionless viscosity, declines from:

$$\text{(6)} \qquad \left(\frac{1}{\mathrm{Re}}\right)_{\mathrm{eff,\,start}} = \left(\frac{1}{\mathrm{Rem}}\right) = \frac{\mu+\varkappa/2}{\varrho L U}$$

to:

$$\text{(7)} \qquad \left(\frac{1}{\mathrm{Re}}\right)_{\mathrm{eff,\,steady}} = \frac{1}{\mathrm{Rem}} + \frac{1}{\mathrm{Rint}}\cdot(-0.5) = \frac{\mu}{\varrho L U} < \frac{1}{\mathrm{Rem}}.$$

The model of many real flows is the axi-symmetrical flow in cylindrical tubes. In cylindrical coordinates system the differential equations of motion in case, when the inertia, components of small radial velocity, the body forces, and the body-couple are neglectedl have the following form:

$$\text{(8)} \qquad \frac{dv}{dt} = -\frac{\partial P}{\partial z} + \frac{1}{\mathrm{Rem}}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{\mathrm{Rint}}\frac{1}{r}\frac{\partial (rw)}{\partial r};$$

$$\text{(9)} \qquad \frac{1}{\mathrm{Ni}}\frac{dw}{dt} = \frac{1}{\mathrm{Rw}}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(rw)}{\partial r}\right] - \frac{1}{\mathrm{Rint}}\left(\frac{\partial v}{\partial r} + 2w\right).$$

Examining the flow, when the pressure gradient pulsates with a frequency ω, the following substitutions were made:

$$\text{(10)} \qquad -\frac{\partial P}{\partial z} = P_c + P_a \sin(\omega t), \quad \frac{dv}{dt} = \frac{\partial v}{\partial t}, \quad \frac{dw}{dt} = \frac{\partial w}{\partial t}.$$

The solution to this problem for a steady flow of micropolar fluid has been given by ERINGEN [1], CONDIFF and DAHLER [2], ARIMAN and CARMAK [3], PENNINGTON and COWIN [4]. In the dimensionless variables introduced here this solution has the form:

$$\text{(11)} \qquad v_{cp} = P_c^*\left[(1-r^2) - \frac{\mathrm{Rem}}{\mathrm{Rint}}\frac{I_0(\gamma) - I_0(\gamma r)}{\gamma I_1(\gamma)}\right],$$

$$\text{(12)} \qquad w_{cp} = P_c^*\left[r - \frac{I_1(\gamma_r)}{I_1(\gamma)}\right],$$

where:

$$(13)\quad \gamma^2 = \frac{\mathrm{Rw}}{\mathrm{Rint}}\left(2-\frac{\mathrm{Rw}}{\mathrm{Rint}}\right) = \frac{2\,\mathrm{Rw}\,\mathrm{Rem}}{\mathrm{Rint}}\left(\frac{1}{\mathrm{Rem}}-\frac{1}{2\,\mathrm{Rint}}\right), \quad P_c^* = \frac{P_c}{4\left(\frac{1}{\mathrm{Rem}}-\frac{1}{2\,\mathrm{Rint}}\right)}.$$

This solution is subject to boundary conditions of "hyperstick" type:

$$(14)\qquad v|_{r=1} = 0, \quad w|_{r=1} = 0, \quad v|_{r=0}, \quad w|_{r=0} \text{ — finite}.$$

Alternative boundary conditions for the microrotation field have been proposed by CONDIFF and DAHLER [2] as well as by REVINDRAN and DEVI [6], who used them for Couette flows. The boundary conditions of "no antisymmetric stress" type result for the Poiseuille flow through a pipe in the following formula:

$$(15)\qquad \left.\frac{1}{r}\frac{\partial(rw)}{\partial r}\right|_{r=1} = 0.$$

COWIN and PENNINGTON [5] suggest that this boundary condition is more consistent with the image of physical phenomena. The solutions to the equations of motion (8) and (9), subject to this boundary conditions, were worked out. The components of velocity and microrotation, which result from the constant pressure gradient P_c, constitute a solution to the steady motion and have the form:

$$(16)\qquad v_{cp} = P_c^*\left[(1-r^2) - \frac{2\,\mathrm{Rem}}{\mathrm{Rint}}\frac{I_0(\gamma)-I_0(\gamma r)}{\gamma^2 I_0(\gamma)}\right];$$

$$(17)\qquad w_{cp} = P_c^*\left[r - \frac{2}{\gamma}\frac{I_1(\gamma_r)}{I_0(\gamma)}\right].$$

The unsteady flow solution, subject to no initial condition, have the following form:

$$(18)\quad v = v_{cp}+v_{pp} = v_{cp}+P_a\sum_{\lambda}\frac{2J_0(\lambda r)}{\lambda J_1(\lambda)}\left\{\left[\frac{(\delta-\alpha)k_m}{2\delta(k_m^2+\omega^2)}+\frac{(\delta+\alpha)k_p}{2\delta(k_p^2+\omega^2)}\right]\sin(\omega t) - \left[\frac{(\delta-\alpha)\omega}{2\delta(k_m^2+\omega^2)}+\frac{(\delta+\alpha)\omega}{2\delta(k_p^2+\omega^2)}\right]\cos(\omega t)\right\};$$

$$(19)\quad w = w_{cp}+w_{pp} = w_{cp}+P_a\sum_{\lambda}\frac{2J_1(\lambda r)}{\lambda J_1(\lambda)}\frac{\lambda\mathrm{Ni}}{2\delta\,\mathrm{Rint}}\left\{\left[\frac{k_m}{k_m^2+\omega^2}-\frac{k_p}{k_p^2+\omega^2}\right]\sin(\omega t) - \left[\frac{\omega}{k_m^2+\omega^2}-\frac{\omega}{k_p^2+\omega^2}\right]\cos(\omega t)\right\};$$

$$\lambda:\ J_0(\lambda) = 0, \quad k_p, k_m:\ k_p = k_0+\delta > k_m = k_0-\delta > 0,$$

$$k_0 = \lambda^2\left(\frac{1}{\mathrm{Rem}}+\frac{\mathrm{Ni}}{\mathrm{Rw}}\right)+\frac{2\mathrm{Ni}}{\mathrm{Rint}}, \quad \delta^2 = \alpha^2+\frac{\lambda^2\mathrm{Ni}}{(\mathrm{Rint})^2},$$

$$\alpha^2 = \lambda^2\left(\frac{1}{\mathrm{Rem}}-\frac{\mathrm{Ni}}{\mathrm{Rw}}\right)-\frac{2\mathrm{Ni}}{\mathrm{Rint}}.$$

The formula of the velocity of Eringen micropolar fluid—subject to the pulsatile pressure gradient — will be compared with a corresponsing formula of Newtonian fluid flow, obtained with the help of the same solving method. The solution given by WOMERSLEY [9] based on Bessel functions of the complex variable (which are ber-function and bei-function) is convenient to the comparison as well as to digital computer calculations. In the following solution of the equation of Newtonian fluid motion, there are distinguished the components of velocity, being subject to constant P_c, and a pulsatile P_a pressure gradient:

$$(20)\quad u = u_{cp}+u_{pp} = \frac{P_c(1-r^2)}{4/\mathrm{Re}} + P_a \sum_{\lambda} \frac{2J_0(\lambda r)}{\lambda J_1(\lambda)} \left[\frac{(\lambda^2/\mathrm{Re})\sin(\omega t)-\omega\cos(\omega t)}{(\lambda^2/\mathrm{Re})^2+\omega^2}\right].$$

The velocity profiles of micropolar fluid, compared with the corresponding profiles of Newtonian fluid, might serve better than others to describe the flow of blood. An application of the polar fluid theory to the description of blood flow has been proposed by ERINGEN [1]. The numerical values of the material constants are derived here from experimental data of BUGLIARELLO and SEVILLA [10] and from the rheological experiments and the theoretical investigations as described by TURK, SYLVESTER and ARIMAN [7]. The experiments of Bugliarello and Sevilla concern the flow of blood red cells (erythrocytes) suspension in plasma through the glass fibres. Such a suspension, as opposed to blood, does not coagulate. According to investigations of ALLEN and KLINE [12], the basic value of viscosity coefficient μ is assumed to be equal to plasma viscosity.

In choosing values

$$(21)\quad \begin{aligned} &\mu = 1.2cP, \quad \beta = 4.8\cdot 10^{-8}\ \mathrm{gcm/s}, \quad L = 20\ \mu m,\\ &\varkappa = 0.8cP, \quad \varrho = 1\ \mathrm{g/cm^3}, \quad U = 6\ \mathrm{mm/s},\end{aligned}$$

which are averaged limits given in works of ARIMAN *et al.* [11], [7], the basic values of similarity numbers are obtained:

$$(22)\quad \frac{1}{\mathrm{Re}} = 10, \quad \frac{1}{\mathrm{Rint}} = 12, \quad \frac{1}{\mathrm{Rem}} = \frac{1}{\mathrm{Re}} + \frac{1}{2\mathrm{Rint}} = 16, \quad \mathrm{Rw} = 0.1.$$

The basic value of microinertia coefficient J is taken to satisfy the condition:

$$(23)\quad \mathrm{Ni} = \frac{\mathrm{Rw}}{\mathrm{Re}} = \mathrm{Rw}\left(\frac{1}{\mathrm{Rem}} - \frac{1}{2\,\mathrm{Rint}}\right) = 1.$$

This restriction is equivalent to the reduction of Eqs. (9) to (8) by substituting $w = -\frac{1}{2}\frac{\partial v}{\partial r}$, that means the coincidence of microrotation and the regional angular velocity, typical of the steady Couette flow.

Numerical computation of the velocity profiles together with microrotation profiles and average velocities have been made. The analysis of these results leads to the following conclusions:

1. In comparison with the parabolic profile of Newtonian fluid, the basic velocity profile of micropolar fluid subject to the constant pressure gradient P_c, is slightly sharper. The velocity is greater on the axis but smaller near the wall. This comparison as well as most of the following ones is made for the same flow rate. This is equivalent to assigning the mean velocity as the corresponding characteristic velocity U. The mean velocity of Newtonian fluid, subject to constant pressure gradient and the same basic coefficient of

viscosity μ, is about 40% greater than the mean velocity of micropolar fluid; the basic values of material constants are as above.

2. The dependence of velocity profiles on pulsation frequency is illustrated in Fig. 2. It can be seen that the decrease of frequency below the value $\omega = 1$ slightly affects the

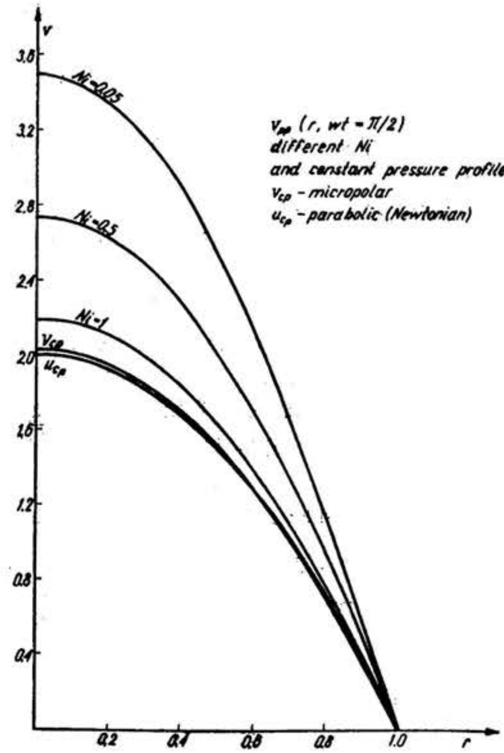

FIG. 1. The velocity profiles for the pulsatile pressure gradient (different Ni) and for the constant pressure gradient.

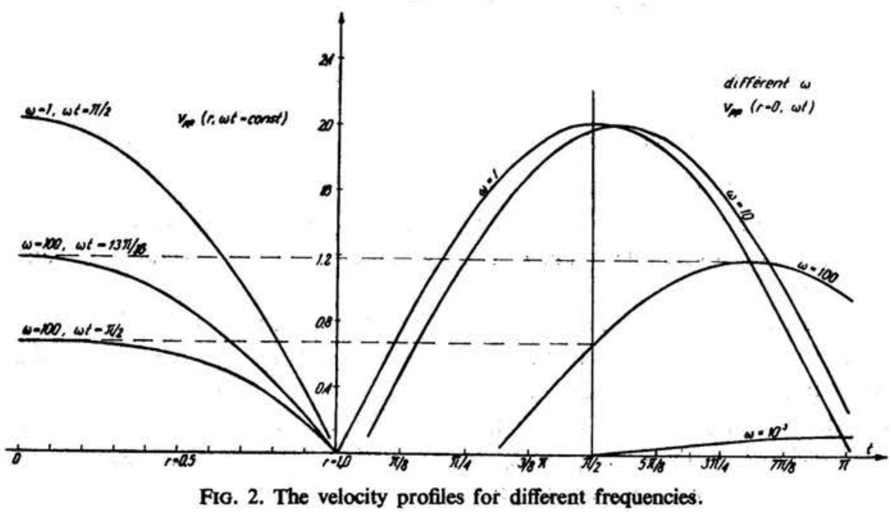

FIG. 2. The velocity profiles for different frequencies.

velocity profiles. (Because of the relation: $\omega = \omega^* \cdot L/U$, the dimensionless frequency equal to 1 corresponds to dimensional frequency of about 45 Hz). The increase of frequency results in flattering velocity profiles. The amplitude of velocity component subject to pulsatile pressure gradient becomes smaller. Together with it the frequency increase effects in the phase lag between moments in which the maximum profile (what designates the profile of biggest volume flow rate) and the greatest pressure gradient occur. For the frequency of range 10^3 this phase lag reaches $\pi/2$.

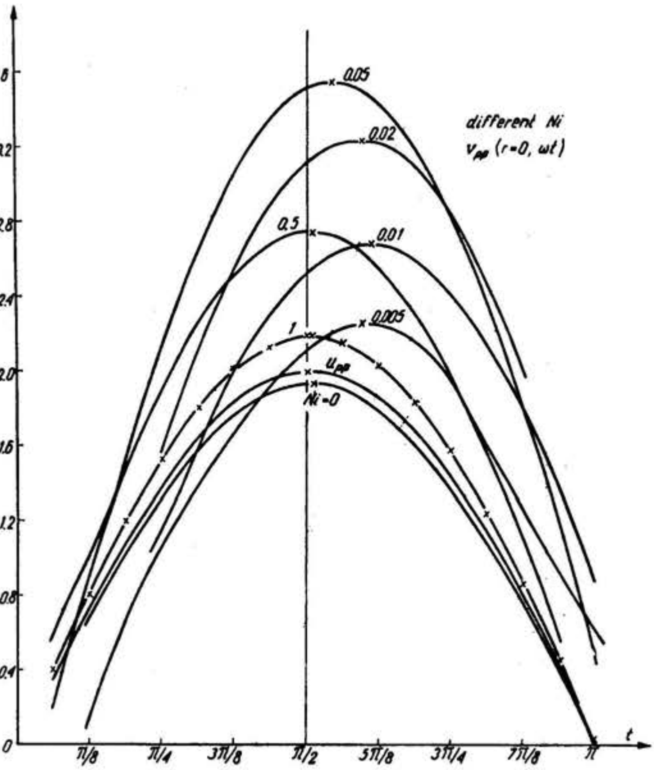

FIG. 3. The axial velocity for different Ni.

3. The influence of parameter Ni upon velocity distributions is shown in Figs. 1 and 3. For the constant value of the parameter Ni it represents the dependence on the ratio of constants Ni and Rint. The increase of the parameter Ni from zero value (i.e., the decrease of microinertia coefficient J) results in an increase of fluid "mobility". A significant variation of velocity profiles represented by the axial velocity is observable for successive moments of a pulsation period. The velocity profiles flatter than the Newtonian profiles for $Ni = 0$ become sharper and the value of velocity on the axis reaches even 3.5 for Ni = 0.05.

The velocity profiles for different moments of pulsation are plotted in Fig. 4; the pressure gradient has the form: $P_c + P_e \sin(t)$. Three cases of velocity distribution are compared: Newtonian (u), micropolar for Ni = 1(vs) and for Ni = 0.5(vm). There occurs a higher "mobility" of the micropolar fluid velocity profiles. The micropolar profiles approaching (vs) or being flatter (vm) than the Newtonian profiles become sharper with an increase of pressure gradient; this is already seen for $t = \pi/4$ and very remarkable for $t = \pi/2$. The micropolar profiles diminish quicker with the decrease of pressure gradient; it is observed for $t = 5\pi/4$.

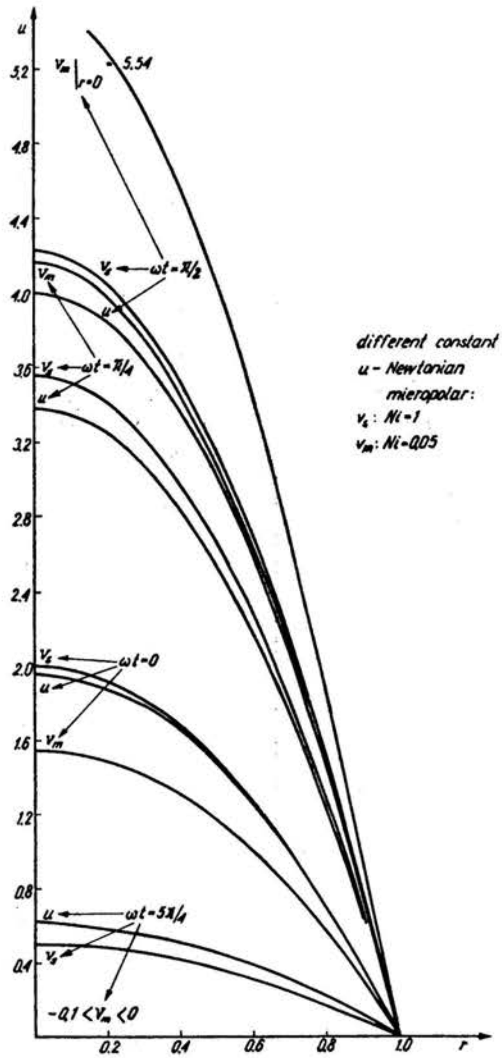

FIG. 4. The velocity profiles for different Ni and different moments.

The great variance of velocity distribution, considerably exceeding that of Newtonian fluid, is observable especially in the case of micropolar fluid with Ni = 0.05.

Together with an increase of fluid "mobility" occurring for the values of parameter Ni within the limits from 0 to 0.05 appears the phase lag between the moments of a maximum profile and a maximum pressure gradient. The greatest value of this delay is noticed for Ni $\approx$ 0.01 and reaches the value of $\pi/8$. The further increase of parameter Ni above 0.05 results in gradual assimilation of profiles to velocity distribution determined for $Ni = 1$. The velocities are about 10% greater than the corresponding velocities of appropriate Newtonian fluid. The phase lag between velocity and pressure gradient is small.

4. The parameter Rw slightly influences the shape of velocity profiles as in Fig. 5, its increase brings a small increment of the velocity on the axis.

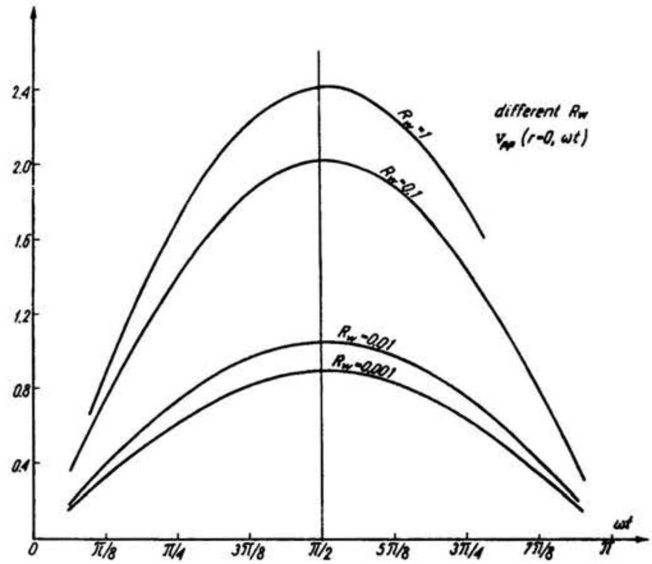

FIG. 5. The axial velocity for different Rw.

The comparison of the presented relations for the micropolar fluid velocity profiles with the experimental data for the suspension of red cells in plasma, as reported by BUGLIARELLO and SEVILLA [10], permits to conclude that the micropolar fluid, due to its sharper velocity profiles and greater "mobility", describes better the motion of blood. Bugliarello and Sevilla have measured the velocities in the neighbourhood of a tube axis and compared them with the velocity profiles of Newtonian fluid. They have found that the latter, practically in every moment of pulsation period, exhibits too slow variance and that the values of velocity calculated for Newtonian model are smaller than the values derived from experiments.

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