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Non-linear waves in gases with weak influence of relaxation

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NEW simplified equations for the propagation of one-dimensional unsteady and two-dimensional steady waves in a relaxing gas with one internal degree of freedom will be derived, which are approximately valid if the wave amplitude is relatively small and if the influence of relaxation remains weak. The influence of relaxation is weak if either the difference between the frozen and equilibrium speeds of sound is small or if the typical frequency involved is large or small as compared to $1/\tau$, τ being the relaxation time.

W pracy badano rozchodzenie się fali jednowymiarowej niestacjonarnej i dwuwymiarowej stacjonarnej, uwzględniając zjawiska relaksacji, w gazie o jednym wewnętrznym stopniu swobody. Otrzymano uproszczone równania propagacji przy założeniach, że amplituda fali jest stosunkowo mała oraz że wpływ relaksacji jest nieznaczny. Ten ostatni warunek jest spełniony w dwóch przypadkach: 1) gdy zamrożona i równowagowa prędkość dźwięku różnią się nieznacznie, 2) gdy częstość charakterystyczna jest duża (lub mała) w porównaniu z $1/\tau$, τ — czas relaksacji.

В работе исследовано распространение одномерной нестационарной и двухмерной стационарной волн, учитывая явления релаксации в газе с одной внутренней степенью свободы. Получены упрощенные уравнения распространения, при предположениях, что амплитуда волны сравнительно мала, а также, что влияние релаксации незначительно. Это последнее условие удовлетворено в двух случаях: 1) когда вмороженная и равновесная скорости звука отличаются незначительно, 2) когда характеристическая частота больше (или мала) по сравнению с $1/\tau$, τ — время релаксации.

1. Introduction

THE AIM of this paper is to develop a general algorithm which would allow to derive simplified equations for a gas flow with relatively small amplitude and with weak influence of internal rate processes.

To fix ideas let us assume the gas to be a mixture of chemically reacting components. In slow flow the chemical composition is always in equilibrium with pressure and entropy. Hence, the fields of pressure and entropy suffice for a description of the thermodynamic state of the gas mixture, and the equations of classical gasdynamics are valid. However, for rapid flows, and therefore rapid changes of the thermodynamic state, the chemical reactions lag behind the changes of pressure and entropy. Then, at least one internal state variable q has to be introduced, and the equations of relaxing gas dynamics describe the flow. If the gas is a mixture of two chemically reacting species, q might be the mass concentration of one of the species. The lag of q behind its equilibrium value $\tilde{q}(p, s)$ satisfies a relaxation equation which, for most applications, may be written in the well-tested form

(1.1) $$\frac{Dq}{Dt} = -\frac{q-\tilde{q}(p, s)}{\tau(p, s, q)}.$$

In this equation p is the pressure and s the specific entropy. D/Dt denotes the material time derivative, and τ is the relaxation time. In chemically reacting gases the equilibrium value $\tilde{q}(p, s)$ of the internal state variable q may be computed from the law of mass action. In many cases the flow of the relaxing gas differs only slightly from the flow of an associated non-relaxing gas. Let us define a dimensionless parameter $\delta > 0$ as a measure of that difference. If δ is small, we shall say that the influence of relaxation on the flow is weak. For slow flow in which the time scale T of typical pressure changes of the particles is large as compared with a characteristic relaxation time τ_0, the small parameter δ may be defined by

$$\delta_e = \tau_0/T \ll 1.$$

In this case the associated non-relaxing gas flow is the equilibrium gas flow. For rapid flow, in which the internal state variable does not change considerably (nearly frozen flow), the small parameter δ may be defined by

$$\delta_f = T/\tau_0 \ll 1$$

and the associated non-relaxing gas flow is the completely frozen gas flow. Finally, the effects of relaxation are also small if the energy or, equivalently, the enthalpy of the gas depends only weakly on the internal state variable. In that case the difference between the frozen and equilibrium values of sound speed, b and a, is small and one can define δ by

$$\delta_r = \frac{b_0 - a_0}{b_0} \ll 1.$$

Here again the completely frozen gas flow may be chosen as an associated non-relaxing gas flow. In this paper we shall try to present a theory which can be applied to each of the three cases, $\delta_e \ll 1$, $\delta_f \ll 1$ and $\delta_r \ll 1$, simultaneously.

Unfortunately, the smallness of δ alone does not suffice to simplify the equations of gasdynamics considerably. But if, in addition, the amplitude ε of the flow is also small, so that non-linear effects can be taken account of with sufficient accuracy in an ε^2-approximation, great simplifications of the equations can be achieved. The main reason for this is that gas flows with small δ and ε are approximately isentropic. Since we neglect the effects of viscosity, heat conduction and diffusion ,entropy is produced only by relaxation processes and in discontinuous (frozen) shocks. The entropy produced by relaxation vanishes with $\delta\varepsilon^2$. In shock waves of the dimensionless amplitude $\varepsilon \ll 1$, the entropy produced is of the order ε^3. Hence, entropy changes can be neglected if terms of the order ε^3 and $\delta\varepsilon^2$ are neglected.

One may suspect that substantial simplifications of the equations for small amplitude gas flow with weak influence of relaxation are possible if one considers the following well-known examples: Unidirectional waves of small amplitude ε in relaxing gases are governed

by "Burger's equation" [1, 2] if the gas is always nearly in equilibrium, that is, if $\delta_e \ll 1$;

by a high frequency approximation due to DUNWOODY [3], if the gas is always close to the completely frozen state, that is, if $\delta_f \ll 1$;

by an equation which was first derived by OCKENDON and SPENCE [4, 5] for gases with small difference of the speeds of sound, that is, with $\delta_r \ll 1$.

In what follows we shall try to develop a quite general algorithm which would allow to derive, apart from the equations already mentioned, a host of other simplified equations for a gas flow with weak influence of relaxation.

2. Outline of the algorithm

To make the following arguments clearer, let us first call back to our minds classical gasdynamics, that is the dynamics of non-relaxing gases. In this case we have the following set of balance equations (note that within an ε^2 approximation the flow is isentropic; knowing this we can overlook the energy balance):

(2.1)
momentum balance (3 scalar equations),
mass balance (1 scalar equation)

for the following unknown fields

velocity $u(y_i, t)$ (3 scalar fields),
pressure $p(y_i, t)$ (1 scalar field),
density $\varrho(y_i, t)$ (1 scalar field).

Here, t is the time and the y_i are coordinates, fixing the position of the particles. The y_i may, e.g., be Cartesian coordinates in a fixed frame of reference. In this case we shall write $y_i = x_i$. In a one-dimensional flow it is useful to identify y with the Lagrangean coordinate X which is defined by

$$X = \frac{1}{\varrho_0} \int\limits_{\alpha}^{x} \varrho(x, t)\, dx,$$

$x_0(t)$ being the position of an arbitrarily chosen particle and ϱ_0 a constant reference density.

Obviously, the number of unknown fields exceeds the number of balance equations by one. But, if we add the isentropic equation of state

$$\varrho = \hat{\varrho}(p) \tag{2.2}$$

the number of equations suffices to solve the problem, provided appropriate initial and boundary conditions are given.

Often it is useful to reduce the set of equations (2.1), (2.2) to one higher order equation for one unknown field. Let us denote the unknown field to be chosen by ψ and let us write, symbolically, the differential equation for ψ in the following form:

$$\mathsf{C}(\psi) = 0.$$

For a one-dimensional flow, for instance, one may choose $y = X$ and $\psi = p$ and eliminate the velocity $\boldsymbol{u}$ and the density ϱ from the equations. The result of this elimination process is the equation

$$\mathsf{C}(p) \equiv \left(p_t/A^2(p)\right)_t - p_{XX} = 0$$

for the pressure p. Here,

$$A(p) = \frac{\varrho}{\varrho_0} a(p) = \frac{\varrho}{\varrho_0} (d\varrho/dp)^{-1/2}$$

is the speed of sound with respect to X. Since X is a material coordinate, we will term $A(p)$ the "material speed of sound".

After this reminder of classical gasdynamics let us turn back to relaxation gas dynamics. As we have already mentioned, changes of entropy in relaxing gases are of the order ε^3 and $\varepsilon^2\delta$. Since we intend to neglect terms of this order, we can neglect changes of entropy as we have already done in the case of classical gasdynamics. Comparing the equations of classical gasdynamics and relaxation gasdynamics we observe that we have one additional unknown field, namely, the field $q(y_i, t)$ of the internal state variable. But we also have an additional balance equation, namely, the relaxation equation (1.1). Further, the equation of state (2.2) is to be replaced by

$$\varrho = \hat{\varrho}(p, q). \tag{2.3}$$

To make the following arguments clearer let us assume for the moment that the field

$$q = q^*(y_i, t)$$

of the internal state variable is already known. Then, we can forget the relaxation Eq. (1.1) and replace Eq. (2.3) by

$$\varrho = \hat{\varrho}\big(p, q^*(y_i, t)\big) = \hat{\varrho}^*(p, y_i, t). \tag{2.4}$$

The flow field may now be computed from the set of balance Eq. (2.1) and the equation of state (2.4). Obviously, the only difference between this system of equations and the equations of classical gasdynamics is that Eq. (2.2) has been replaced by Eq. (2.4). If we go again through the same elimination process, as we did in classical gasdynamics (carefully paying attention to the fact that $\hat{\varrho}^*$ now depends explicitly on y_i and t) we must end up with a differential equation of the form

$$\mathsf{R}\big(\psi, q^*(y_i, t)\big) = 0.$$

Finally, let us drop the assumption that q is a given function of y_i and t. Then, obviously the system

$$\mathsf{R}(\psi, q) = 0, \tag{2.5}$$

$$\frac{Dq}{Dt} = -\frac{q-\tilde{q}(p, s_0)}{\tau(p, s_0, q)} \tag{2.6}$$

governs the flow. For one-dimensional waves, for instance, we choose $y = X$, $\psi = p$. Equation (2.5) reads in this case

$$\mathsf{R}(p, q) = -\big(B^{-2}(p, q)p_t\big)_t + p_{XX} + \varrho_0^2\big((\hat{\varrho}^{-1})_q q_t\big)_t = 0, \tag{2.7}$$

whereas Eq. (2.5) can be written in the form

$$\frac{\partial q(X, t)}{\partial t} = -\frac{q-\tilde{q}(p, s_0)}{\tau(p, s_0, q)}.$$

In Eq. (2.7), $B = b\varrho/\varrho_0 = (\partial\hat{\varrho}/\partial p)^{-1/2}\varrho/\varrho_0$ is the material frozen speed of sound.

The set of Eqs. (2.5), (2.6) is useful only if $\tau(p, s_0, q)$ and $\tilde{q}(p, s_0)$ can be expressed in terms of q and of ψ and its derivatives:

$$\tau = \hat{\tau}(\psi, \partial\psi/\partial y_i, \partial\psi/\partial t, \ldots, q),$$

$$\tilde{q} = \tilde{q}(\psi, \partial\psi/\partial y_i, \partial\psi/\partial t, \ldots).$$

This is clearly the case if ψ is a thermodynamic state variable. Furthermore, it is sometimes possible if ψ involves the velocity. To give an example let us consider steady irrotational flows (*).

In this particular case it is useful to identify ψ with the velocity potential $\phi(x_1, x_2, x_3)$ of the flow. Now, in steady flow the energy balance can be integrated, the result being the relation

$$\hat{h}(p, q, s)+\frac{1}{2}\mathbf{u}^2 = h_0+\frac{1}{2}U_0^2 \tag{2.8}$$

between the velocity $\mathbf{u} = \operatorname{grad}\phi$ and the enthalpy $\hat{h}(p, q, s)$ of the gas. In our case, the flow is isentropic: $s = s_0$. Hence, we can use Eq. (2.8) to compute p in terms of q and the components $\partial\phi/\partial x_i$ of $\mathbf{u}$, the result being

$$p = \hat{p}(q, \partial\phi/\partial x_i).$$

Inserting this in $\tau(p, s_0, q)$, we get τ in terms of q and $\partial\phi/\partial x_i$. Similarly, using the thermodynamic relation (law of mass action) $\hat{h}_q(p, \tilde{q}, s) = 0$, we obtain

$$\hat{h}_q(\hat{p}(\tilde{q}, \partial\phi/\partial x_i), \tilde{q}, s_0) = 0 \tag{2.9}$$

from which we can compute $\tilde{q}$ in terms of the velocity components $\partial\phi/\partial x_i$.

The aim of the next steps is to reduce the two Eqs. (2.5), (2.6) to one equation for ψ. Since the influence of relaxation is weak, δ being a small dimensionless parameter measuring this influence, there must exist a non-relaxing gas flow such that

$$\mathsf{R}(\psi, q)-\mathsf{C}_a(\psi) = O(\delta), \tag{2.10}$$

where

$$\mathsf{C}_a(\psi) = 0$$

is the differential equation for ψ in the associated non-relaxing "classical" gas flow. In slow flow ($\delta_e \ll 1$) we identify $\mathsf{C}_a(\psi) = 0$ with the equation for ψ in equilibrium flow:

$$\mathsf{C}_a(\psi) = \mathsf{R}\big(\psi, \tilde{q}(\psi, \partial\psi/\partial y_i, \partial\psi/\partial t, \ldots)\big).$$

The left hand side of Eq. (2.10) vanishes with δ_e in this case, since the difference between q and $\tilde{q}$ vanishes with δ_e. If the flow is nearly frozen ($\delta_f \ll 1$), the associated flow is the completely frozen flow and we choose

$$\mathsf{C}_a(\psi) = \mathsf{R}(\psi, q_0).$$

The left hand side of Eq. (2.10) vanishes with δ_f in this case, since $(q-q_0)$ vanishes with δ_f. Finally, if the difference between the frozen and equilibrium speeds of sound is small ($\delta_r \ll 1$), the operator C_a may be identified either with $\mathsf{R}(\psi, \tilde{q})$ or with $\mathsf{R}(\psi, q_0)$. Let us choose in this case

$$\mathsf{C}_a(\psi) = \mathsf{R}(\psi, q_0).$$

The left hand side of Eq. (2.10) vanishes with δ_r in this case, since the dependence of $\mathsf{R}(\psi, q)$ on q vanishes with δ_r. Note that in each of the three cases $\delta_e \ll 1$, $\delta_f \ll 1$ and $\delta_r \ll 1$, C_a is of the form

$$\mathsf{C}_a(\psi) = \mathsf{R}\big(\psi, q_a(\psi, \partial\psi/\partial y_i, \partial\psi/\partial t, \ldots)\big),$$

where $q_a = \tilde{q}$ if $\delta_e \ll 1$ and $q_a = q_0$ if $\delta_f \ll 1$ or $\delta_r \ll 1$.

(*) Note that, due to the fact that δ and ε are small, it follows from the Croccos theorem that the flow is irrotational for all times, if it was so initially.

Let us take as an example the one-dimensional unsteady flow of a relaxing gas. In this case, from Eq. (2.7) we get

$$\mathsf{R}(p,\tilde{q}) = -(A^{-2}(p)p_t)_t + p_{xx}, \tag{2.11}$$

$$\mathsf{R}(p,q_0) = -(B^{-2}(p)p_t)_t + p_{xx}, \tag{2.12}$$

where $A(p) = (\varrho/\varrho_0)a = \{B^{-2} - \varrho_0^2(\varrho^{-1})_q|_{q=\tilde{q}}\tilde{q}_p\}^{-1/2}$ has the meaning of the material equilibrium speed of sound.

Next, let us make use of the fact that the amplitude ε, by which the flow differs from a reference flow, is small. We assume that the reference flow is a parallel flow with constant velocity U_0 of a gas, which is homogeneous and in thermodynamic equilibrium. Let the values of ψ and q in the reference flow be ψ_0 and $q_0 = \tilde{q}_0$, respectively. Note that as a consequence of these properties, the reference flow is simultaneously a possible flow of both the relaxing gas and the associated non-relaxing gas:

$$\mathsf{R}(\psi_0, -q_0) = 0,$$

$$\mathsf{R}(\psi_0, q_a(\psi_0, \partial\psi_0/\partial y_i, \ldots)) = 0.$$

Therefore, $\mathsf{R}(\psi, q) - \mathsf{R}(\psi, q_a)$ does not only vanish with δ, but also with the amplitude ε:

$$\mathsf{R}(\psi, q) - \mathsf{R}(\psi, q_a) = 0(\varepsilon)\cdot O(\delta).$$

Since we neglect terms of order, $\varepsilon^2\delta$, $\mathsf{R}(\psi, q) - \mathsf{R}(\psi, q_a)$ may be simplified, linearizing in $(q - q_a)$ and its derivatives:

$$\mathsf{R}(\psi, q) - \mathsf{R}(\psi, q_a) = \frac{b_0 - a_0}{b_0}\mathsf{L}(q - q_a), \tag{2.13}$$

where L is a linear differential operator. Due to the fact that the reference flow is a steady parallel flow of a homogeneous gas, the coefficients of L are constant. Note that

$$\frac{b_0 - a_0}{b_0}\mathsf{L}(q - q_a)$$

vanishes both with δ and ε. If $\delta_e \ll 1$ or $\delta_f \ll 1$, this is true, because $(q - q_a)$ vanishes with $\delta\varepsilon$. If $\delta_r = (b_0 - a_0)/b_0 \ll 1$, it is also true, since then the dependence of R on q vanishes with δ, while $(q - q_a)$ vanishes with ε. To stress this fact, we have taken the factor $(b_0 - a_0)/b_0$ out of the linear operator L. Making use of Eq. (2.13) we can rewrite Eq. (2.5) in the form

$$\mathsf{R}(\psi, q_a) + \frac{b_0 - a_0}{b_0}\mathsf{L}(q - q_a) = 0. \tag{2.14}$$

The next few steps will be made separately for the cases $\delta_f \ll 1$ or $\delta_r \ll 1$ and $\delta_e \ll 1$, respectively. To make the following arguments clearer, let us introduce for the moment a dimensionless time t^* by

$$t^* = t/T,$$

where T is the time scale of typical pressure changes for the particles. Then, the relaxation Eq. (2.6) may be given in the form

$$\frac{\tau}{\tau_0}\frac{\tau_0}{T}\frac{Dq}{Dt^*} = -(q - \tilde{q}). \tag{2.15}$$

Note that, due to the particular choice of T, Dq/Dt^* and $D\psi/Dt^*$ are, at the most, of the order ε, whereas, by definition, $T/\tau_0 = \delta_f \ll 1$ for nearly frozen flow, and $\tau_0/T = \delta_e \ll 1$ for very slow flow. Further, let us denote the linearized form of the material time derivative by D_0/D_0t^*. Since the reference flow is a steady parallel flow, D_0/D_0t^* commutes with the operator L.

Let us now reduce the two Eqs. (2.14) and (2.15) to one equation for ψ, first taking the case in which either $(b_0-a_0)/b_0 = \delta_r \ll 1$ or $T/\tau_0 = \delta_f \ll 1$. In both cases we have chosen $q_a = q_0$. If we apply the operator $D_0/D_0t^*+T/\tau_0$ to Eq. (2.14) we get, taking advantage of the fact that the operators L and D_0/D_0t^* commute,

$$\left(\frac{D_0}{D_0t^*}+\frac{T}{\tau_0}\right)\mathsf{R}(\psi, q_0)+\frac{b_0-a_0}{b_0}\mathsf{L}\left(\frac{D_0q}{D_0t^*}+\frac{T}{\tau_0}(q-q_0)\right)=0.$$

Now, in both cases $\delta_r \ll 1$ and $\delta_f \ll 1$, the term $\frac{b_0-a_0}{b_0}\mathsf{L}\left(\frac{D_0q}{D_0t^*}\right)$ differs only by terms of order $\varepsilon^2\delta$ from $\frac{b_0-a_0}{b_0}\mathsf{L}\left(\frac{\tau}{\tau_0}\frac{Dq}{Dt^*}\right)$. Therefore, in both these cases we can write, using the relaxation Eq. (2.15)

$$\frac{D_0}{D_0t^*}\mathsf{R}(\psi, q_0)+\frac{T}{\tau_0}\left\{\mathsf{R}(\psi, q_0)+\frac{b_0-a_0}{b_0}\mathsf{L}(\tilde{q}-q_0)\right\}=0.$$

According to Eq. (2.13), the second term of this equation equals $T/\tau_0\cdot\mathsf{R}(\psi,\tilde{q})$. Therefore, transforming t^* back to the dimensional time t, we arrive at the following result:

$$\tau_0\frac{D_0}{D_0t}\mathsf{R}(\psi, q_0)+\mathsf{R}(\psi,\tilde{q})=0. \tag{2.16}$$

Next, let us show that the same equation also holds, if $\delta_e \ll 1$. In this case we apply the operator $\left(\tau_0/T\frac{D_0}{D_0t^*}+1\right)$ to Eq. (2.14), what leads to

$$\left(\frac{\tau_0}{T}\frac{D_0}{D_0t^*}+1\right)\mathsf{R}(\psi,\tilde{q})+\frac{b_0-a_0}{b_0}\mathsf{L}\left(\frac{\tau_0}{T}\frac{D_0}{D_0t^*}(q-\tilde{q})+(q-\tilde{q})\right)=0.$$

Within an error of order $\varepsilon^2\delta_e$, we have

$$\frac{\tau_0}{T}\frac{D_0}{D_0t^*}(q-\tilde{q})=\frac{\tau_0}{T}\frac{\tau}{\tau_0}\frac{D}{Dt^*}(q-\tilde{q})=-(q-\tilde{q})-\frac{\tau_0}{T}\frac{D_0\tilde{q}}{D_0t^*}.$$

Therefore,

$$\frac{\tau_0}{T}\frac{D_0}{D_0t^*}\mathsf{R}(\psi,\tilde{q})+\frac{b_0-a_0}{b_0}\mathsf{L}\left(-\frac{\tau_0}{T}\frac{D_0}{Dt^*}(\tilde{q}-q_0)\right)+\mathsf{R}(\psi,\tilde{q})=0$$

from which, using Eq. (2.13) we get

$$\frac{\tau_0}{T}\frac{D_0}{D_0t^*}\mathsf{R}(\psi, q_0)+\mathsf{R}(\psi,\tilde{q})=0.$$

Going back to the dimensional time t we find again Eq. (2.16).

Finally, let us sum up the conditions used and the results which have been obtained. The conditions, which allow for considerable simplifications of the equations, are the following:

1. Effects of viscosity, heat conduction and diffusion can be neglected.

2. The gas has only one internal state variable.

3. The relaxation equation is of the form $Dq/Dt = -(q-\tilde{q}(p,s))/\tau(p,q,s)$.

4. The influence of relaxation is weak and the amplitude, by which the flow differs from a reference flow, is small; terms of order ε^3 and $\varepsilon^2\delta$ are neglected.

5. The unperturbed reference flow of the gas is a steady parallel flow of the homogeneous equilibrium gas.

If these conditions are satisfied, the following algorithm can be employed to derive a simplified equation for a field variable ψ which we choose:

1. Derive the differential equation $\mathsf{R}(\psi, q) = 0$, eliminating the other field variables from the balance equations, taking advantage of the fact that the flow is nearly isentropic.

2. Express the equilibrium value $\tilde{q}$ of q in terms of ψ, $\partial\psi/\partial y_i$, $\partial\psi/\partial t$, $\cdot$.... Derive the differential operators $\mathsf{R}(\psi, \tilde{q})$ and $\mathsf{R}(\psi, q_0)$.

Remark: $\mathsf{R}(\psi, \tilde{q}) = 0$ and $\mathsf{R}(\psi, q_0) = 0$ are the differential equations for ψ in strict equilibrium flow and strict frozen flow, respectively.

3. Then, $\tau_0 \dfrac{D_0}{D_0 t} \mathsf{R}(\psi, q_0) + \mathsf{R}(\psi, \tilde{q}) = 0$ is the simplified differential equation for ψ in the relaxing gas flow, where $\tau_0 = \tau(p_0, s_0, q_0)$ is the relaxation time in the reference state of the gas, and where $D_0/D_0 t$ is the linearized material time derivative.

3. Examples: One-dimensional unsteady flow

Let us, finally, apply the algorithm to some special gas flows. For *one-dimensional small amplitude flow* we have already derived expressions (2.11) and (2.12) for $\mathsf{R}(p, \tilde{q})$ and $\mathsf{R}(p, q_0)$, respectively. Therefore, the algorithm leads to the following equation for the pressure:

$$\tau_0 \frac{\partial}{\partial t} \{(p_t/B^2(p))_t - p_{XX}\} + \{(p_t/A^2(p))_t - p_{XX}\} = 0. \tag{3.1}$$

This equation is asymptotically correct in each of the three cases $\delta_e \ll 1$, $\delta_f \ll 1$ and $\delta_r \ll 1$.

We will term a wave $p(X, t)$ *a "right-running"* wave if

$$p = f(X - Vt, t) = f(\xi, t), \tag{3.2}$$

where a typical value of $\partial f(\xi, t)/\partial t$ is smaller by a factor of the order of δ than a typical value of $\partial f/\partial \xi$. We expect that Eq. (3.1) allows for right-running waves as solutions. The material speed $V > 0$ of these waves should be approximately

$$V = A \quad \text{if} \quad \delta_e \ll 1,$$
$$V = B \quad \text{if} \quad \delta_f \ll 1.$$

In the case of a small difference of speeds of sound, $\delta_r \ll 1$, we may choose either $V = A$ or $V = B$. To be definite, let us take

$$V = B \quad \text{if} \quad \delta_r \ll 1.$$

Inserting Eq. (3.2) into Eq. (3.1) and neglecting higher order terms, we obtain, after some straightforward manipulations,

$$\tau_0 \frac{\partial}{\partial t}\{p_t + B(p)p_X\} + \{p_t + A(p)p_X\} = 0, \quad \text{if} \quad \delta_r \ll 1, \tag{3.3}$$

$$p_t + A(p)p_X - \frac{1}{2}\tau_0 A_0^2\left(1 - \frac{A_0^2}{B_0^2}\right)p_{XX} = 0, \quad \text{if} \quad \delta_e \ll 1, \tag{3.4}$$

$$p_t + B(p)p_X + \frac{1}{2\tau_0}\frac{B_0^2 - A_0^2}{A_0^2}(p - p_0) = 0, \quad \text{if} \quad \delta_f \ll 1. \tag{3.5}$$

Equation (3.3) is essentially the equation of OCKENDON and SPENCE [4]. Equation (3.4) is the BURGER's equation [1, 2] for low-frequency waves and (3.5) is the well-known kinematic wave equation which governs high-frequency processes (see, e.g., BECKER [5]). The kinematic wave Eq. (3.5) can be used, for instance, to study acceleration waves.

4. Examples: Two-dimensional steady flow

In a similar way the algorithm leads to the following equation for *small amplitude two-dimensional irrotational steady flows*:

$$U_0\tau_0 \frac{\partial}{\partial x_1}\{(1 - \phi_{x_1}^2/b^2)\phi_{x_1x_1} + \phi_{x_2x_2} - 2\phi_{x_1x_2}\phi_{x_1}\phi_{x_2}/b^2\}$$
$$\{(1 - \phi_{x_1}^2/a^2)\phi_{x_1x_1} + \phi_{x_2x_2} - 2\phi_{x_1x_2}\phi_{x_1}\phi_{x_2}/a^2\} = 0. \tag{4.1}$$

In this equation x_1 and x_2 Cartesian coordinates, ϕ is the velocity potential and U_0 is the speed of the unperturbed reference flow which is in x_1-direction. The frozen and equilibrium speeds of sound, $b(p) = (\partial\hat{\varrho}(p, q_0)/\partial p)^{-1/2}$ and $a(p) = (d\hat{\varrho}(p, \tilde{q}(p))/dp)^{-1/2}$, can be computed as functions of ϕ_{x_1}, ϕ_{x_2}, making use of the Eqs. (2.8) and (2.9). Equation (4.1) is again correct in each of the three cases $\delta_e \ll 1$, $\delta_f \ll 1$, $\delta_r \ll 1$. It can be used, for instance, to study the flow around a slender profile. Its classical counterpart can be found in many text books (see, e.g., [6, 7]).

If the difference of speeds of sound is small ($\delta_r \ll 1$), Eq. (4.1) may be simplified further for *trans-sonic* flow. This is due to the fact that in the trans-sonic flow of a gas with $(b_0 - a_0)/b_0 \ll 1$ both the frozen and the equilibrium Mach number

$$M_{f0} = U_0/b_0 \quad \text{and} \quad M_{e0} = U_0/a_0$$

of the reference flow are close to one, and both the frozen and equilibrium characteristics are nearly parallel to the x_2-axis. Using these facts we get, after some straightforward manipulations,

$$b_0\tau_0\frac{\partial}{\partial x_1}\{(1 - M_{f0}^2)\phi_{x_1x_1} + \phi_{x_2x_2} - 2(1 + \varrho_0 b_0 b_0')(\phi_{x_1}/b_0 - M_{f0})\phi_{x_1x_1}\}$$
$$+ \{(1 - M_{e0}^2)\phi_{x_1x_1} + \phi_{x_2x_2} - 2(1 + \varrho_0 a_0 a_0')(\phi_{x_1}/a_0 - M_{e0})\phi_{x_1x_1}\} = 0,$$

where b_0' is the following derivative of $b(p)$:

$$b_0' = (db/dp)|_{p=p_0}$$

and where a_0' is defined similarly.

Steady right-running waves are the steady two-dimensional equivalent of unsteady one-dimensional right-running waves discussed in the previous chapter. If $\delta_r \ll 1$, such waves are governed asymptotically by the equation

$$U_0\tau_0\frac{\partial}{\partial x_1}\{\phi_{x_1}+\phi_{x_2}/\sqrt{M_{f0}^2-1}+\phi_{x_1}^2(1+\varrho_0 b_0 b_0')M_{f0}^4/(2U_0(M_{f0}^2-1))\}$$
$$+\{\phi_{x_1}+\phi_{x_2}/\sqrt{M_{e0}^2-1}+\phi_{x_1}^2(1+\varrho_0 a_0 a_0')M_{e0}^4/(2U_0(M_{e0}^2-1))\}=0.$$

This result can be derived from Eq. (4.1), employing essentially the same method which leads from Eq. (3.1) to Eq. (3.3) in the one-dimensional unsteady case.

Finally, for *stream tube approximation of steady flows*, the algorithm leads to the following simplified differential equation:

$$U_0\tau_0\frac{d}{dx}\left\{\vartheta\left(1-\frac{b}{u}\right)\frac{du}{dx}+\frac{b^2}{u}\frac{d\vartheta}{dx}\right\}-\left\{\vartheta\left(1-\frac{a}{u}\right)\frac{du}{dx}+\frac{a^2}{u}\frac{d\vartheta}{dx}\right\}=0, \tag{4.2}$$

where U_0 is the speed of the unperturbed flow and where $\vartheta=\varrho_0U_0F_0/F(x)$ is the mass flow, $F(x)$ being the cross section of the stream tube. Note that the speeds of sound, b and a, can be expressed in terms of the velocity u. This is a consequence of the Eqs. (2.8) and (2.9). Equation (4.1) is again correct in each of the three cases $\delta_e \ll 1$, $\delta_f \ll 1$, $\delta_r \ll 1$. The condition that the amplitude be small implies that the relative change of the cross section of the stream tube, $(F(x)-F_0)/F_0$, must be sufficiently small.

The examples given in this and the previous section demonstrate the utility of the method which has been proposed in this paper. As far as the author knows, Eq. (3.1) of Section 3 and all the equations of Section 4 have been published for the first time in this paper.

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