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On the motion of a turbulent vortex ring

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THE mathematical model for a description of the motion of turbulent vortex ring is constructed. The problem with initial conditions of special form is formulated. This problems turned out to be self-similar. The obtained law of the motion of vortex ring is in good agreement with the results of the experiments. The proper boundary problem for the vorticity distribution to be determined is obtained. The limit case as turbulent viscousity approaches to zero is considered. The question about the transfer of admixture by the turbulent vortex ring is concerned.

Skonstruowano matematyczny model służący do opisu ruchu turbulentnego pierścienia wirowego. Problem początkowy sformułowano przy założeniu szczególnej postaci warunków początkowych. Otrzymane prawo ruchu pierścienia wirowego zgadza się dobrze z wynikami doświadczeń. Rozkład zawirowań określono za pomocą otrzymanego odpowiedniego problemu brzegowego. Rozważono przypadek graniczny, gdy turbulentny współczynnik lepkości dąży do zera. Rozważany jest również problem nośnika domieszki w turbulentnym pierścieniu wirowym.

Построена математическая модель служащая для описания движения турбулентного вихревого кольца. Начальная задача сформулирована при предположении частного вида начальных условий. Как оказалось это автомодельная задача. Полученный закон движения вихревого кольца хорошо совпадает с экспериментальными результатами. Распределение завихренностей определено при помощи полученной соответствующей краевой задачи. Рассмотрен предельный случай, когда турбулентный коэффициент вязкости стремится к нулю. Рассматривается тоже задача носителя добавки в турбулентном вихревом кольце.

VORTEX or smoke rings have been well known for a long time. They can be produced for example by filling a box having a flexible back wall and a circular hole in the front wall with smoke and striking the back wall or, by ejecting a puff of smoke suddenly from the mouth through rounded lips. This intriguing, easily produced and observed phenomenon has attracted the attention of many investigators who obtained important and interesting results mainly within the framework of the theory of inviscid incompressible fluid.

Some time ago a publication appeared carrying an information on attempts to make practical use of vortex rings at industrial enterprises to remove smoke and noxious gases and on the possibility of using such rings in rain-making experiments to project seeding material into clouds. It is obvious that the elucidation of the possibility of using vortex rings in a practical way requires the answer to such questions as the distance covered by the vortex ring, the dependence of vortex ring parameters on the way of formation, the amount of admixture transferred by the vortex ring, the losses of admixture during motion, the dependence of the amount of lost admixture on the way of filling the vortex ring by admixture in the moment of formation etc. It is quite evident that these questions should not be answered within the framework of the theory of inviscid fluid and we must take into account the dissipation of energy due to viscosity and the turbulent character of fluid motion.

In the present work the mathematical model for a description of the motion of a turbulent vortex ring and the transfer passive admixture by the turbulent vortex ring is constructed. The model is based on the analyses and summarizing of the results of experiments carried out in a wide range of Reynolds number (from 10^3–10^7). The size and velocity of the vortex ring varied between a few centimetres and two metres (vortex ring radius) and between a few centimetres per second and a hundred metres per second, respectively. A comparison of experimental results with theoretical conclusion allows to state that the motion, the structure and admixture transfer by the turbulent vortex ring is well described by the suggested theory. This theory is based on the supposition that the turbulent character of fluid motion in a vortex ring may be described by introducing the turbulent coefficient of viscosity. As we know such an approximation gives good results in the turbulent jets theory and in some other cases.

Let us assume that the coefficient of turbulent viscosity $\nu_*(t)$ is a function of time and is independent of space coordinates. The value of $\nu_*(t)$ is defined by a characteristic scale of motion (size and velocity of the vortex ring)

$$\nu_*(t) = aR(t)V(t), \tag{1}$$

where the coefficient a is constant and its value should be defined in the comparison of the predictions of the theory with experimental data. The assumption on the independence of the turbulent viscosity coefficient of the space coordinates is obviously not at a sufficiently long distance from the vortex ring as there, this coefficient should approach zero. It is evident, however, that terms with viscosity are significant only where vorticity is far from zero. As the vorticity in the vortex ring decreases very rapidly with the distance, it is hoped that the suggested assumption will yield no great error. A similar situation takes place in the theory of turbulent jets that was in good agreement with experiment. Under the condition that Reynolds numbers are sufficiently large over the greater part of the distance of vortex ring motion the turbulent viscosity is rather greater than the molecular one and the latter may be neglected.

Using the preceding assumption, we find the following set of governing equations for the description of the turbulent vortex ring:

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \operatorname{rot}[\mathbf{\Omega}\times\mathbf{V}] - \nu_*(t)\operatorname{rot}\operatorname{rot}\mathbf{\Omega}, \tag{2}$$

$$\operatorname{rot}\mathbf{V} = \mathbf{\Omega}, \quad \operatorname{div}\mathbf{V} = 0. \tag{3}$$

These equations have a conservation law that is essential for the present consideration as we shall see later on. Namely, the value for vorticity impulse

$$\mathbf{P} = \frac{1}{2}\varrho\int \mathbf{r}\times\mathbf{\Omega}dv \tag{4}$$

is a constant of motion (independent of the time).

For this set of governing equations it is nessesary to set up initial conditions, i.e., the initial distribution of vorticity defined by the way of formation of the vortex ring. However, the results of experiments provide evidence to the following. When Reynolds numbers are small, a laminar vortex ring with fine spiral structure is formed. The structure appears to be stable and maintains itself practically up to the complete stop of the vortex ring. In this case vorticity distribution is defined essentially by the initial field of velocity, the form of the nozzle which produces the vortex ring, a law of plunger motion, etc. This picture of flow field takes place when Reynolds number (defined by the radius and velocity of the vortex ring) $\lesssim 10^3$.

In the range of Reynolds numbers 10^3–10^4 the spiral structure becomes unstable and soon after the formation is destroyed. After all, from Reynolds number 10^4 and larger, the motion of fluid in the vortex ring becomes turbulent. On account of instability the spiral structure is immediately destroyed after vortex ring formation. The turbulent mixing of sheet vorticity takes place and, consequently, a vortex core is formed. In this case, as experiment proves, vorticity distribution is not dependent or depends very little on the details of the way of vortex ring formation. After the vortex ring covers the distance of some hole diameters which serve to form it, some vorticity distribution independent of the way of vortex ring formation is produced. To be more exact, average motion in a turbulent vortex ring in the first approximation is defined only by the size and velocity of the vortex ring and some constant the value of wich contains all the information on the way of vortex ring formation.

All this suggests that vorticity distribution in a turbulent vortex ring is described by a self-similar solution of governing equations. According to this a problem with the following initial conditions of special form is formulated:

$$\mathbf{\Omega}(0, \mathbf{r}) = -\mathbf{P}_0 \times \nabla \delta(\mathbf{r}), \tag{5}$$

where $\mathbf{P}_0$ is the vortex impulse, assigned to fluid density, $\delta(\mathbf{r})$ — delta-function.

Henceforth, the formulated problem appears to be self-similar, as the only dimensional parameter is the value of vortex impulse with the dimension $[P_0] = L^4/T$. From the analysis of dimensions it appears that the functions to be found, for which it is convenient to take vorticity Ω and stream function Ψ and the coefficient of turbulent viscosity $\nu_*(t)$, should have the following form [1]:

$$\begin{gathered} \Omega = \frac{1}{t}\omega(x, y), \quad \Psi = \frac{P_0^{3/4}}{t^{1/4}}\psi(x, y), \quad \nu_*(t) = \lambda\frac{P_0^{1/2}}{t^{1/2}}, \\ x = \frac{z}{P_0^{1/4}t^{1/4}}, \quad y = \frac{r}{P_0^{1/4}t^{1/4}}, \end{gathered} \tag{6}$$

where λ is a constant the value of which should be defined from the experiment. Self-similarity specifies the law of vortex ring motion. From (6) we know that the radius of the vortex ring $R(t)$ and the distance $L(t)$ covered by the vortex ring is

$$R(t) = P_0^{1/4}t^{1/4}y_0(\lambda), \quad L(t) = P_0^{1/4}t^{1/4}x_0(\lambda), \tag{7}$$

where $x_0(\lambda)$ and $y_0(\lambda)$ are coordinates of the point, where vorticity ω (x, y) have a maximum. From (7) we find that

$$R(t) = \alpha L(t), \tag{8}$$

where $\alpha(\lambda) = \dfrac{y_0(\lambda)}{x_0(\lambda)}$, i.e., the radius of the vortex ring increases in a linear way with the distance covered by the vortex ring.

In comparison with experiment the formulas (7) take the following form [1]:

$$l(t) = \frac{R_0}{\alpha}\left[\left(1+\frac{4\alpha U_0}{R_0}t\right)^{1/4}-1\right], \tag{9}$$

$$R(t) = R_0+\alpha l(t), \tag{10}$$

where R_0 and U_0 are initial values of the radius and the velocity of vortex ring for which it takes the values these parameters at some distance ahead of exit hole (some ten hole diameters) this point accepts for origin for time and distance $l(t)$.

This is associated with the fact that just after vortex ring formation vorticity distribution distinguishes itself from self-similar one and some time is needed to produce self-similar distribution.

The linear increase of the vortex ring radius distance is in good agreement with the results of experiments. The value α of measurements in experiments appears to be very small (about 10^{-2}–10^{-3}). It is a one-valued function of λ which depends on the way and the conditions of vortex ring formation and, as it will be seen further, completely specifies the structure of vorticity distribution in the vortex ring.

According to self-similar law of motion (9) the vortex ring covers an infinite distance, while experiments point at the opposite. In reality, the self-similar law of motion is valid until turbulent viscosity is rather larger than molecular viscosity. When the motion finishes the influence of molecular viscosity becomes essential and the self-similarity is destroyed. However, at a considerable part of the distance covered by the vortex ring the formulas (9) and (10) agree well with experimental results. As Reynolds number increases this stage of motion increases as well.

For specification of vorticity and stream function we have the following set of equations (in the cylindrical system of coordinates, axis x is the axis of symmetry, axis y is the distance from the symmetry axis)

$$\lambda\left(\omega_{xx}+\omega_{yy}+\frac{1}{y}\omega_y-\frac{1}{y^2}\omega\right)+\frac{1}{4}x\omega_x+\frac{1}{4}y\omega_y+\omega = \frac{1}{y}\left(\psi_y\omega_x-\psi_x\omega_y+\frac{1}{y}\psi_x\omega\right), \tag{11}$$

$$\psi_{xx}+\psi_{yy}-\frac{1}{y}\psi_y = -y\omega \tag{12}$$

with the boundary conditions

$$\omega\to 0,\quad \psi\to 0 \quad \text{as} \quad x^2+y^2\to\infty, \quad \omega=\psi=0 \quad \text{as} \quad y=0 \tag{13}$$

and the condition of normalization which follows from the conservation law of vortex impulse

$$\pi\int_{-\infty}^{\infty}\int_{0}^{\infty}\omega y^2 dy dx = 1. \tag{14}$$

The small parameter λ at main derivatives in Eq. (11) is the principal source of difficulties which arise when attempting to investigate both analytically and numerically the problem under consideration.

In connection with this it is natural to study the limit case that corresponds to the approach to zero of turbulent viscosity (vanishing viscosity). It this limit solution is found, then, it will provide a good base for the numerical solution of the problem at finite λ.

Let us substitute the variables, assuming

$$(15)\qquad \begin{aligned} &\xi = \frac{1}{\lambda^{1/2}}\left(x - \frac{1}{\lambda^{3/2}}\xi_0\right), \quad \eta = \frac{1}{\lambda^{1/2}}y, \quad \mu = \frac{1}{\lambda^2}, \\ &\omega = \frac{1}{\lambda^2}\bar{\omega}(\xi, \eta), \quad \psi = \frac{1}{\lambda^{1/2}}\left(\bar{\psi}(\xi, \eta) + \frac{1}{8}\xi_0\eta^2\right). \end{aligned}$$

This substitution, except for spreading, corresponds to the pass into the coordinate system associated with the vortex ring.

Equation (11) will be as follows:

$$(16)\qquad \bar{\omega}_{\xi\xi} + \bar{\omega}_{\eta\eta} + \frac{1}{\eta}\bar{\omega}_\eta - \frac{1}{\eta^2}\bar{\omega} + \frac{1}{4}\xi\bar{\omega}_\xi + \frac{1}{4}\eta\bar{\omega}_\eta + \bar{\omega} = \mu\left[\bar{\psi}_\eta\frac{\partial}{\partial_\xi}\left(\frac{\bar{\omega}}{\eta}\right) - \bar{\psi}_\eta\frac{\partial}{\partial\eta}\left(\frac{\bar{\omega}}{\eta}\right)\right]$$

and Eq. (12) and the normalization (14) will have the same form with the substitution $x \to \xi$ and $y \to \eta$. The boundary conditions for $\bar{\omega}$ remain the same while for $\bar{\psi}$ at infinity we have

$$(17)\qquad \frac{1}{\eta}\bar{\psi}_\eta \to -\frac{1}{4}\xi_0, \quad \frac{1}{\eta}\bar{\psi}_\xi \to 0 \quad \text{as} \quad \xi^2 + \eta^2 \to \infty.$$

Value ξ_0 is defined by the requirement that the maximum of $\bar{\omega}$ is to lie on the line of $\xi = 0$.

The approximate solution obtained in the process of investigating the formulated problem [2] suggests in new variables the solution at $\lambda \to 0$ (or $\mu \to \infty$ accordingly $\lambda \to 0$) approaches at some restricted limit solution. With such a suggestion from Eq. (16) it follows that in the limit $\mu \to \infty$ we have

$$(18)\qquad \frac{\bar{\omega}}{\eta} = \bar{\Omega} = \bar{\Omega}(\bar{\Psi})$$

where, however, the form of functional dependence $\bar{\Omega}(\bar{\Psi})$ remains indefinite.

In the considered flow the streamline $\bar{\Psi} = 0$ divides the region into two parts: external, in which streamlines run from infinity to infinity and internal (the atmosphere of the vortex ring) where streamlines are closed. Owing to the boundary condition on infinity for $\bar{\omega}$, from Eq. (18) it follows that in the external region in limit we have $\bar{\Omega} = 0$. The form $\bar{\Omega}(\bar{\Psi})$ in the internal region is to be defined.

An analogous problem was considered in paper [3] for the steady laminar flow of viscous fluid with closed streamlines at large Reynolds number (Reynolds number approach to infinity). Keeping in line with the idea of this work, let us integrate Eq. (16) in a region with the boundary defined by some closed streamline at the finite value dl. The term on the right-hand side vanishes identically, as this can easily be verified and, consequently, at any μ we have the exact integral condition

$$(19)\qquad \oint \eta\nabla\bar{\Omega}\mathbf{n}dl + 2\oint \bar{\Omega}n_\eta dl + \frac{1}{4}\oint \xi\eta\bar{\Omega}n_\xi dl + \frac{1}{4}\oint \eta^2\bar{\Omega}n_\eta dl + \frac{1}{2}\iint \eta\bar{\Omega}d\xi d\eta = 0$$

to be satisfied for all closed streamlines. Here, dl is a line element, $\mathbf{n}$ — the unit outward normal to the streamline. Now, making the limit operation $\mu \to \infty$ and taking into account that at this we have $\bar{\Omega} \to \bar{\Omega}(\bar{\Psi})$, we find from Eq. (19)

$$P(\bar{\Psi})\frac{d\bar{\Omega}}{d\bar{\Psi}} = \frac{1}{2}\Gamma(\bar{\Psi}) + \frac{3}{4}S(\bar{\Psi})\bar{\Omega}(\bar{\Psi}), \tag{20}$$

where

$$P(\bar{\Psi}) = \iint \eta^3\bar{\Omega}d\xi d\eta, \quad S(\bar{\Psi}) = \iint \eta d\xi d\mu, \quad \Gamma(\bar{\Psi}) = \iint \eta\bar{\Omega}d\xi d\mu$$

is the circulation round the streamline. Here, the integrals are taken over the region defined by the closed streamline.

Thus, in the limit $\mu \to \infty$ the problem of finding the vortex ring structure reduces to that of matching the irrotational (in the external region) and vortex (in the internal region) flow of inviscid fluid under the condition of continuity of $\bar{\Psi}$ and $\nabla\bar{\Psi}$ at the boundary [4] function form $\bar{\Omega}(\bar{\Psi})$ is defined by ordinary (for variable $\bar{\Psi}$) differential Eq. (20).

It can be proved that the assumption of restricting the limit solution entails its continuity. Therefore, the boundary condition $\bar{\Omega}(0) = 0$ is to be added to Eq. (20). A numerical solution of the formulated problem (i.e., the solution in the limit case $\mu \to \infty$) is much simpler than the initial problem. The results of the numerical solution are in good agreement with experimental data.

The question about the transfer of the passive admixture by the turbulent vortex ring can be considered by the analogy with the foregoing [2]. The ability of the vortex ring to transfer admixture is due to the mass of fluid carried along with the vortex ring. Under condition that the flow is laminar the passive admixture contained in this mass of fluid in not lost practically, without taking into account the slow process of molecular diffusion. With turbulent motion losses of admixture rise sharply due to turbulent diffusion but some admixture is transferred by the vortex ring over a great distance.

The consideration of the process of passive admixture transfer is based on the assumption that the turbulent diffusion can be described by introducing the turbulent coefficient of diffusion $D_*(t)$. Then, basing on known experimental results of turbulent jets where it was found that the coefficient of turbulent diffusion coincided with the coefficient of turbulent viscosity with a factor on the order of unity, it is assumed that

$$D_*(t) = \gamma\nu_*(t),$$

where γ is constant on the order of unity.

Neglecting molecular diffusion we have the following equation that describes average distribution of passive admixture in the turbulent vortex ring [2]:

$$\frac{\partial C}{\partial t} + (\mathbf{v}\nabla)C = \gamma\nu_*(t)\Delta C, \tag{21}$$

where C represents admixture concentration. Velocity field defined the coefficients in Eq. (21) is to be found from problem solution on vortex ring motion formulated above.

Equation (21) has the evident law of conservation: the complete amount of admixture Q in the whole volume

$$Q = \int C dv \tag{22}$$

is independent of time.

Experiment shows that the distribution of admixture concentration, like the distribution of vorticity, rather quickly approaches to some distribution independent of initial conditions; some amount of admixture is lost quickly and forms a characteristic train (smoke in particular) streching beyond the vortex ring. The admixture placed near the boundary of the fluid volume which moves together with the vortex ring loses the most quickly. This explains the fact that very quickly after the vortex ring is produced by the box filled with smoke we see a smoke ring (toroidal region of great concentration admixture, approximately coinciding with the region of maximum vorticity) though, in fact, together with the vortex ring the fluid volume moves in the form of an oblate spheroid.

Let us assume that the limit concentration distribution of admixture like vorticity distribution, describes the self-similar solution of Eq. (21). Then, we have

$$C = \frac{Q}{P_0^{3/4} t^{3/4}} c(x, y) \tag{22'}$$

and for specification of concentration of admixture we have the following equation:

$$\gamma\lambda\left(c_{xx} + c_{yy} + \frac{1}{y} c_y\right) + \frac{1}{4} x c_x + \frac{1}{4} y c_y + \frac{3}{4} c = \frac{1}{y} (\psi_y c_x - \psi_x c_y) \tag{23}$$

with the boundary conditions

$$c \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty, \quad c_y = 0 \quad \text{as} \quad y = 0 \tag{24}$$

and normalization

$$2\pi \int_{-\infty}^{\infty} \int_{0}^{\infty} c y dx dy = 1. \tag{25}$$

This problem may be solved after finding a solution to the problem of vortex ring motion.

At early stages of vortex ring motion when the self-similar distribution of admixture is not yet produced, the admixture losses are described by the following formula:

$$Q = Q_0(\lambda) + \frac{Q_*}{\left(1 + \alpha \frac{l(t)}{R_0}\right)^n}, \tag{26}$$

where $Q_0(\lambda)$ and Q_* are constants, n is a constant defined from the experiment (that gives $n \simeq 14$). The value Q_0 is defined by the self-similar distribution of the admixture. The value Q_* depends on the way of filling the vortex ring at the moment of its formation. In particular, if one fills the vortex ring at the moment of formation so that the admixture gets only into the core of the vortex ring, then, the losses of admixture decrease essentially.

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