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Are there any stagnation points of an ideal fluid within a three-dimensional toroidal geometry?

E. MARTENSEN (KARLSRUHE)

THE VELOCITY vector of an ideal fluid within a toroidal (i.e., double connected) three-dimensional region, the boundary of which has a continuous curvature, is defined uniquely by well-known theorems of potential theory, when its normal component vanishes on the boundary and its circulation has a fixed non-vanishing value. An open question which recently arised and got some actuality especially in magneto-hydrodynamics is whether this velocity field may have stagnation points (zeroes) or not.

LET $\Omega \subset \mathsf{R}^3$ be a toroidal, i.e., double-connected, bounded open set and let the boundary $\partial\Omega$ of Ω have at least a continuous curvature. Then, by well-known theorems of potential theory [4], there exists an uniquely defined vector field v in the closure $\overline{\Omega}$ of Ω which is continuous in $\overline{\Omega}$ and harmonic ([1]) in Ω; furthermore v has vanishing normal components on $\partial\Omega$ and a fixed non-vanishing toroidal circulation. If we, for convenience, assume the boundary $\partial\Omega$ to be analytic, the harmonic vector v is analytic everywhere in $\overline{\Omega}$.

The harmonic field v can be realised by the velocity of an (incompressible) ideal fluid within a torodidal geometry as well as by a stationary current-free magnetic field bounded by a toroidal supra-conductor. Now we shall deal with the question whether this toroidal ideal fluid or the corresponding magnetic field has stagnation points, that is, zeroes or not. The latter version of the problem has been of some interest lately for stability problems in magneto-hydrodynamics, which arose from the efforts on controlled nuclear fusion; here one hopes that the vector field has no zeroes (as a special consequence of this the minimum principle would be applicable to v, resulting from the fact that $|v|^{-2}$ is a subharmonic function, if v is a non-vanishing harmonic vector). If our toroidal geometry especially has cylindrical symmetry as, for instance, a circle torus has, we can give at once the required harmonic field v as a potential vortex field around the symmetry axis, and from this it follows that v cannot have any stagnation points. For arbitrary toroidal geometries in R^3 this however is an open question at present. But as the analogous problem in R^2 could be solved a long time ago in the sense that there are no zeroes, it should be useful to discuss the methods in R^2 and see what they can do in R^3. In order to simplify our problem in R^2 as well as in R^3 we shall restrict ourselves to the case where there are no zeroes on the boundary.

In R^2 we have a double-connected bounded open set Ω the boundary $\partial\Omega$ of which

([1]) This means that v is continuously differentiable and satisfies the equations div $v = 0$, rot $v = 0$.

consists of two sufficiently smooth disjunct curves. If the vector field $v = (v_x, v_y)$ is continuous in $\bar{\Omega}$ and harmonic in Ω, the complex function

$$w := v_x - iv_y \tag{1}$$

is continuous in $\bar{\Omega}$ and holomorphic in Ω. From our assumption $w \neq 0$ on $\partial\Omega$ it follows that we can, at the most, have a finite number of zeroes $P_1, \ldots, P_n \in \Omega$ with positive orders $m_1, \ldots, m_n$; otherwise the zeroes would have an accumulation point in Ω which, by a fundamental theorem of function theory, would lead to $w = 0$ in $\bar{\Omega}$. Now, we shall describe three methods by which the problem in R^2 can be solved.

1. *Conformal mapping*. As $\bar{\Omega} \subset \mathsf{R}^2$ by means of Riemann's mapping theorem can be mapped conformally on the closure of a circle-ring and as an ideal fluid with non--vanishing circulation within this special plane geometry obviously possesses no stagnation points, it follows by transformation of the complex velocity function (1) that the ideal fluid in $\bar{\Omega}$ has no stagnation points either.

2. *The method of the logarithmic residue*. If we assume $\partial\Omega$ to have continuous derivatives of the third order, the harmonic field v has continuous derivatives of the first order in $\bar{\Omega}$. Then, for the complex function (1) we can write down the logarithmic residue theorem

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{w'}{w}\, dz = \sum_{\nu=1}^{n} \frac{1}{2\pi i} \int_{\partial\Omega_\nu} \frac{w'}{w}\, dz, \tag{2}$$

where $\Omega_1, \ldots, \Omega_n \subset \Omega$ are sufficiently small vicinities of the zeroes $P_1, \ldots, P_n \in \Omega$ and the boundaries $\partial\Omega, \partial\Omega_1, \ldots, \partial\Omega_n$ are orientated in such a way that the corresponding sets $\Omega, \Omega_1, \ldots, \Omega_n$ lie to the left. As v lies tangential to $\partial\Omega$ the logarithmic residue with respect to $\partial\Omega$ may be computed in the well-known manner by counting the circulations of origin by the values of w; thus, with Eq. (1) we get

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{w'}{w}\, dz = -(1-p), \tag{3}$$

where p is the first Betti number for Ω, i.e., the number of topological invariant closed curves in Ω, which cannot be contracted in a point of Ω. Because of $p = 1$ in our case the logarithmic residue (3) and therefore the sum on the right hand side in Eq. (2) vanishes. If we at last think of the logarithmic residues belonging to $\partial\Omega_1, \ldots, \partial\Omega_n$ to have the values $m_1, \ldots, m_n$ of the orders of the zeroes, Eq. (2) reduces to

$$\sum_{\nu=1}^{n} \frac{1}{2\pi i} \int_{\partial\Omega_\nu} \frac{w'}{w}\, dz = \sum_{\nu=1}^{n} m_\nu = 0 \tag{4}$$

and such all $m_1, \ldots, m_n$ must vanish.

3. *The method of the degree of mapping*. Under the same assumption for $\partial\Omega$ as before we consider the harmonic field v as a continuous differentiable mapping $\bar{\Omega} \to \mathsf{R}^2$ and are now interested in the degree $d[v; \Omega, 0]$ of this mapping with respect to the value 0. If, furthermore, $d[v; \Omega_1, 0], \ldots, d[v; \Omega_n, 0]$ denote the local degrees of mapping belonging to the zeroes $P_1, \ldots, P_n \in \Omega$, the concept of the degree of mapping [3] gives the identity

$$d[v; \Omega, 0] = \sum_{\nu=1}^{n} d[v; \Omega_\nu, 0]. \tag{5}$$

Using now Kronecker's integral representation for the degree

$$d[v;\Omega,0]=\frac{1}{2\pi}\int\limits_{\partial\Omega}\frac{v_x\dot{v}_y-v_y\dot{v}_x}{|v|^2\sqrt{\dot{x}^2+\dot{y}^2}}\,ds, \tag{6}$$

where $x(t)$, $y(t)$, $[\dot{x}(t)]^2+[\dot{y}(t)]^2>0$, is a parameter representation for $\partial\Omega$, running $\partial\Omega$ such that Ω lies to the left, we find by a simple computation that $d[v;\Omega,0]$ is the negative real part of the logarithmic residue (3) and so we get

$$d[v;\Omega,0]=1-p. \tag{7}$$

Because of $p=1$ and a same conclusion for the right hand side in Eq. (5) it follows

$$\sum_{\nu=1}^{n}d[v;\Omega_\nu,0]=-\sum_{\nu=1}^{n}m_\nu=0 \tag{8}$$

and from this the same result as before.

If we now look for the situation in R^3, we cannot exclude accumulation points of zeroes lying in Ω as we did before in R^2. But in R^3 we can exclude zeroes lying all over some surface $S\subset\Omega$. For if the harmonic field v were to vanish on S, it could be locally represented as the gradient of a harmonic function φ, which has constant values and vanishing normal derivatives on S; thus, by the theorem of Cauchy-Kowalewski, it would follow that φ is a constant and v therefore would first vanish locally and then, as it is a harmonic vector, in the whole closure $\bar{\Omega}$.

Now, in R^3 we shall only discuss the case when there is, at the most, a finite number of zeroes $P_1,\dots,P_n\in\Omega$ of the harmonic vector v. If we further assume the surface $\partial\Omega$ to have continuous derivatives of the third order, all occurrent derivatives will exist and be continuous. Now, looking back to the three methods which work in R^2, we see at once that we cannot make use of conformal mappings and complex methods based on the logarithmic residue. But for the degree of mapping [3] we also have in R^3 the identity

$$d[v;\Omega,0]=\sum_{\nu=1}^{n}d[v;\Omega_\nu,0], \tag{9}$$

where $\Omega_1,\dots,\Omega_n\subset\Omega$ are sufficiently small vicinities of $P_1,\dots,P_n$. Now, it can be shown in two ways that the degree $d[v;\Omega,0]$ vanishes. First, we use the homotopy character of the degree of mapping, which through this remains unaltered if we continuously turn the non-vanishing tangential vector v on $\partial\Omega$ into the direction of the exterior normal of $\partial\Omega$. The degree of mapping of the vector then obtained has the value of the Gaussian curvatura integra of $\partial\Omega$ divided by 4π and hence by a well-known theorem of differential geometry [1], we get

$$d[v;\Omega,0]=1-p, \tag{10}$$

where p denotes the first Betti number for Ω. So, $d[v;\Omega,0]$ vanishes as we have $p=1$.

The second way to show this is a more formal one. Let the vector $x(u^1,u^2)$ be a parameter representation for $\partial\Omega$ where the surface parameters u^1, u^2 are orientated in such a way that they form a positive system together with the exterior normal of $\partial\Omega$ and let furthermore, g_{ik}, g^{ik} and b_{ik} be the first, the inverse first and the second fundamental tensor for $\partial\Omega$, where $g := \det(g_{ik})$ is positive, and $|_i$ resp. $||_i$ denote the partial resp. covariant derivatives by u^i, $i = 1, 2$. Besides this we will use Einstein's summation convention. Then, the degree of mapping can be expressed by Kronecker's integral

$$d[v;\Omega,0] = \frac{1}{4\pi}\iint_{\partial\Omega} \frac{(v, v|_1, v|_2)}{|v|^3\sqrt{g}}\,df, \tag{11}$$

where $(v, v|_1, v|_2)$ is the spat-product of v and its derivatives. Now, by means of the Mainardi-Codazzi equations of differential geometry [1] one can show by some calculations that the integrand in Eq. (11) is equal to the divergence expression

$$\frac{(v, v|_1, v|_2)}{|v|^3\sqrt{g}} = \left(\frac{b^i_k v^k - b^k_k v^i}{|v|}\right)\Bigg|\Bigg|_i, \quad b^i_k = g^{ij}b_{jk}, \quad v = v^i x|_i \tag{12}$$

and from this it immediately follows that the degree (11) vanishes.

Therefore in any case we get from Eq. (9)

$$\sum_{\nu=1}^{n} d[v;\Omega_\nu,0] = 0, \tag{13}$$

in other words, the sum of the local degrees of mapping referring to the zeroes $P_1, \ldots, P_n$ vanishes. Now, the question is what do these local degrees mean. Here we again leave the analogy to the plane since these degrees have no relationship to the order of the zeroes. We first note that in R^3 the local degree of mapping referring to an isolated zero of even order vanishes by reasons of symmetry. Furthermore an isolated zero of odd order m does not need to have the degree of mapping m or $-m$. For this we consider the harmonic vector field

$$v = \operatorname{grad}\left\{\frac{3}{4}(x^2+y^2)^2 - 6(x^2+y^2)z^2 + 2z^4\right\}, \tag{14}$$

which has only one isolated zero of the third order in the origin. By some computation we find that the local degree of Eq. (14) has the value 1; replacing v by $-v$, we see that the degree -1 is possible, too. Recently, HALTER [2] showed, that an isolated zero of the third order in case of a harmonic vector can also possess the degrees $\pm$ 3 and $\pm$ 5.

Bearing in mind the negative results just mentioned, Eq. (13) allows us to make only a few statements. There cannot be exactly one zero of the first order in Ω because otherwise, according to the theorem on implicit functions, the mapping v in this point would have a non-vanishing Jacobian from which the local degree of mapping 1 or -1 would follow [3]. There also cannot be at the most a finite number of zeroes in Ω, when their local degrees of mapping differ from 0 with the same sign. We with finally should remark that these statements are only based on the fact that v does not vanish on $\partial\Omega$ and has no normal component on $\partial\Omega$; especially the harmonicity of v has not been used here.

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MATHEMATISCHES INSTITUT II
UNIVERSITÄT KARLSRUHE.

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