75.

ON THE NUMBER OF PROPER VULGAR FRACTIONS IN THEIR LOWEST TERMS THAT CAN BE FORMED WITH INTEGERS NOT GREATER THAN A GIVEN NUMBER.

[*Messenger of Mathematics*, XXVII. (1898), pp. 1—5.]

A SLIGHT reflexion will show that the number of such fractions $\left(\frac{1}{1}\text{ counting as one of them}\right)$ with the limit n is the sum of the totients of all the numbers from 1 to n.

Let us use Ej as usual to denote the integer part of j, τEj to denote the totient (or number of numbers not exceeding and prime) to Ej, and JEj to denote the sum of such totients for all numbers from 1 to j. Then we may establish the following exact equation given by the author of this article, but without proof and with some slight inaccuracy, in the *Phil. Mag.* for April, 1883 [p. 102, above]. The equation is

$$JEj + JE\left(\tfrac{1}{2}j\right) + JE\left(\tfrac{1}{3}j\right) + \text{etc.},$$

or, more shortly,

$$\sum_1^\infty JE\frac{j}{i} = \tfrac{1}{2}\{(Ej)^2 + (Ej)\}. \qquad (1)$$

The proof is as follows. Remarking that $E(j-1) = Ej - 1$, the right-hand side of equation (1), when j is reduced to $j-1$ obviously suffers a diminution equal to Ej.

On the left-hand side of the equation any term $JE\frac{j}{i}$ remains unaltered, when for j is written $(j-1)$, unless Ej is divisible by i, in which case the term undergoes a diminution JEj. Thus for example $J\frac{100}{11} - J\frac{99}{11} = 0$, but $J\frac{100}{5} - J\frac{99}{5} = J(20)$. And, as in the case supposed, $\frac{Ej}{i}$ is a factor of Ej, the total diminution, when $j-1$ replaces j, will be the sum of the totients of the factors of Ej, which by a known theorem equals Ej. Hence equation (1) is satisfied for j if it is satisfied for $j-1$, and as it is true when $Ej=1$ it is true for all values of j, as was to be proved. From equation (1) it follows that JEj is of the order $(Ej)^2$, and making

$$JEj = \tfrac{1}{2}\mu\,(Ej)^2 + \epsilon j,$$

where ϵj is zero when $j=\infty$, we obtain

$$\mu(1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}+\ldots)=1,$$

or $$\mu = \frac{6}{\pi^2}, \text{ or approximately } Jj = \frac{3j^2}{\pi^2}.$$

In the tables in the *Phil. Mag.* for April and September, 1883*, the value of Jj is computed up to $j=1{,}000$ and compared with the mean value $\frac{3}{\pi^2}j^2$. From this table it appears that Jj is always intermediate between $\frac{3}{\pi^2}j^2$ and $\frac{3}{\pi^2}(j+1)^2$, and much nearer to their mean, which to an insignificant fraction *près* is the same as $\frac{3}{\pi^2}(j^2+j)$, than it is to either extreme. The first, at least, of these statements ought to be susceptible of proof.

As a matter of philosophical interest as embodying a principle applicable to other cases, I will show how I originally found the value $\frac{3}{\pi^2}j^2$ for the number of proper vulgar fractions in their lowest terms that can be formed by means of the first integers.

It is obvious that the probability of any unknown number being divisible by a prime number i is $\frac{1}{i}$, and of any two numbers, being each so divisible, is $\frac{1}{i^2}$, so that the probability of two unknown numbers being each *not* divisible either by 2, 3, 5, 7, n, or any other prime, will be

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{n^2}\right)\ldots,$$

which we know is equal to the sum of the reciprocal of the squares of the natural numbers, that is, is equal to $\frac{6}{\pi^2}$. Hence the number of fractions in their lowest terms that can be got by combining each of j integers with each of i others, found *roughly* by adding together the probable expectation of any such combination consisting of two relative primes, will be $\frac{6}{\pi^2}j^2$, and the number of *proper* fractions in their lowest terms so capable of being formed will be the half of this or $\frac{3j^2}{\pi^2}$. It appears incidentally from this

[* p. 103, above.]

that the average or mean value of the totient of any number is $\frac{3}{\pi^2}$ into, or rather more than, $\frac{3}{10}$ths of that number.

In like manner, if we define a mid-prime to the number $2n$ to be one which is greater than $\frac{1}{2}n$ and less than $\frac{3}{2}n$, the *range* of numbers amongst which such primes are to be found will, to a unit *près*, be n. Let us call the number of such mid-primes μ. Then the probability of any number and its complement in respect to $2n$ being each of them primes will be $\frac{\mu^2}{n^2}$. If now we seek the number of solutions of the equation in prime numbers $x + y = 2n$, which will be an even or an odd number, according as n is a composite number or a prime, we may suppose a row of n white balls and n black balls, each series being marked with all the numbers from 1 to n inclusive. It follows from what has been said that the sum of the expectation of x being inscribed on any one of the white balls being itself a prime, and its complement $2n - x$ upon one of the black balls being so likewise, will be $n \cdot \frac{\mu^2}{n^2}$, that is $\frac{\mu^2}{n}$,* and as the same will be true when x is a figure on a black ball and $2n - x$ on a white, the total value of the expectation of the equation in *primes* $x + y = 2n$ being satisfied will be the double of this, or $\frac{2\mu^2}{n}$. I have had tables constructed for determining the number of the solutions of this equation (x and y being primes) from $2n = 2$ up to $2n = 500$.

Call the number of solutions for any value of n, $\theta \frac{\mu^2}{n}$; on taking the average value of θ for all values of $2n$ on the 1st, 2nd, 3rd, 4th, 5th, centuries respectively, it will be found that

$$\begin{aligned} \tfrac{1}{2}\theta &= \cdot 96344 \\ &= \cdot 99349 \\ &= 1\cdot 00603 \\ &= \cdot 98281 \\ &= \cdot 99764, \end{aligned}$$

of which the sum is 4·94341 and the average is ·98868, agreeing with wonderful nearness to the rough estimate of the number of solutions being $\frac{2\mu^2}{n}$.

* μ is of the order of, and ultimately in a ratio of equality with, $\frac{n}{\log n}$, in the sense that, however small ϵ be taken, a limit $L\epsilon$ can be found such that for all values of n beyond it, μ will be limited on the two sides by $(1 \pm \epsilon)\frac{n}{\log n}$; this follows demonstrably from a known theorem proved within the last few years, and as a consequence we see that the number of solutions in "mid-primes" of the equation $x + y = 2n$ will necessarily be of the same order as $\frac{n}{(\log n)^2}$ and *presumably* in a ratio of equality with it in the sense explained above, but this, of course, awaits demonstration.

I ought not, however, to suppress the fact that, from another point of view, this number might be expected to eventuate as $\frac{\mu^2}{n}$ instead of $\frac{2\mu^2}{n}$.

In equation (1) we may write $F(j)$ for the sum of the totients of all the numbers not exceeding j, and it then takes the form

$$\phi j = \tfrac{1}{2}\{Ej + (Ej)^2\} = Fj + F(\tfrac{1}{2}j) + F(\tfrac{1}{3}j) + F(\tfrac{1}{4}j) + \text{etc.},$$

which, by the well-known formula of reversion (see *Phil. Mag.*, December, 1884*), gives

$$Fj = \phi j - \phi(\tfrac{1}{2}j) - \phi(\tfrac{1}{3}j) - \phi(\tfrac{1}{5}j) + \phi(\tfrac{1}{6}j) - \text{etc.}$$

Thus for example the number of terms in a Farey series with 17 as a limit should be equal to

$$\tfrac{1}{2}(17-8-5-3+2-2+1-1-1+1+1-1)$$
$$+\tfrac{1}{2}(289-64-25-9+4-4+1-1-1+1+1-1)$$

that is $\tfrac{1}{2}(1)+\tfrac{1}{2}(191)$ or 96, which is right†.

* I do not know whether the annexed important case of reversion has been noticed or not: i being greater than unity, let σ_i denote the sum of the *negative* ith powers of the prime numbers 2, 3, 5, 7, etc., and s_i the *logarithm* of the sum of the negative ith powers of the natural numbers 1, 2, 3, 4, etc. (which, when i is an even integer, is a known quantity), then it is easily shown that

$$s_i = \sigma_i + \tfrac{1}{2}\sigma_{2i} + \tfrac{1}{3}\sigma_{3i} + \tfrac{1}{4}\sigma_{4i} + \tfrac{1}{5}\sigma_{5i} + \text{etc.},$$

and therefore by reversion

$$\sigma_i = s_i - \tfrac{1}{2}s_{2i} - \tfrac{1}{3}s_{3i} - \tfrac{1}{5}s_{5i} + \tfrac{1}{6}s_{6i} - \tfrac{1}{7}s_{7i} + \tfrac{1}{10}s_{10i} + \text{etc.}$$

A very general case for reversion arises when $\phi i = \Sigma \frac{1}{n^2}\phi(n^s . i)$. In this last application of the formula $r=1$, $s=1$; in the case considered in the text relating to Farey series $r=0$, $s=-1$.

† And so in general, since by a well-known theorem

$$Ej - E(\tfrac{1}{2}j) - E(\tfrac{1}{3}j) + E(\tfrac{1}{6}j) + \text{etc.}$$

is always equal to unity, so that

$$(Ej)^2 - 2JEj + 1 = E(\tfrac{1}{2}j)^2 + E(\tfrac{1}{3}j)^2 - E(\tfrac{1}{6}j)^2 + \text{etc.},$$

we have always

$$2JEj - 1 = (Ej)^2 - E(\tfrac{1}{2}j)^2 - E(\tfrac{1}{3}j)^2 + E(\tfrac{1}{6}j)^2 + \text{etc.}$$

a very convenient, and, I believe, new formula for calculating the number of fractions in their lowest terms where neither numerator nor denominator exceeds j.

To this E theorem there exists a pendant which may be called the H theorem, namely let Hx mean the nearest integer (when there is one) to x, but when x is midway between two integers Hx is to denote the first integer above x; let $p, q, r, \ldots$ be the primes not exceeding the integer n, and call

$$H_n = n - \Sigma H\frac{n}{p} + \Sigma H\frac{n}{pq} - \Sigma H\frac{n}{pqr} + \text{etc.};$$

then H_n will be the number of primes greater than n and less than $2n$, so that H_n is always greater than zero; and if $\epsilon(x)$ means unity or zero according as x is a prime or not, we shall always have

$$H_n - H_{n-1} = \epsilon(2n-1) - \epsilon(n).$$

I do not know whether this theorem has been previously noticed. It may be obtained by the Eratosthenes sieve process applied to the progression $n+1, n+2, n+3, \ldots, 2n$, replacing therein every prime number by unity.

If not already known, it may be worth while to register the two following additional theorems concerning $E_1 n$ and $H_1 n$, by which I mean what E_n and H_n become when the even prime 2 does not count among the primes p, q, r, which are less than n, namely

$$E_1 n = E\left(\frac{n}{2}\right) - \Sigma E\ \frac{n}{2p} + \Sigma E\ \frac{n}{2pq} + \text{etc.} = E\left(\frac{\log n}{\log 2}\right),$$

$$H_1 n = H\frac{n}{2} - \Sigma H\ \frac{n}{2p} + \Sigma H\frac{n}{2pq} + \text{etc.} = 1.$$

This paper was sent by Professor Sylvester to the editor on Feb. 12th, 1897, with a letter in which he wrote "I could subsequently send you the valuable table referred to in the text, giving the number of solutions of the equation $x + y = 2n$ in prime numbers for all values of n up to 500." In subsequent letters he made several slight additions to the paper. He corrected the proof sheets about the end of the month, and then added the first footnote and the last paragraph of the third note. His death took place on March 15th.