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A Navier-Stokes analysis of developing slip flow

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VELOCITY distribution and pressure drop are presented for incompressible slip flow in the inlet section of a parallel plate channel. In contrast to boundary layer theory, the Navier-Stokes profiles show maximum velocities off the centerline. This phenomenon is explained as following from the high pressure gradients perpendicular to the channel axis. The dependence on Reynolds number and slip coefficient is discussed. The pressure decrease is compared for slip coefficients between 0.0 and 1.99. It is shown that by first-order boundary layer theory the overall pressure drop in the inlet region is given as too low.

W pracy przedstawiono rozkład prędkości i spadku ciśnienia dla nieściśliwego przepływu z poślizgiem u wlotu płaskiego kanału złożonego z płyt równoległych. W przeciwieństwie do teorii warstwy przyściennej profile maksymalnych prędkości obliczone z równań Naviera-Stokesa występują nie na linii środkowej. Jak wyjaśniono, zjawisko to wynika z dużych gradientów ciśnienia prostopadłych do osi kanału. Przedyskutowano wpływ na rozwiązanie liczby Reynoldsa i współczynnika poślizgu. Spadek ciśnienia porównano dla współczynnika lepkości wziętego z przedziału 0–1,99. Pokazano, że całkowity spadek ciśnienia w obszarze wlotu, obliczony według teorii warstwy przyściennej pierwszego rzędu, jest zbyt zaniżony.

В работе представлено распределение скорости и падения давления для несжимаемого течения со скольжением на входе плоского канала, состоящего из параллельных пластин. В противовес к теории пограничного слоя профили максимальных скоростей, вычислены из уравнений Навье-Стокса, выступают не на срединной линии. Как выяснено это явление следует из больших градиентов давления перпендикулярных к оси канала. Обсуждено влияние на решение числа Рейнольдса и коэффициента скольжения. Падение давления сравнено для коэффициента вязкости вятого из интервала 0–1,99. Показано, что полное падение давления в области входа, расчитанное по теории пограничного слоя первого порядка, слишком занижено.

1. Introduction

IN FLUID dynamics the wall shear stress τ_w is obtained from the velocity gradient

$$\tau_w = \eta \frac{\partial u}{\partial y}\bigg|_{y=0} \tag{1.1}$$

(y — dimensionless coordinate normal to the channel axis, u — dimensionless axial velocity component, asterisks omitted, see Eq. (2.5)). In the nonslip case the velocity gradient follows from the Navier-Stokes equation subject to the boundary condition

$$u|_{y=0} = 0. \tag{1.2}$$

Generally, gas flow can be treated as continuum flow with no slippage of the fluid over the bounding wall. If the pressure level is reduced or the solid surface "polished", perhaps in addition to rarefaction, slip has to be taken into account. In the slip flow regime which is confined to the Knudsen-number region

$$0.01 \bigg/ \frac{2-\alpha}{\alpha} \leqslant \mathrm{Kn} \leqslant 0.1 \tag{1.3}$$

the equations of continuum theory are retained [1]. The fluid surface interaction comes into play via the generalized boundary conditions

$$u|_{y=0} = \zeta \frac{\partial u}{\partial y}\bigg|_{y=0}. \tag{1.4}$$

In Eq. (1.4) constant wall temperature is assumed. We have for the slip-coefficient ζ [2]

$$\zeta = \frac{2-\alpha}{\alpha}\,\mathrm{Kn} \tag{1.5}$$

($\mathrm{Kn} = l/h$, l — mean free path, h — channel half-width). While Kn depends on the physical state of the gas, α is characteristic of the interaction system made up of a solid surface and a gaseous phase. The so-called accomodation coefficient of tangential momentum α is just that fraction of the incident momentum flux lost to the wall. Studies of the slip flow field have been carried out primarily for external flows, see SCHAAF [3]. For inner flows it appears that most authors have been investigating the Hagen-Poiseuill regime.

The incompressible entrance flow in a parallel-plate channel was investigated for slip by SPARROW, LUNDGREN and LIN [4] and by QUARMBY [5] who integrated the first-order boundary layer equation. Because of the relationship

$$\mathrm{Kn} \sim \mathrm{Ma}/\mathrm{Re}, \qquad \mathrm{Re} = \frac{u_m h}{\nu} \tag{1.6}$$

the flow field is incompressible for comparatively low Re-numbers only. So actually we cannot change from the Navier-Stokes to the first-order boundary layer equations. Moreover, for $\alpha \approx 1$ slippage is a second-order boundary layer effect and becomes of first order only if α is small as compared with unity; the slip condition, Eq. (1.4), is generally not compatible with first-order boundary layer approximation.

In the present paper the steady laminar slip flow of an incompressible Newtonian fluid in the inlet section of a two-dimensional channel is analysed by integrating the Navier-Stokes equations. The numerical method of solution developed is exact in the sense that no terms in the momentum equation which are not identically zero have been neglected.

2. Formulation of the problem

Introducing the stream function ψ defined by

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \tag{2.1}$$

and the vorticity, which in a two-dimensional flow can be written

$$\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \tag{2.2}$$

we reduce the Navier-Stokes equation and the continuity equation to a parabolic vorticity transport equation

$$\frac{\partial \omega}{\partial t} + \frac{\partial}{\partial x}(u\omega) + \frac{\partial}{\partial y}(v\omega) - \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) = 0 \tag{2.3}$$

and an elliptic Poisson equation for the stream function

(2.4) $$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\omega.$$

Equation (2.3) is the "conservation form" of the vorticity transport equation since it may be shown that the transport property ω is conserved. After rendering the terms of Eqs. (2.3) and (2.4) dimensionless it is possible to apply Reynolds principle of similarity. With h and u_m (mean velocity) being the characteristic length and velocity respectively, we define

(2.5) $$p^*=\frac{p}{\varrho u_m^2},\quad x^*=\frac{x}{h\mathrm{Re}},\quad y^*=\frac{y}{h},\quad t^*=\frac{tu_m}{h\mathrm{Re}},$$
$$u^*=u/u_m,\quad v^*=v\mathrm{Re}/u_m,\quad \omega^*=\omega h/u_m.$$

As the velocity approaches the fully-developed profile only asymptotically we change from the infinite x-region $x\in[0,\infty]$ to the finite downstream boundary $\xi\in[0,1]$ applying the transformation [6]:

(2.6) $$\xi=1-\frac{1}{1+1.2\mathrm{Re}x^*},\quad \frac{\partial\xi}{\partial x^*}=n(\xi),\quad \frac{\partial n}{\partial\xi}=\dot{n},$$

which results in the dimensionless conservation vorticity transport equation (we omit the asterisks!)

(2.7) $$\frac{\partial\omega}{\partial t}=\frac{1}{\mathrm{Re}^2}n\dot{n}\frac{\partial\omega}{\partial\xi}+\frac{1}{\mathrm{Re}^2}n^2\frac{\partial^2\omega}{\partial\xi^2}+\frac{\partial^2\omega}{\partial y^2}-h\frac{\partial}{\partial\xi}(u\omega)-\frac{\partial}{\partial y}(v\omega).$$

The dimensionless vorticity and stream function are related by the Poisson equation

(2.8) $$\frac{1}{\mathrm{Re}^2}\left[n^2\frac{\partial^2\psi}{\partial\xi^2}+n\dot{n}\frac{\partial\psi}{\partial\xi}\right]+\frac{\partial^2\psi}{\partial y^2}=\omega.$$

Various inlet conditions were considered by WILSON [7]. On the assumption of uniform velocity component u and zero vorticity ω we have, at the channel entrance,

(2.9) $$\xi=0:\quad \psi=y,\quad \omega=0.$$

It was shown, see VAN DYKE [8], that Eq. (2.9) does not replace the inlet condition $u=0$, $v=0$ which introduces weak vorticity into the inviscid core.

Because of the small boundary layer thickness near the very beginning of the plate, for outer flows and $\alpha\approx 1$ it might be doubted whether the Navier-Stokes equation is appropriate there. But for inner flows the channel half-width may be considered as the characteristic length. Moreover, the mean free path is small as far as we confine ourselves to small Kn-numbers (0.01, or even lower) and ζ values resulting from small α values.

At the axis of symmetry we have

(2.10) $$y=1:\quad \psi=1,\quad \omega=0.$$

The channel wall is at $y=0$. Higher order boundary layer theory, which has to assume ζ to be small in order to get convergence of the power series, shows that for slip flow $v_w\neq 0$ is a third-order effect only. This argument might be reconsidered for large ζ values. In this paper the slip boundary conditions are (see Eq. (1.4)):

(2.11) $$y=0:\quad u|_{y=0}=\zeta\left.\frac{\partial u}{\partial y}\right|_{y=0}=\zeta\omega|_{y=0},\quad \psi=\psi_{y=0}.$$

The pressure gradient in the channel only weakly contributes to the slip velocity. The related term has been neglected in Eq. (2.11).

Far downstream the fully-developed condition must be fulfilled:

$$(2.12) \qquad \xi = 1: \quad \psi = \frac{3}{1+3\zeta}\left(-\frac{1}{6}y^3+\frac{1}{2}y^2+\zeta y\right), \quad \omega = \frac{3}{1+3\zeta}(1-y).$$

The solution of the large system of algebraic equations approximating the elliptic Poisson-equation generally consumes the overwhelming amount of time. By using a method based on Fast-Fourier technique, which solves the system of equations directly in contrast to iterative procedures, the time needed has been reduced. The method works if one has constant mesh sizes and the partial differential equation has no first derivative in the Fourier-analysis direction. Because of the last restriction and Eq. (2.8) we can apply the Fourier transformation only in y-direction. In order to maintain sufficient accuracy, especially near the wall finite difference formulae which follow the Mehrstellen approach have been set up introducing a truncation error of $0(\Delta y^4)$ only, without using more than three netpoints in each equation.

3. Results

In Table 1 the Re = 75 nonslip solution of this paper is compared with the Navier-Stokes solution given by WANG and LONGWELL [6], who solved the nonlips case for Re = 75, and the results from boundary layer theory presented by SCHLICHTING [9]. The velocity profiles for $\xi = 0.05$ are given. This cross-section is located in the critical region very near the entrance. The present results and WANG and LONGWELL [6] are in good accordance.

Table 1. Velocity-profile for $\xi = 0.05$

Y	u (SCHLICHTING)	u (WANG and LONGWELL)	u (this paper)
0	0	0	0
0.1	1.007	0.9958	1.0010
0.2	1.040	1.0324	1.0208
0.3	1.040	1.0265	1.0203
0.4	1.040	1.0214	1.0177
0.5	1.040	1.0178	1.0154
0.6	1.040	1.0155	1.0138
0.7	1.040	1.0140	1.0127
0.8	1.040	1.0131	1.0120
0.9	1.040	1.0126	1.0116
1.0	1.040	1.0124	1.0114

In contrast to boundary layer theory the Navier-Stokes profiles show maximum velocities off the centerline. They are not flat but saddle-shaped with a velocity overshoot at $y_{max} = 0.2$.

In Fig. 1 $u(y)$ is given for several channel cross-sections. Fluid moving from left to right enters the channel. With ξ increasing, the velocity maximum moves from the wall towards the centerline while the peak is reduced and the profiles gradually become parabolic. No Reynolds-number dependence and no explanation for the saddle-shape of the velocity profiles were given in WANG and LONGWELL'S paper.

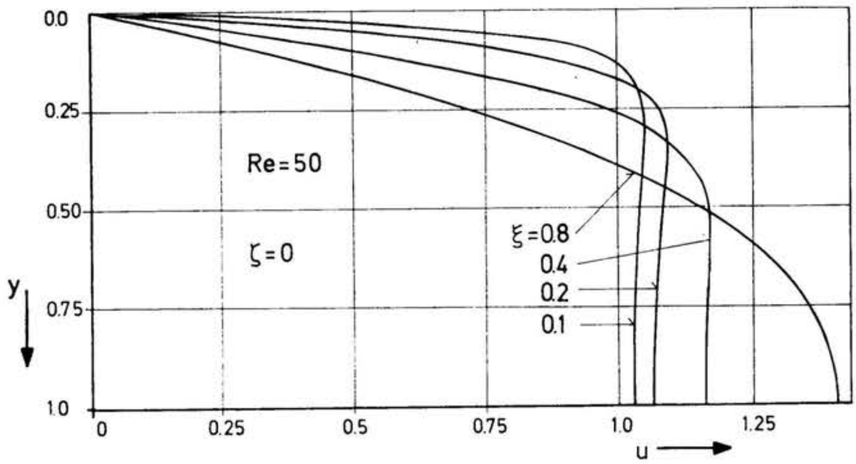

FIG. 1.

In Fig. 2 the pressure drop $\Delta p(y)$ between $\xi = 0$ and $\xi = 0.2$, as well as between $\xi = 0$ and $\xi = 0.4$ is presented. There is a remarkable transverse gradient in the pressure field near the entrance. This is in contradiction to boundary layer theory which assumes the pressure to be constant with respect to y.

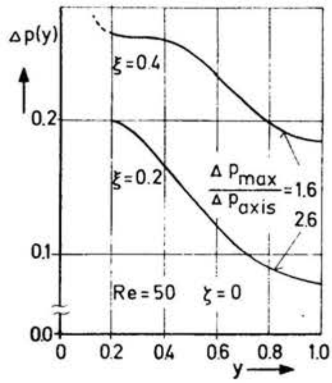

FIG. 2.

For small ξ large velocity gradients at the wall yield high shear stress values there. The boundary layer near the entrance is thin. This results in only a small amount of mass which must be transported to the channel center and only little acceleration of the central core is necessary. In accordance with this physical reasoning, in the vicinity of the entrance the pressure drop is largest near the wall. The pressure values themselves are higher near the axis, mass is prevented from getting to the central core which results in a velocity overshoot. Comparison with Fig. 1 shows that with ξ increasing, y_{max} does move towards the channel axis in common with the region of largest pressure gradients $\partial p/\partial y$.

In ξ-direction the thickness of the boundary layer at the channel wall increases. The reduced central core must be accelerated more because of the continuity equation, while the velocity gradient at the wall simultaneously becomes smaller and smaller. This at last results in constant pressure values in the y-direction. For $\xi = 0.6$ ($x/h \approx 1.25$) already, the maximum pressure drop is only 1.1 times the pressure drop at the axis. Accordingly, one expects the velocity overshoot to decrease.

In Fig. 3, u_{max}/u_{axis} is shown as a function of ξ for Re = 50, Re = 75 and several slip coefficients ζ. For the initially flat velocity profile and for the fully-developed profile we have $u_{max}/u_{axis} = 1$. The curves show maxima inbetween. Comparing Fig. 3 with Fig. 2 we find that the increase in central concavity is due to the developing gradient in the y-direction while u_{max}/u_{axis} in Fig. 3 approaches 1 with $\partial p/\partial y$ decreasing in Fig. 2.

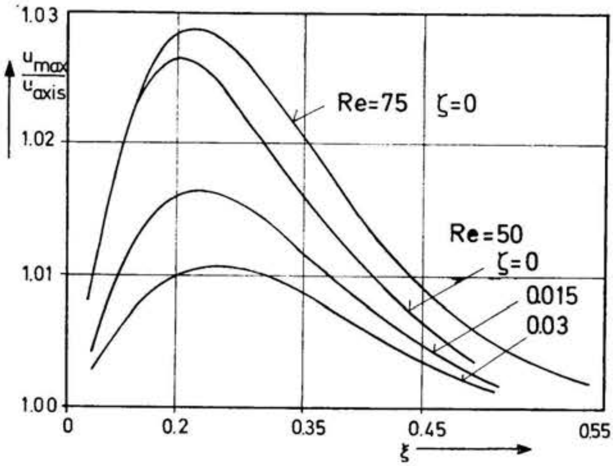

FIG. 3.

For Re = 75 we have higher shear stress values at the wall, the pressure gradient in y-direction is increased as compared with the Re = 50 flow field. The boundary layer thickness for a given ξ is reduced, less mass has to be transported towards the channel axis. As the results of the mathematical analysis presented in Fig. 3 show, the pressure gradient influence which favours the saddle-shape dominates and the velocity overshoot is higher for Re = 75.

The gradient $\partial p/\partial x$ is negative. Referring to GLAUERTS formula [10] we conclude that the shear stress for channel slip flow is lower than in the nonslip case. From this fact a reduction of $\partial p/\partial y$ with increasing ζ results and the velocity overshoot is diminished as the numerical analysis shows.

For Re = 75, $\zeta = 0$ and Re = 50, $\zeta = 0.03$ we find the maxima in Fig. 3 changed to higher x/h values as compared with the Re = 50, $\zeta = 0$ curve. This is due to the fact that the boundary layer and the pressure gradient between channel wall and channel axis is developed slower with respect to x/h for higher Re-numbers and higher slip-coefficients.

The related overshoot in the velocity profiles of the flat plate Navier-Stokes solution is presented by several authors, e.g., GERKING [11]. The given explanation for this profile shape to be necessary in order to fulfill the continuity equation is often refused because of the infiniteness of the flow field in direction perpendicular to the flat plate direction. The results of this paper seem to favour the existence of the overshoot effect in the flat plate velocity profiles.

The quantity K shown in Fig. 4 is equal to the pressure decrease with reference to entrance conditions minus that which would occur in the stated length for fully-developed parabolic flow

$$K(x) = \frac{p-p_0}{\varrho \bar{u}_m^2} - \frac{3}{1+3\zeta} \frac{x/h}{\mathrm{Re}_h}. \tag{3.1}$$

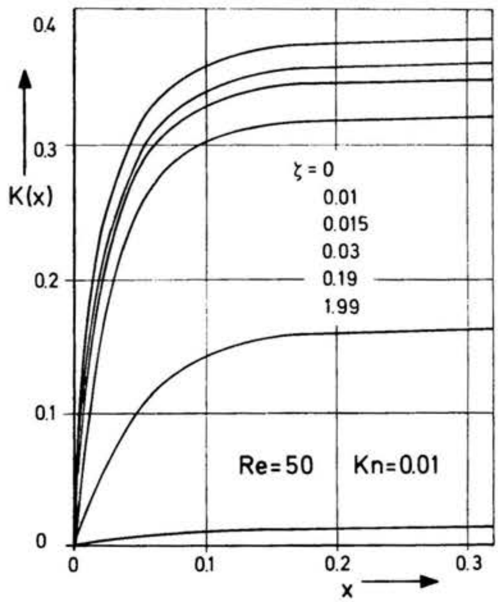

FIG. 4.

This quantity can be considered as an entrance loss in terms of dimensionless pressure. First-order boundary layer theory neglects pressure gradients in y-direction and yields (SCHLICHTING [9])

$$\zeta = 0: \quad K(x \to \infty) = 0.301.$$

Actually more work has to be done as the Navier-Stokes analysis shows; from this analysis

$$\zeta = 0: \qquad K(x \to \infty) = 0.385,$$

$$\zeta = 0.19: \quad K(x \to \infty) = 0.163$$

has been obtained.

Numerical results for $\zeta = 0.19$ and $\zeta = 1.99$ are presented. These ζ-values can be obtained in the slip flow regime only if the tangential momentum accomodation coefficient

(TAC) is smaller than 1. Nature produces TAC-values of the order of unity. Because of advances in solid state physics activities have been started in order to reduce TAC's and the related friction drag by a special treatment of the solid surface.

SEIDL and STEINHEIL [12] measured TAC's in the gold-helium system. After several cleaning procedures consisting of argon ion bombardement and vacuum annealing, the surface was almost completely cleaned and the TAC was reduced to below 0.1. From Fig. 4 the reduction of pressure drop gained by slippage can be obtained for the entrance region. Equation (3.1) yields the reduction of frictional drag for fully-developed parabolic velocity profiles.

For TAC = 0.1 and Kn = 0.01, for example, the pressure drop reduction is highest near the entrance ($\approx 70\%$) and is lowered to $\approx 36\%$ in the Hagen-Poiseuille region.

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