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Some problems of the theory of conical gas flows

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TWO-DIMENSIONAL unsteady problems of gas dynamics describing decompositions of two-dimensional discontinuities are considered. There are: the expansion of a wedge of gas into a vacuum, the instantaneous with drawal at constant velocity of two semi-infinite intersecting, straight, rigid walls which initially contain gas at rest within the dihedral angle formed by these walls, the expansion of gas from the tube. The existence of continuous solutions of the above mentioned problems is proved.

Rozważane są dwuwymiarowe zagadnienia dynamiki gazów opisujące rozkład dwuwymiarowych nieciągłości. Są to: ekspansja klina gazu w próżnię, nagłe wyjęcie ze stałą prędkością dwóch nieskończonych przecinających się prostych sztywnych ścian, które w chwili początkowej zawierają gaz w spoczynku wewnątrz kąta dwuściennego, utworzonego przez te ściany, wypływ gazu z rury. Udowodniono istnienie ciągłych rozwiązań wyżej wspomnianych zagadnień.

Рассмотрены двумерные задачи динамики газов, описывающие распад двумерных разрывов. Это: разлет клина газа в вакуум, внезапное удаление с постоянной скоростью двух полубесконечных, пересекающихся, плоских жестких стенок, которые в начальный момент содержат покоящийся газ внутри двугранного угла образованного этими стенками, истечение газа из трубы. Доказано существование непрерывных решений вышеупомянутых задач.

TWO-DIMENSIONAL unsteady problems of gas dynamics describing decompositions of discontinuities are considered. There are the expansion of a wedge of gas into a vacuum, the gas flow after a piston of a wedge form moving with constant velocity, the initial stage of the expansion of gas from the tube. These problems were studied in papers [1-4] where some numerical results were obtained. The purpose of this paper is to prove existence to the solutions of above mentioned problems. Proved theorems may be apllied to some other conical flow problems.

1. Equations of the motion

Two-dimensional potential conical flows of a polytropic gas are described by the quasi-linear second-order equation

$$
\begin{gathered}
((\varphi_\xi-\xi)^2-c^2)\varphi_{\xi\xi}+2(\varphi_\xi-\xi)(\varphi_\eta-\eta)\varphi_{\xi\eta}+\big((\varphi_\eta-\eta)^2-c^2\big)\varphi_{\eta\eta}=0,\\
c^2=(\gamma-1)(\xi\varphi_\xi+\eta\varphi_\eta-\varphi-{}^1/_2(\varphi_\xi^2+\varphi_\eta^2)),
\end{gathered}
\tag{1.1}
$$

where φ is potential, $\varphi_\xi = u$, $\varphi_\eta = v$, $\mathbf{u} = (u, v)$ is velocity vector, c is sound speed, γ is polytropic index, $\xi = x/t$, $\eta = y/t$, x, y — coordinates at a plane, t — time. At all points where $|\nabla\varphi-\xi| > c > 0$ Eq. (1.1) is hyperbolic, if $|\nabla\varphi-\xi| < c$ Eq. (1.1) is elliptic. Eq. (1.1) parabolically degenerates at points where $|\nabla\varphi-\xi| = c$ or $c = 0$. We shall consid-

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er Eq. (1.1) in the domain of hyperbolicity. Introducing characteristic variables α, β by the relations

$$\eta_\alpha = \mathrm{tg}(\theta+A)\xi_\alpha, \quad \eta_\beta = \mathrm{tg}(\theta-A)\xi_\beta,$$
$$\theta = \mathrm{arctg}\,\eta - v/(\xi-u), \quad A = \arcsin c/|\mathbf{u}-\boldsymbol{\xi}|)$$

we obtain characteristic system of differential equations for five quantities A, θ, u, v, c

$$(1.2)\qquad \begin{aligned} &A_\alpha+\theta_\alpha+f(A)c_\alpha/c = 0, && A_\beta-\theta_\beta+f(A)c_\beta/c = 0,\\ &(\gamma-1)u_\alpha+2\sin(\theta-A)c_\alpha = 0, && (\gamma-1)u_\beta-2\sin(\theta+A)c_\beta = 0,\\ &(\gamma-1)v_\alpha-2\cos(\theta-A)c_\alpha = 0, && (\gamma-1)v_\beta+2\cos(\theta+A)c_\beta = 0. \end{aligned}$$

Here $(\gamma-1)f(A) = \mathrm{tg}A(4\cos^2A-\gamma-1)$. Variables ξ, η may be expressed in the form

$$(1.3)\qquad \xi = u+c\cos\theta/\sin A, \quad \eta = v+c\sin\theta/\sin A.$$

Differentiating of Eqs. (1.2) gives the system of two second-order equations for functions A and $C = \ln c$

$$(1.4)\qquad 2A_{\alpha\beta}+(f(A)C_\alpha)_\beta+(f(A)C_\beta)_\alpha = 0, \quad C_{\alpha\beta}+\psi(A)C_\alpha C_\beta = 0,$$

where $\psi(A) = 1+\mathrm{tg}A\cdot f(A)$. The quantities θ, u, v may be found by integrating of Eqs. (1.2) after we have determined A, C. Thus the boundary value problems for Eqs. (1.1) may be reduced to the corresponding problems for system (1.4). If formulas (1.3) define one-to-one mapping from the domain of definition of the solution in α, β plane to the corresponding domain in ξ, η plane, one can find the solution $\varphi(\xi, \eta)$.

2. Formulation of the problems

2.1. The expansion of gas into vacuum

Polytropic gas at rest is contained in the domain $x > |y|\,\mathrm{tg}\,\delta\,(\delta > 0)$. The wall $x = |y|\,\mathrm{tg}\,\delta$ disappears at the moment $t = 0$. It is necessary to describe expansion of gas into vacuum for $t > 0$. Motion will be one-dimensional out of some neighbourhood of the vertex angle. It will be described by simple Riemann waves. The domain of interaction of the simple waves is bounded by the characteristics of opposite families which pass through the point of intersection of the wave front lines bounding gas at rest in ξ, η plane. We must satisfy the conditions of continuous joining of the unknown solution to the simple waves. Thus we obtained the Goursat problem for Eq. (1.1).

2.2. The piston problem

We consider the instantaneous with drawal at constant velocity $\mathbf{U}_0 = (U_0, 0)$ $(U_0 < 0)$ of two semi-infinite intersecting straight rigid walls which initially contain gas at rest within the dihedral angle $x > |y|\,\mathrm{tg}\,\delta$ formed by these walls. As in the previous case it is necessary to find the solution in the domain of interaction of two one-dimensional solutions. We obtain mixed boundary problem for Eq. (1.1): the conditions of continuous joining of unknown solution to the simple waves and to the constant solution on characteristic and the condition $(u-U_0)\cos\delta \pm v\sin\delta = 0$ on the piston.

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2.3. Initial stage of the expansion of gas from a tube

Polytropic gas was contained at rest in the tube $x > 0$, $-1 < y < 0$. The wall $x = 0$ disappears at the instant of time $t = 0$ and gas is expanding into vacuum. For small values of t, expansion is described by one-dimensional law at all points out of some neighbourhoods of points $M(0, 0)$, $N(0, -1)$. One-dimensional flow is disturbed by waves of Prandtl-Mayer type centred at points M, N. To find the solution in the neighbourhood of point M, we must solve Goursat problem with singularity. On the characteristic passing point M we have conditions of continuous joining of unknown solution to the simple Riemann wave. At the point M the solution must have the singularity of the Prandtl-Mayer type.

3. Solvability of the Goursat type problems

After integrating of Eq. (1.4) we have

$$2A+\int_0^\alpha f(A)C_\alpha d\alpha+\int_0^\beta f(A)C_\beta d_\beta = (A+\theta)(0,\beta)+(A-\theta)(\alpha,0),$$

$$C_\alpha(\alpha,\beta) = C_\alpha(\alpha,0)\exp\Big(-\int_0^\beta \psi(A)C_\beta d_\beta\Big), \tag{3.1}$$

$$C_\beta(\alpha,\beta) = C_\beta(0,\beta)\exp\Big(-\int_0^\alpha \psi(A)C_\alpha d_\alpha\Big).$$

The boundary conditions of the problems 2.1.–2.3. determine functions $(A+\theta)(0,\beta)+$ $+(A-\theta)(\alpha,0)$, $C_\alpha(\alpha,0)$, $C_\beta(0,\beta)$. In the case of the problem 2.1

$$(A+\theta)(0,\beta)+(A-\theta)(\alpha,0) = \pi-2\delta,\quad C_\alpha(\alpha,0)<0,\quad C_\beta(0,\beta)<0,$$

$$\Big(A_\alpha+\frac{1}{2}f(A)C_\alpha\Big)(\alpha,0) = 0,\quad \Big(A_\beta+\frac{1}{2}f(A)C_\beta\Big)(0,\beta) = 0.$$

We shall formulate existence theorem for Goursat problem. The solvability of problems 2.1. and 2.2. for some parameters γ, δ, U_0 follows from this theorem. Let us introduce the following designations

$$A_0 = \arcsin\Big(\frac{1}{2}\sqrt{3-\gamma}\Big),\quad A_1 = \arccos\Big(\frac{1}{2}(1+\sqrt{2\gamma+3})^{\frac{1}{2}}\Big)\quad (1<\gamma<3),$$

$$m(\alpha,\beta,\alpha_0,\beta_0) = (A+\theta)(\alpha_0,\beta)+(A-\theta)(\alpha,\beta_0),$$

$$U = \frac{2}{\gamma-1}\operatorname{tg}A\Big(\frac{\gamma+1}{2}-\cos^2A\Big)C_\alpha-A_\alpha,\quad P = A_\alpha+f(A)C_\alpha,$$

$$V = \frac{2}{\gamma-1}\operatorname{tg}A\Big(\frac{\gamma+1}{2}-\cos^2A\Big)C_\beta-A_\beta,\quad Q = A_\beta+f(A)C_\beta.$$

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THEOREM 1. *Let* $c(\alpha, 0)$, $c(0, \beta)$, $m(\alpha, \beta, 0, 0) \in C^1(\Omega)$ $(\Omega = [0, \alpha_1] \times [0, \beta_1])$, $c(\alpha_1 0) = c(0, \beta_1) = 0$ *and one of three groups of conditions are satisfied*:

1) $$1 < \gamma < 3, \quad A(0,0) > A_0, \quad m(\alpha, \beta, 0, 0) < \pi - \sigma \quad (\sigma > 0),$$
$$c_\alpha(\alpha, 0) < 0, \quad c_\beta(0, \beta) < 0, \quad \phi(\alpha, 0) > 0, \quad Q(0, \beta) > 0;$$

2) $$1 < \gamma < 3, \quad 0 < A(0,0) < A_0, \quad m(\alpha, \beta, 0, 0) > 2A_1,$$
$$c_\alpha(\alpha, 0) < 0, \quad c_\beta(0, \beta) < 0, \quad \phi(\alpha, 0) < 0, \quad Q(0, \beta) < 0,$$
$$U(\alpha, 0) < 0, \quad V(0, \beta) < 0;$$

3) $$\gamma > 3, \quad A(0,0) > 0, \quad m(\alpha, \beta, 0, 0) < \pi - \sigma \quad (\sigma > 0),$$
$$U(\alpha, 0) < 0, \quad V(0, \beta) < 0, \quad c_\alpha(\alpha, 0) < 0, \quad c_\beta(0, \beta) < 0.$$

Then the classical solution of the Goursat problem exists in the domain $\mathcal{D}$ *boundary of which consist of characteristics* $\alpha = 0, \beta = 0$ *and the line, where* $c(\alpha, \beta) = 0$. *Mapping* $(\alpha, \beta \to (\xi, \eta)$ *is univalent in* $\mathcal{D}$.

P r o o f. Local existence theorem is valid under the conditions $|m| < \pi - \sigma_1$, $|C_\alpha| < K$, $|C_\beta| < K$. We prove that monotony conditions of Theorem 1 are valid in the domain of definition of the solution. For example, in the case 1 $c_\alpha(\alpha, \beta) < 0$, $c_\beta(\alpha, \beta) < 0$, $\phi(\alpha, \beta) > 0$, $Q(\alpha, \beta) > 0$. It follows from (3.1) and the linear system of differential equations for ϕ and Q. Integrating of monotony conditions gives inequality $A(\alpha, \beta) > A_0$ at all points where $c > 0$. Eq. (3.1) gives the bound $A_0 < A < (\pi - \sigma)/2$. This allows to get the following estimate

$$2A_0 < m(\alpha, \beta, \alpha_0, \beta_0) < \frac{1}{2}(\pi - \sigma) \quad (\alpha \geqslant \alpha_0, \quad \beta \geqslant \beta_0).$$

From the second equation (1.4) we obtain lower bound of

$$c(\alpha, \beta) \geqslant \big(c(\alpha_0, \beta)^s + c(\alpha, \beta_0)^s - c(\alpha_0, \beta_0)^s\big)^{1/s},$$

where $s = s(\gamma) > 1$, $\alpha > \alpha_0$, $\beta > \beta_0$. With the use of these bounds we prove that the conditions of local theorem are valid in the rectangle $0 \leqslant \alpha \leqslant \alpha_0(\varepsilon)$, $0 \leqslant \beta \leqslant \beta_1(\varepsilon)$ $(c(0, \beta_1(\varepsilon) = 2\varepsilon,\ c(\alpha_0(\varepsilon), \beta_1(\varepsilon)) \geqslant \varepsilon)$. Then we consider the rectangle $\alpha_0(\varepsilon) \leqslant \alpha \leqslant 2\alpha_0(\varepsilon)$, $0 < \beta < \beta_2(\varepsilon)$ $(c(\alpha_0(\varepsilon), \beta_2(\varepsilon)) = 2\varepsilon,\ c(2\alpha_0(\varepsilon), \beta_2(\varepsilon)) \geqslant \varepsilon)$ and so on. If the domain $\mathcal{D}_\varepsilon$ is the union of the obtained rectangles, the domain $\mathcal{D} = \lim\limits_{\varepsilon \to 0} \mathcal{D}_\varepsilon$. Univalence of the mapping (1.3) is proved in the analogous manner, as it will be done in Theorem 2.

From this theorem we obtain solvability of *Problem 1* with parameters $1 < \gamma < 3$, $0 < \delta < \frac{1}{2}\pi - A_1$ and $\gamma > 3$, $0 < \delta < \pi/2$.

Problem 2. From the properties of the boundary conditions of Goursat problem follows the existence of constant U_*, for which inequality $U_0 \leqslant U_*$ involves that the domain of interaction of simple waves contains points where $c = 0$. In this case the mixed boundary problem may be reduced to two Goursat problems, for which existence of the solution may be obtained from Theorem 1. Notice, that continuous solution was constructed in the numerical calculations only in the case of vacuum zone appearance in the neighbourhood of the vertex angle.

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Problem 3. Point $\xi = 0, \eta = 0$ maps into interval $\alpha = 0$, $0 \leqslant \beta \leqslant \beta_1$ on the characteristic plane. The Prandtl-Mayer conditions are satisfied on this interval. On characteristic $\beta = 0$ we have the condition of continuous joining of the simple Riemann wave.

THEOREM 2. *Let* $\gamma > 3$, $c(\alpha, 0)$, $m(\alpha, \beta, 0, 0) \in c^1(\Omega)$, *and following inequalities are valid*

$$c_\alpha(\alpha, 0) < 0, \quad U(\alpha, 0) < 0, \quad 0 < m(\alpha, \beta, 0, 0) < \pi.$$

Then the Goursat problem with singularity has continuosly differentiable solution in the domain $\mathscr{D}$ *bounded by characteristics* $\alpha = 0, \beta = 0$ *and the line* $c(\alpha, \beta) = 0$. *Mapping* $(\alpha, \beta) \to \to (\xi, \eta)$ *is univalent in* $\mathscr{D}$.

Scheme of the proof is analogous to the corresponding Theorem 1. We shall prove univalence of the mapping (1.3). From the existence theorem we obtain the following properties of the solution

$$(3.2) \qquad 0 < m(\alpha, \beta, \alpha_0, \beta_0) < \pi - \sigma \quad (\alpha \geqslant \alpha_0 > 0, \beta \geqslant \beta_0 > 0), \quad (A+\theta)_\alpha < 0,$$
$$(A-\theta)_\beta < 0, \quad U(\alpha, \beta) < 0, \quad V(\alpha, \beta) < 0 \quad (\alpha > 0, \beta > 0).$$

We have relations

$$(3.3) \qquad \xi_\alpha = \cos(\theta + A)U, \quad \eta_\alpha = \sin(\theta + A)U,$$
$$\xi_\beta = \cos(\theta - A)V, \quad \eta_\beta = \sin(\theta - A)V.$$

Notice that turning of the coordinate system in the ξ, η plane changes quantity θ into corresponding constant. According to the properties (3.2), the coordinate system may be chosen so that

$$|A + \theta| < {}^1/_2(\pi - \sigma), \quad |A - \theta| < {}^1/_2(\pi - \sigma).$$

In this case $\xi_\alpha < 0$, $\xi_\beta < 0$. Let $(\alpha_0, \beta_0) \in \mathscr{D}$. We shall prove that $\boldsymbol{\xi}(\alpha_0, \beta_0) \neq \boldsymbol{\xi}(\alpha, \beta)$ if $(\alpha, \beta) \neq (\alpha_0, \beta_0)$. Introduce the coordinate system in α, β plane with the centre in the point (α_0, β_0) and coordinate axes parallel to original axes. It follows from the monotonicity of ξ that points where $\boldsymbol{\xi}(\alpha, \beta) = \boldsymbol{\xi}(\alpha_0, \beta_0)$ may lie only in the second or fourth quadrants of the plane α, β. Let (α_1, β_1) belong to the fourth quadrant. If the point (α_1, β_0) belongs to domain $\mathscr{D}$ we choose coordinate system ξ, η so that $\theta(\alpha_1, \beta_0) = 0$. In this case $(A+\theta)(\alpha, \beta_0) > 0$, $(\theta - A)(\alpha_1, \beta) < 0$. According to (3.3) we see that change of η from point (α_0, β_0) into point (α_1, β_1) is non-zero. If point (α_1, β_0) does not belong to $\mathscr{D}$ characteristics $\alpha = \alpha_1$, $\beta = \beta_0$ intersect line $c = 0$ at points (α_2, β_0) and (α_1, β_2). We choose coordinate system ξ, η so that $\theta(\alpha_2, \beta_0) = 0$. From (3.2) we obtain $0 = \theta(\alpha_2, \beta_0) > \theta(\alpha_1, \beta_2)$. From (1.3) we have $\eta(\alpha_2, \beta_0) > \eta(\alpha_1, \beta_2)$. Change of η into $\beta = \beta_0$ and $\alpha = \alpha_1$ is considered as in the previous case. Univalence of mapping (1.3) in the domain $\mathscr{D}$ is proved.

Solvability of *Problem 3* follows from Theorem 2 for $\gamma > 3$.

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