72.

ON ARITHMETICAL SERIES.

[*Messenger of Mathematics*, XXI. (1892), pp. 1—19, 87—120.]

THE first part of this article relates to the prime numbers (or so to say latent primes) contained as factors of the terms of given arithmetical series; the second part will deal with the actual or, say, visible primes included among such terms. Both investigations repose alike upon certain elementary theorems concerning the "index-sums" (relative to any given prime) of arithmetical series, whether simple and continuous as in the case of series ordinarily so called or compound and interstitial as such before named series become when subjected to certain periodic and uniform interruptions.

PART I.

§ 1. *Preliminary Notions.*

Consider any given sequence

$$m+1,\ m+2,\ m+3,\ \ldots,\ m+n,$$

in relation to any given prime number q.

Let r be the sum of the indices of the highest powers of q which are contained in the several terms of the natural sequence

$$1,\ 2,\ 3,\ \ldots,\ n,$$

s the sum of the indices of the highest powers of q contained in the given sequence.

Then it is almost immediately obvious that $s =$ or $> r$, that is $s > r-1$.

For the index-sum of the natural sequence will be represented by

$$r = E\left(\frac{n}{q}\right) + E\left(\frac{n}{q^2}\right) + E\left(\frac{n}{q^3}\right) + \ldots,$$

and the index-sum of the given sequence by

$$s = E\left(\frac{m+n}{q}\right) + E\left(\frac{m+n}{q^2}\right) + E\left(\frac{m+n}{q^3}\right) + \ldots$$
$$-E\left(\frac{m}{q}\right) - E\left(\frac{m}{q^2}\right) - E\left(\frac{m}{q^3}\right) - \ldots$$

and this is at least equal to

$$E\left(\frac{n}{q}\right) + E\left(\frac{n}{q^2}\right) + E\left(\frac{n}{q^3}\right) + \ldots,$$

that is $s =$ or $> r$.

But there is another and more important theorem, less immediately obvious, and more germane to the subject-matter of the following section, which I proceed to explain.

Suppose $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n$ to be the several exponents of the highest powers of q which are contained in

$$x+1,\ x+2,\ x+3,\ \ldots,\ x+n,$$

and let σ be one of these n exponents which is not less than any other of them.

Call any term in the sequence

$$x+1,\ x+2,\ x+3,\ \ldots,\ x+n$$

which contains q^σ, say P, a principal q-term.

On one side of P the terms are less, on the other greater than P; in lieu of any term substitute the difference between it and P, then I say that the q-index of such altered term will be the same as when it was unaltered.

For let the principal term, or the chosen principal term if there are more than one, be λq^σ, and let μq^ρ be any other term.

If $\rho < \sigma$, $\lambda q^\sigma \sim \mu q^\rho$ will obviously have ρ for its q-index; also if $\rho = \sigma$ the same will be true, that is supposing $\mu q^\rho - \lambda q^\rho$ to be positive, ρ will be its q-index: for if we write $\lambda = aq + b$ and $\mu = cq + d$, where $b < q$ and $d < q$, a and c must be equal, since otherwise between λq^ρ and μq^ρ there would be a term $(a+1)\,q\,.\,q^\rho$ containing a higher power of q than the principal term: hence $\mu - \lambda = d - b$ and does not contain q. In like manner if $\lambda q^\rho - \mu q^\rho$ is positive, ρ is its q-index for the same reason as before.

Hence the index-sum, *quâ* any prime q, of the two sequences

$$m+1,\ m+2,\ \ldots,\ P-1;\quad P+1,\ P+2,\ \ldots,\ m+n-1,\ m+n$$

is the same as the sum of the index-sums of

$$1,\ 2,\ 3,\ \ldots,\ P-m-1,$$
$$1,\ 2,\ \ldots,\ m+n-P.$$

Call the sum of these two index-sums s', then

$$s' = E\left(\frac{P-m-1}{q}\right) + E\left(\frac{P-m-1}{q^2}\right) + E\left(\frac{P-m-1}{q^3}\right) + \ldots$$
$$+ E\left(\frac{m+n-P}{q}\right) + E\left(\frac{m+n-P}{q^2}\right) + E\left(\frac{m+n-P}{q^3}\right) + \ldots$$

and this is

$$= \text{or} < E\left(\frac{n-1}{q}\right) + E\left(\frac{n-1}{q^2}\right) + E\left(\frac{n-1}{q^3}\right) + \ldots$$
$$= \text{or} < E\left(\frac{n}{q}\right) + E\left(\frac{n}{q^2}\right) + E\left(\frac{n}{q^3}\right) + \ldots$$
$$= \text{or} < r.$$

Hence $s' =$ or $< r$. But the original index-sum of the sequence is diminished by σ on account of P being omitted.

Hence $s - \sigma$ or $s' =$ or $< r$.

Thus we have $s > r - 1$, $s - \sigma < r + 1$.

But this is not all: we may for certain relative values of m, n, and q (without regard to the situation of the principal term) establish the inequality $s - \sigma < r$.

I premise the obviously true statement that if $f + g < h$, then

$$f + E\left(\frac{f}{q}\right) + E\left(\frac{f}{q^2}\right) + \ldots + g + E\left(\frac{g}{q}\right) + E\left(\frac{g}{q^2}\right) + \ldots$$
$$< h + E\left(\frac{h}{q}\right) + E\left(\frac{h}{q^2}\right) + \ldots.$$

Let now h be the number of terms in the natural sequence from 1 to n which contain q.

Then in the given sequence the number will be

$$h + E\left(\frac{m+n}{q}\right) - E\left(\frac{m}{q}\right) - E\left(\frac{n}{q}\right), \text{ say } h + e,$$

and the sum of the number of terms divisible by q in the partial sequences on each side of P will be $h + e - 1$, where $e = 1$ or 0; let the respective numbers be f, g. Then $f + g = h - 1 + e$, where $e = 0$ or 1, and, using the same notation as before,

$$s - \sigma = f + E\left(\frac{f}{q}\right) + E\left(\frac{f}{q^2}\right) + \ldots$$
$$+ g + E\left(\frac{g}{q}\right) + E\left(\frac{g}{q^2}\right) + \ldots,$$

and
$$r = h + E\left(\frac{h}{q}\right) + E\left(\frac{h}{q^2}\right) + \ldots.$$

Hence if $e = 0$, $s - \sigma < r$,

if $e = 1$, $s - \sigma < r + 1$,

the former inequality subsisting whenever

$$E\left(\frac{m+n}{q}\right) - E\left(\frac{m}{q}\right) - E\left(\frac{n}{q}\right) = 0.$$

If for example $m = n$, then $s - \sigma < r$ when

$$E\left(\frac{2n}{q}\right) - 2E\left(\frac{n}{q}\right) = 0.$$

which it is easily seen happens whenever $E\left(\frac{2n}{q}\right)$ is an even number.

§ 2. *Proof that* $(m+1)(m+2)\dots(m+n)$ *when* $m>n-1$ *contains a prime not contained in* $1.2.3\dots n$*.

The universal condition independent of the relation between m, n, q, above found, namely, $s-\sigma=$ or $<r$ will be found sufficient to establish the theorem which constitutes the object of this section and which is as follows:—

"If the first term of a sequence is greater than the number of terms in it, then one term at least must be a prime or a multiple of a prime greater than that number."

When the first term exceeds by unity the number of terms, the sequence takes the form $m+1, m+2, \dots, 2m-1$, and since no term in this sequence can be a multiple of m, the theorem for such case is tantamount to affirming that one term at least is a prime number which is in accord with and an easy inference from the well-known "postulate of Bertrand," that between m and $2m-2$ there must always be included some prime number when $m>\frac{7}{2}$.

Suppose if possible that $m+1, m+2, \dots, m+n$ contains no other primes than such as are not greater than n, and which therefore divide some of the numbers from 1 to n.

Let q be any such prime, and P_q a principal term of the sequence

$$m+1, m+2, \dots, m+n, \quad \text{quâ } q.$$

Then, by virtue of the proposition above established,

$$\frac{(m+1)(m+2)\dots(m+n)}{P_q}$$

will contain no higher power of q than does $1.2.3\dots n$, and consequently if P be the least common multiple of the principal terms in respect to the several primes, say ν in number (unity not being reckoned one of them), none greater than n, we may infer that

$$\frac{(m+1)(m+2)\dots(m+n)}{P}$$

will be wholly contained in, and therefore not greater than $1.2.3\dots n$, if the sequence $m+1, m+2, \dots, m+n$ contains no prime or multiple of a prime greater than n. To fix the ideas let us agree to consider that term in the sequence which contains the highest power of q, and is the greatest of all that do the same (if there be more than one), *the* principal q-term. The least common multiple cannot be greater than the product of the principal terms which are *distinct* from each other, and since even if they are all distinct, their number cannot exceed ν (the number of primes other than

* It will readily be seen that, if this theorem is true, for n any prime, it will be so *à fortiori* when n is a composite number.

unity less than $n+1$), it follows that P cannot be greater than the product of the *highest* ν terms in the given sequence. Hence we may infer that unless

$$(m+1)(m+2)\dots(m+n-\nu)$$

is less than $1.2.3\dots n$, some prime greater than n must divide one term at least of the sequence

$$m+1,\ m+2,\ \dots,\ m+n.$$

We might go further and say that unless $1.2.3\dots n$ is greater than

$$(m+1)(m+2)\dots(m+n-\nu)D,$$

where
$$D=\Pi q^{1+E\left(\frac{m}{q}\right)+E\left(\frac{n}{q}\right)-E\left(\frac{m+n}{q}\right)},$$

(q being made successively each of the ν primes between 2 and n inclusive and Π being used in the ordinary sense of indicating products), this same conclusion must obtain.

Conversely the theorem is true when either of these inequalities is denied. The denial of the first of them, which is sufficient for the object in view, is implied in the inequality

$$(m+1)(m+2)\dots(m+n-\nu)>1.2.3\dots n,$$

which, since ν depends only on n, may be written under the form

$$F(m,n)>1.2.3\dots n.$$

This will be referred to hereafter, in this section, as the *fundamental inequality**.

Since $F(m,n)$ increases with m, the theorem if true for m must be true for any greater value of m, when n remains constant.

From this it will be seen at once that the theorem must be true when m has any value exceeding n^2 and $n>7$.

For when $n=8$ the number of primes in the range from 1 to 8 is 4 and is equal to $\frac{1}{2}n$: but as n increases the number of new primes being less than the number of odd numbers must be less than $\frac{1}{2}n$.

Hence if $n>7$ and $m>n^2$,

$$F(m,n)>m^{n-\nu}>(n^2)^{\frac{1}{2}n}>n^n>1.2.3\dots n.$$

This result enables us to prove that the theorem is true when

$$13<n<3000.$$

The theorem it will be borne in mind is true if some prime number occurs in the sequence $m+1, m+2, \dots, m+n$, or in other words if the above sequence does not consist exclusively of composite numbers. But

* The subsistence of the fundamental inequality for any given value of n implies for that value of n the truth of the theorem to be established: but the converse does not necessarily hold. The theorem may be true when the fundamental inequality is *not* satisfied.

Dr Glaisher has found* that the highest sequence of composite numbers within the first 9000000 contains only 153 terms, namely, the sequence 4652354 to 4652506 (both inclusive). Hence if the theorem is not true when $n < 3000$, in which case $n^2 + n < 9000000$, we must have $n =$ or < 153, and there ought to be a sequence of n composite numbers in which the first term is less than $(153)^2$ which is 23409. But the longest sequence of composite numbers under 23409 is that which extends from 19610 to 19660 containing 51 terms, the square of 51 is 2601 and the longest sequence under this number is that which extends from 1328 to 1360 comprising 33 terms. The square of 33 is 1089, the longest sequence below which is from 888 to 906 comprising 19 terms: the square of 19 is 361, the longest sequence below which stretches from 114 to 126 comprising 13 terms. Hence the theorem is true for all values of n not greater than 3000 and not less than 13.

It is easy to show that the theorem is true for all values of n not greater than 13.

(1) Suppose $n = 13$, which gives $\nu = 6$.

The theorem must be true when m is taken so great that

$$(m+1)(m+2)(m+3)(m+4)(m+5)(m+6)(m+7) > 1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9 . 10 . 11 . 12 . 13,$$

which is easily seen to be satisfied when $m =$ or > 100.

But there is no sequence of 13 composite numbers till we come to the sequence 114 to 126, so that when $m < 100$ the theorem must be true as well as when $m =$ or > 100.

(2) Suppose $n = 11$, for which value of n, $\nu = 5$.

The theorem is true if

$$(m+1)(m+2)(m+3)(m+4)(m+5)(m+6) > 1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9 . 10 . 11,$$

which is obviously satisfied as before when $m = 100$, but there is no sequence of 11 which precedes the sequence before named from 114 to 126. Hence the theorem is true generally for $n = 11$.

When $n = 7$, $\nu = 4$ and the theorem is true for all values of m which make

$$(m+1)(m+2)(m+3) > 1 . 2 . 3 . 4 . 5 . 6 . 7, \text{ that is, } > 5040,$$

which is obviously the case if $m =$ or > 20, but there is no sequence of 7 composite numbers till we come to 89 to 97. Hence the theorem is proved for $n = 7$.

When $n = 5$, $\nu = 3$ and the condition of the theorem is satisfied if

$$(m+1)(m+2) > 2 . 3 . 4 . 5, \text{ that is, } > 120,$$

* See table at the end of this section.

as is the case if $m =$ or > 10, but the first composite sequence of 5 terms is 24 to 28. In like manner when $n = 3$, $\nu = 2$ and the theorem is true when $m + 1 =$ or $> 1 . 2 . 3$, that is, $m =$ or > 5, but 8, 9, 10 is the first composite sequence of 3 terms. Similarly, when $n = 2$, $\nu = 1$ and the condition $m + 1 =$ or > 2 is necessarily satisfied since $m =$ or $> n$ by hypothesis.

Finally, the theorem is obviously true when $n = 1$, because $m + 1$, whatever m may be, contains a factor greater than 1.

Being true for the prime numbers not exceeding 13, the slightest consideration will serve to prove that, as previously remarked in a footnote, it must be true *à fortiori* for all the composite numbers between them. Hence the theorem is verified for all values of n not greater than 3000, and it only remains to establish it for values of n exceeding that limit.

To prove it for this case we must begin with finding a superior limit to ν, when $n > 3000$, under the convenient form of a multiple of $\frac{n}{\log n}$.

If we multiply together the first 9 prime numbers from 2 to 23 and divide their product by that of the natural numbers up to 9 increased in the ratio of 1 to 2^9, the quotient will be found to exceed unity; and since the following primes are all more than twice the corresponding natural numbers, if we denote by $p_1, p_2, p_3, \ldots$, the prime numbers 2, 3, 5, ..., we must have

$$p_1 . p_2 . p_3 \ldots p_\nu > 2^\nu (1 . 2 . 3 \ldots \nu),$$

(provided that $\nu > 22$, as is the case if $n =$ or > 89),

or $$\log (1 . 2 . 3 \ldots \nu) + (\log 2) \nu < \log (p_1 . p_2 . p_3 \ldots p_\nu).$$

But by Stirling's theorem (Serret, *Cours d'Alg. Sup.*, ed. 4, vol. II. p. 226),

$$\nu \log \nu - \nu - \tfrac{1}{2} \log \nu + \tfrac{1}{2} \log 2\pi < \log (1 . 2 . 3 \ldots \nu),$$

and by Tchebycheff's theorem (Serret, vol. II. p. 236)*,

$$\log (p_1 . p_2 . p_3 \ldots p_\nu) < n',$$

where $n' = \tfrac{6}{5} A n + \dfrac{5}{4 \log 6} (\log n)^2 + \tfrac{5}{2} \log n + 2$, and $A = \cdot 921292 \ldots$.

Hence $(\log \nu)(\nu - \tfrac{1}{2}) - (1 - \log 2)(\nu - \tfrac{1}{2}) + (\tfrac{1}{2} \log 2\pi - \tfrac{1}{2} \log \tfrac{1}{2} e) < n'$,

and *à fortiori* $$\log (\nu - \tfrac{1}{2})(\nu - \tfrac{1}{2}) - (\log \tfrac{1}{2} e)(\nu - \tfrac{1}{2}) < n',$$

or $$\frac{2}{e} (\nu - \tfrac{1}{2}) \log \left\{ \frac{2}{e} (\nu - \tfrac{1}{2}) \right\} < \frac{2}{e} n'.$$

Hence, if we write $$\mu \log \mu = \frac{2}{e} n' = n_1$$

we shall have $$\nu - \tfrac{1}{2} < \tfrac{1}{2} e \mu.$$

* For greater simplicity I have left out the term $-An^{\frac{1}{2}}$, and thereby *increased* the superior limit.

But
$$\mu = \frac{n_1}{\log \mu},$$
and therefore
$$\log \mu = \log n_1 - \log\log \mu = \log n_1 - \log(\log n_1 - \log\log \mu) > \log n_1 - \log\log n_1.$$

Hence
$$\mu < \frac{n_1}{\log n_1 - \log\log n_1}$$
$$< \frac{2}{e}\,\frac{n'}{\log n' - \log\log n' + \log\frac{2}{e}}$$
and
$$\nu < \tfrac{1}{2} + \frac{n'}{\log n' - \log\log n' - (1 - \log 2)}\text{*}.$$

Hence, observing that $\frac{1}{u}$, $\frac{\log u}{u}$, $\frac{(\log u)^2}{u}$, $\frac{\log\log u}{\log u}$ *all* decrease as the denominators increase (provided as regards the second of these fractions that $u > e$, as regards the third that $u > e^2$, and as regards the fourth that $u > e^e$), we may find a superior limit to ν in the case before us, where $n > 3000$, by writing in the numerator of $\nu - \frac{1}{2}$,
$$\frac{(\log 3000)^2}{3000}\,n, \quad \frac{\log 3000}{3000}\,n, \quad \frac{2}{3000}\,n,$$
for
$$(\log n)^2, \quad \log n, \quad 2,$$
and in its denominator, first, $\log n - \log\log n$ for $\log n' - \log\log n'$, and then
$$\frac{\log\log 3000}{\log 3000}\log n \quad\text{and}\quad \frac{1 - \log 2}{\log 3000}\log n,$$
for
$$\log\log n \quad\text{and}\quad 1 - \log 2 \text{ respectively.}$$

Making the calculations it will be found that we shall get
$$\nu - \tfrac{1}{2} < 1{\cdot}606\,\frac{n}{\log n}.$$

With the aid of this limit it will now be easy to prove the truth of the theorem when $n =$ or > 3000.

Let us suppose $n =$ or > 3000.

(1) Suppose $m < 2n$, then $m + n > \frac{3}{2}m$ and the theorem will be proved for this case, if it can be shown that in the range of numbers from m to $\frac{3}{2}m$, there is at least one prime number when $m =$ or > 3000.

* From this it will be seen that the asymptotic ratio of ν to $\frac{n}{\log n}$ is less than the asymptotic ratio which any superior limit to the sum of the logarithms of the primes not exceeding n bears to n: this perhaps is a new result, at all events it is not to be found in Serret nor indeed is it wanted for Tchebycheff's proof of the famous postulate which Serret has so lucidly expounded. The correlative theorem that the asymptotic ratio of ν to $\frac{n}{\log n}$ is always *greater* than the asymptotic ratio which any *inferior* limit to the sum aforesaid bears to n is of course an obvious and familiar fact.

This will be the case (Serret, vol. II. p. 239), if (on that supposition) $\frac{5}{6}\cdot\frac{3}{2}n - n$, that is, if

$$\frac{n}{4} > 2\sqrt{(\tfrac{3}{2}n)} + \frac{25(\log\frac{3}{2}n)^2}{16A\log 6} + \frac{125}{24A}(\log\tfrac{3}{2}n) + \frac{25}{6A},$$

where $A = \cdot 92129202\ldots$.

But when $n = 3000$, it will be found that the terms on the second side of the inequality are respectively less than

$$134\cdot 1641,\quad 66\cdot 9773,\quad 47\cdot 5546,\quad 4\cdot 5227,$$

whose sum is less than 750.

Hence, the inequality is satisfied, and accordingly the theorem is true when $m < 2n$ and n is equal to or *greater* than 3000; for when n satisfies that condition the derivative in respect to n of the right-hand side of the above inequality will be always less than $\frac{1}{4}$.

(2) Suppose $m =$ or $> 2n$, then it is only necessary to prove that

$$\log(2n+1)(2n+2)\ldots(3n-\nu) > \log(1.2.3\ldots n),$$

or, what is the same thing, that

$$\log\{1.2.3.4\ldots(3n-\nu)\} > \log(1.2.3\ldots n) + \log(1.2.3\ldots 2n),$$

ν being the number of primes not greater than n, and n being at least 3000.

Call the two sides of the inequality P and Q. Then (Serret, vol. II. p. 226)

$$\begin{aligned}P &> \log\sqrt{(2\pi)} + (3n-\nu)\log(3n-\nu) - (3n-\nu) - \tfrac{1}{2}\log(3n-\nu)\\ &> \log\sqrt{(2\pi)} + (3n-\nu)\log 3n + (3n-\nu)\log\left(1-\frac{\nu}{3n}\right) - 3n + \nu - \tfrac{1}{2}\log 3n\\ &> \log\sqrt{(2\pi)} + 3(\log n)n + (3\log 3 - 3)n - (\log n)\nu\\ &\qquad + (1-\log 3)\nu - \tfrac{1}{2}\log 3 - \tfrac{1}{2}\log n - \nu,\end{aligned}$$

for $-(3n-\nu)\log\left(1-\dfrac{\nu}{3n}\right) = \nu\left\{1 - \tfrac{1}{2}\left(\dfrac{\nu}{3n}\right) - \tfrac{1}{6}\left(\dfrac{\nu}{3n}\right)^2 - \tfrac{1}{12}\left(\dfrac{\nu}{3n}\right)^3 - \ldots\right\} < \nu.$

On the other hand,

$$\begin{aligned}Q &< \log\sqrt{(2\pi)} + n\log n - n + \tfrac{1}{2}\log n + \tfrac{1}{12}\\ &\quad + \log\sqrt{(2\pi)} + 2n\log 2n - 2n + \tfrac{1}{2}\log 2n + \tfrac{1}{12}\\ &< \{2\log\sqrt{(2\pi)} + \tfrac{1}{2}\log 2 + \tfrac{1}{6}\} + 3(\log n)n + (2\log 2 - 3)n + \log n.\end{aligned}$$

Hence

$$\begin{aligned}P - Q &> (3\log 3 - 2\log 2)n - (\log n)\nu - \tfrac{3}{2}\log n - (\log 3)\nu - \{\tfrac{1}{2}\log(12\pi) + \tfrac{1}{6}\}\\ &> (3\log 3 - 2\log 2)n - \log n(\nu - \tfrac{1}{2})\\ &\qquad - 2\log n - \log 3(\nu - \tfrac{1}{2}) - \{\tfrac{1}{2}\log(36\pi) + \tfrac{1}{6}\}\end{aligned}$$

where $\nu - \frac{1}{2} < 1\cdot 606\,\dfrac{n}{\log n}$.

But $3 \log 3 - 2 \log 2 = 1{\cdot}9095415 > 1{\cdot}909.$

Hence*

$$P - Q > (\cdot 303)\, n - (1{\cdot}606 \log 3) \frac{n}{\log n} - 2 \log n - \{\tfrac{1}{2} \log (36\pi) + \tfrac{1}{6}\},$$

say $P - Q > f(n) > 0$ when $n = 3000$.

Also the derivative with respect to n of $(\log n) f(n)$ being

$$(\cdot 303)(1 + \log n) - 1{\cdot}606 \log 3 - \frac{4 \log n}{n} - \frac{\frac{1}{2} \log (36\pi) + \frac{1}{6}}{n},$$

$P - Q$ will increase as n increases and will remain positive for all values of n superior to 3000.

Hence the theorem is true, whatever m may be, when $n = \text{or} > 3000$, and since it has been proved previously for the case of $n < 3000$, it is true universally.

I subjoin the valuable table, kindly communicated to me by Dr Glaisher, referred to in the text above.

Table of Increasing Sequences of Composite Numbers interposed between Consecutive Primes included in the first nine million numbers.

Limits to sequence	Number of terms
7 to 11	3
23 „ 29	5
89 „ 97	7
113 „ 127	13
523 „ 541	17
887 „ 907	19
1129 „ 1151	21
1327 „ 1361	33
9551 „ 9587	35
15683 „ 15727	43
19609 „ 19661	51
31397 „ 31469	71
155921 „ 156007	85
373261 „ 373373	111
492113 „ 492227	113
1349533 „ 1349651	117
1357201 „ 1357333	131
2010733 „ 2010881	147
4652353 „ 4652507	153

* It will now be seen why I take separately the two cases of m greater and m less than $2n$. If we were to take *simpliciter* $m =$ or $> n$ and were to attempt to prove

$$\log \{1 \,.\, 2 \,.\, 3 \ldots (2n - \nu)\} > 2 \log (1 \,.\, 2 \,.\, 3 \ldots n)$$

the inferior limit to the difference between these two quantities would then have for its principal term, not $(3 \log 3 - 2 \log 2 - 1{\cdot}606)\, n$ but $(2 \log 2 - 1{\cdot}606)\, n$, which would be *negative*.

Of course there is no special reason except of convenience (in dealing with an integer instead of a fraction) for making $2n$ the dividing point between the two suppositions separately considered in the text; κn where κ as far as regards the second inequality does not fall short of some

The table is to be understood as follows. The lowest sequence of as many as 3 consecutive composite numbers is that included between 7 and 11: the lowest of as many as 5 is that included between 23 and 29, of as many as 7 that included between 89 and 97; between 13 and 17 there is a break—this indicates that the lowest sequence of as many as 15, or as many as 17 first occurs in the sequence of 17 interposed between 523, 541. Similarly the break between 21 and 33 indicates that the lowest sequence containing 23 or 25 or 27 or 29 or 31 or 33 terms first occurs in the sequence of 33 composite numbers interposed between the primes 1327, 1361.

It is also necessary to add that in the first nine million numbers there is no succession of *more* than 153 consecutive composite numbers.

§ 3. *Relating to irreducible arithmetical series in general**.

Let P be a principal term quâ q in any *irreducible* arithmetical series whose common difference is i, N any other term greater or less than P, and D their difference. If q is not prime to i, no term in the series will be divisible by q.

Just as in the case of a natural sequence when there is only one principal term in the series it may be shown that the index of D quâ q will be the same as that of N; when there is more than one principal term it appears by the same reasoning as before that the index of N cannot be greater than that of D: (it will not now necessarily be equal unless q is greater than the common difference i).

The index-sum quâ q is zero when q has a common measure with i, and we may therefore consider only the case where q is relatively prime to i;

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certain limit, would have served as well: this inferior limit to κ would be some quantity a little greater (how much exactly would have to be found by trial) than the quantity θ which makes $\theta \log \theta - (\theta - 1) \log (\theta - 1)$ equal to the coefficient of $\frac{n}{\log n}$ in the superior limit to ν. As regards the first inequality κ would have to be a quantity somewhat less (how much less to be found by trial) than the quantity η which makes $\frac{\eta + 1}{\eta} = \frac{6}{5}$, that is, $\eta = 5$. This is on the supposition made throughout of using Tchebycheff's own limits, but if we use the more general, but less compact, limits indicated in my paper in vol. IV. of the *American Journal of Mathematics* ‡, any fraction not less than $\frac{6}{5}$ and not so great as $\frac{59595}{51072}$ would take the place of $\frac{6}{5}$, and the extreme value of η would be $\frac{51072}{8523}$, which is a trifle under 6. By a judicious choice of the value given to κ, a value of n could be found considerably less than 3000, which would satisfy both inequalities, and this in the absence of Dr Glaisher's table would have been a matter of some practical importance, but is of next to none when we have that table to draw upon. How low down in the scale of number, n may be taken, without interruption of the existence of the fundamental inequality for the minimum value of n in the case treated of in this section, it has not been necessary for the purpose in hand to ascertain. That it holds good for all values of n above a certain limit follows from the fact that $2 \log 2$ is greater than the coefficient of the leading term in the superior functional limit to the sum of the logarithms of the primes not greater than n.

* An irreducible arithmetical series is one whose terms are prime to their common difference.

[‡ Vol. III. of this Reprint, p. 530.]

on this supposition, by virtue of what has been stated above, the index-sum quâ q of the series whose first term is $m+i$, and number of terms n, will be equal to or less than

$$E\left(\frac{P-m-i}{iq}\right)+E\left(\frac{P-m-i}{iq^2}\right)+E\left(\frac{P-m-i}{iq^3}\right)+\ldots$$
$$+E\left(\frac{m+ni-P}{iq}\right)+E\left(\frac{m+ni-P}{iq^2}\right)+E\left(\frac{m+ni-P}{iq^3}\right)+\ldots;$$

and therefore *à fortiori*

$$< \text{or} = E\left(\frac{(n-1)\,i}{iq}\right)+E\left(\frac{(n-1)\,i}{iq^2}\right)+E\left(\frac{(n-1)\,i}{iq^3}\right)+\ldots$$
$$< \text{or} = E\left(\frac{n}{q}\right)+E\left(\frac{n}{q^2}\right)+E\left(\frac{n}{q^3}\right)+\ldots,$$

that is, not greater than the index-sum of 2, 3, ..., n quâ q.

Consequently, by the same reasoning as that employed in the last section, the theorem now to be proved, namely, that if m (prime to i) $=$ or $>n$, then $(m+i)(m+2i)\ldots(m+ni)$ must contain some one or more prime numbers greater than n, must be true whenever

$$(m+i)(m+2i)(m+3i)\ldots\{m+(n-\nu_1)\,i\} > 1.2.3\ldots n \qquad (\Theta)^*$$

where ν_1 is the number of prime numbers not exceeding n, and not contained in i, and *à fortiori* when for ν_1, we substitute, as for the present we shall do, ν the *entire* number of primes not greater than n. This I term the *fundamental inequality* for the general case now under consideration.

Suppose $n=$ or >3000. The logarithm of the first side of the fundamental inequality when we write ν for ν_1 is obviously greater than the ith part of the logarithm of

$$(m+1)(m+2)\ldots(m+i)(m+i+1)\ldots\{m+(n-\nu)\,i\};$$

and the inequality (subject to certain suppositions) to be established will be satisfied, if on the same suppositions,

$$\frac{1}{i}\log[1.2.3\ldots\{m+(n-\nu)\,i\}] > \log(1.2.3\ldots n)+\frac{1}{i}\log(1.2.3\ldots m).$$

Suppose $m=n$, and make

$$\log[1.2.3\ldots\{(i+1)\,n-i\nu\}] = T,$$
$$(i+1)\log(1.2.3\ldots n) = U,$$
$$F(n,i) = T-U.$$

* If it had been necessary the condition in the text might have been stated in the more stringent form that *some aliquot part* of the factorial of n (namely, this factorial divested of all powers of prime numbers contained in i) would have to be greater than

$$(m+i)(m+2i)\ldots\{m+(n-\nu_1)\,i\}$$

if the theorem were not true for any specified values of m, n, i.

It will be noticed that when i is relatively prime to n, ν_1 is less than ν so that $n-\nu_1>n-\nu$: some use will be made of the formula in the text when dealing with certain small values of n and $m-n$ towards the end of the section.

Then $T > \log(2\pi) + \{(i+1)n - i\nu\}\log\{(i+1)n - i\nu\}$
$$-\{(i+1)n - i\nu\} - \tfrac{1}{2}\log\{(i+1)n - i\nu\},$$
$$U < (i+1)\log\sqrt{(2\pi)} + (i+1)n\log n - (i+1)n + \tfrac{1}{2}(i+1)\log n + \tfrac{1}{12}(i+1).$$

Hence $F(n, i) > -i\log\sqrt{(2\pi)} + \{(i+1)n - i\nu\}\log\{(i+1)n\}$
$$+\{(i+1)n - i\nu\}\log\left\{1 - \frac{i\nu}{(i+1)n}\right\}$$
$$+ i\nu - (i+1)n\log n - \tfrac{1}{2}\log\{(i+1)n - i\nu\} - \tfrac{1}{2}(i+1)\log n - \tfrac{1}{12}(i+1)$$
$$> \{(i+1)\log(i+1)\}n - i\log\{(i+1)n\}\nu - \tfrac{1}{2}\log\{(i+1)n\} - \tfrac{1}{2}(i+1)\log n$$
$$- \tfrac{1}{2}i\log(2\pi) - \tfrac{1}{12}(i+1),$$

that is $> \{(i+1)\log(i+1)\}n - i\log\{(i+1)n\}\nu - \tfrac{1}{2}(i+2)\log n - \tfrac{1}{2}\log(i+1)$
$$- \tfrac{1}{2}i\log(2\pi) - \tfrac{1}{12}(i+1) \quad \text{(H)},$$

so that when $n > 3000$ and consequently $\nu < \frac{1}{2} + (1{\cdot}606)\frac{n}{\log n}$, the inequality to be established will be true *à fortiori* if

$$F(n, i) > \left\{(i+1)\log(i+1) - (1{\cdot}606)\,i\left[1 + \frac{\log(i+1)}{\log n}\right]\right\}n - (i+1)\log n$$
$$- \left[\tfrac{1}{2}(i+1)\log(i+1) + \tfrac{1}{2}\{i\log(2\pi)\} + \tfrac{1}{12}(i+1)\right].$$

When $i = 1$ or 2 or 3 the coefficient of n is negative; consequently the limit to ν before found is no longer applicable to bring out the desired result.

The case of $i = 1$ has been already disposed of; that of $i = 2$ may be disposed of, as I shall show, in a similar manner; when $i = 3$, I shall raise the limit n from 3000 to 8100 of which the logarithm is so near to 9 that it may, for the purpose of the proof in hand, be regarded as equal to 9 without introducing any error in the inequality to be established, as the error involved will only affect the result in a figure beyond the 4th or 5th place of decimals, whereas the inequality in question depends on figures in the first decimal place. When this is done the theorem will be in effect demonstrated for the case of $i = 3$ and $n > 8100$. For all values of n not greater than 8100 I shall be able to show that the fundamental inequality (Θ) is satisfied by employing the actual value of ν_1 or ν instead of a limiting value of the latter.

Thus the fundamental inequality will be shown to subsist for all values of n when $i = 3$ and $m = n$, and *à fortiori* therefore for all values of m and i *not less* than n and 3 respectively.

Case of $i = 2$.

Suppose $n =$ or > 3000, and take separately the cases $m <$ or $= 2n$, $m > 2n$.

(1) Let m be not greater than $2n$ so that $m + 2n$ is greater than $2m - 1$.

By hypothesis m must be odd, and by Bertrand's Postulate

$$m+2,\ m+3,\ m+4,\ \ldots,\ 2m,$$

and therefore $$m+2,\ m+4,\ m+6,\ \ldots,\ (2m-1)$$

(seeing that the interpolated terms are all even) must contain a prime, and thus the first case is disposed of.

(2) Since the fundamental inequality has been shown to be satisfied when $n>3000$, $m>2n$, $i=1$, it will *à fortiori* be so when $n>3000$, $m>2n$, $i=2$.

Hence the theorem is established for $i=2$ when $n>3000$. Finally as regards values of n inferior to 3000, the reasoning employed for the case of $i=1$ applies *à fortiori* to the case of $i=2$.

To see this let us recall the first step of the reasoning applicable to the supposition of $i=1$.

Because in the first nine million numbers there is no sequence of 3000 composite numbers, from Dr Glaisher's Table of Sequences (taken in conjunction with the fact that when $m>n^2$, the theorem has been proved to be true whatever n may be), we were able to infer that it must be true when n does not exceed 153: in the present case, if the theorem were not true when $3000>n>153$, there would be a sequence of 153 composite odd numbers and therefore of over 305 composite consecutive numbers in the first 9000000 numbers, whereas there are not more than 153, and so we may proceed step by step till we arrive at the conclusion that the theorem must be true when $n>13$; and when $n=13$, 11, 7, 5, 3, 2, 1 a like method of disproof (but briefer) will apply as for the case of $i=1$.

Case of $i=$ or >3.

Let $n=$ or >8100. Then we may without ultimate error write

$$\nu-\tfrac{1}{2}<\frac{1{\cdot}1056+\dfrac{5}{4\log 6}\,\dfrac{81}{8100}+\frac{5}{2}\,\dfrac{9}{8100}+\dfrac{2}{8100}}{1-\dfrac{\log 9}{9}-\dfrac{1-\log 2}{9}}\,\frac{n}{\log n}<1{\cdot}546\,\frac{n}{\log n},$$

and accordingly

$$F(n,3)>\left\{4\log 4-(3\times 1{\cdot}546)\left(1+\frac{\log 4}{9}\right)\right\}n$$
$$-\,4\log n-(2\log 4+\tfrac{3}{2}\log 2\pi+\tfrac{1}{3})$$

and $$F(8100,3)>(5{\cdot}545-5{\cdot}352)(8100)-36-5{\cdot}863>0.$$

Hence the Fundamental Inequality is satisfied when $n=$ or >8100.

To prove that it is satisfied for values inferior to 8100, observe that by virtue of the formula (H) it will be so, *ex abundantiâ*, for all values of n not less than $'n$ and not greater than n', provided that, calling n'_ν the number of primes not exceeding n',

$$(5{\cdot}545)\,'n - 3\log(4n')\,n'_\nu - \tfrac{5}{2}\log n' - C > 0,$$

where $$C = \tfrac{1}{3} + \log 2 + \tfrac{3}{2}\log(2\pi) = 3{\cdot}783.$$

On trial it will be found that the above inequality is satisfied when we successively substitute for $'n$, n', and for n'_ν (found from any Table for the enumeration of primes) the values given in the annexed table:

n'	n'_ν	$'n$
8100	1018	5725
5724	753	4096
4095	564	2967
2966	427	2172
2171	326	1604
1603	252	1200
1199	196	903
902	154	687
686	124	535
534	99	415
414	80	325
324	66	260
259	55	210
209	46	171
170	39	141
140	34	111
110	29	99
98	25	84
83	23	76
75	21	68
67	19	62
61	18	57
56	16	50
49	15	46
45	14	42
41	13	39
38	12	36
35	11	32
31	11	31
30	10	30
29	10	29

The fundamental theorem is therefore established when $i > 2$ for all values of n down to 29 inclusive.

It remains to consider the case where n is any prime number less than 29.

Calling μ the difference between n and the number of primes (exclusive of 1) not greater than n, to

$$n = 2,\ 3,\ 11,\ 17,\ 23$$

will correspond

$$\mu = 1,\ 1,\ \ 6,\ 10,\ 14$$

and for each combination of these corresponding numbers it will be found that

$$1 . 2 . 3 \ldots n = \text{or} < (n+3)(n+6) \ldots (n+3\mu).$$

Hence the theorem is proved for these values of n, whatever n may be, when $i = \text{or} > 3$. To

$$n = 13, \quad n = 19$$

corresponds

$$\mu = 7, \quad \mu = 11,$$

and for these combinations of n and μ it will be found that

$$1 . 2 . 3 \ldots n < (n+4)(n+7) \ldots (n+1+3\mu),$$

so that the theorem is true for

$$n = 13, \ 19,$$

except in the case where

$$m = 13, \ 19.$$

That it is true in these excepted cases follows from inspection of the series,

16, 19, 22, 25, &c.,

22, 25, 28, 31, &c.,

where $19 > 13$, $31 > 19$: or it might be proved, but more cumbrously, by the same method as that applied below to the only two values of n remaining to be considered, namely

$$n = 5, \quad n = 7,$$

for which we have respectively

$$\mu = 2, \quad \mu = 3.$$

If $n = 5$ and i has no common measure with $2 . 3 . 4 . 5$, i must be not less than 7, but $1 . 2 . 3 . 4 . 5 < 12 . 19$.

On the other hand, if i has a common measure with $2 . 3 . 4 . 5$, then what we have called ν_1, in formula (Θ), is less than ν, so that $n - \nu_1 > 2$, but

$$1 . 2 . 3 . 4 . 5 < 8 . 11 . 14.$$

These two inequalities combined serve to prove that, whatever i may be, the inequality (Θ) is satisfied, and the theorem is consequently proved for $n = 5$.

So again, when $n = 7$, if i has no common measure with $2 . 3 . 4 . 5 . 6 . 7$ it must be 11 at least. In that case the inequality $2 . 3 . 4 . 5 . 6 . 7 < 18 . 29 . 40$, and in the contrary case the inequality $2 . 3 . 4 . 5 . 6 . 7 < 10 . 13 . 16 . 19$ serves to prove the theorem.

When $n = 1$ the truth of the theorem is obvious: hence combining the results obtained in this and the preceding section, it will be seen we have proved that whatever n and whatever i may be, provided that m is relatively prime to i and not less than n, the product

$$(m+i)(m+2i) \ldots (m+ni)$$

must contain some prime number by which $2.3 \ldots n$ is not divisible, and the wearisome proof is thus brought to a close. It will not surprise the author of it, if his work should sooner or later be superseded by one of a less piece-meal character—but he has sought in vain for any more compendious proof. He has not thought it necessary to produce the figures or refer in detail to the calculations giving the numerical results inserted in various places in the text: had he done so the number of pages, already exceeding what he had any previous idea of, would probably have been more than doubled*.

PART II†.

Explicit Primes.

In this part I shall consider the asymptotic limits to the number of primes of certain *irreducible* linear forms $mz + r$ comprised between a number x and a given fractional multiple thereof kx, the method of investigation being such that the asymptotic limits determined will be unaffected by the value of r, and will be the same for all values of m which

* The author was wandering in an endless maze in his attempts at a general proof of his theorem, until in an auspicious hour when taking a walk on the Banbury road (which leads out of Oxford) the Law of Ademption flashed upon his brain: meaning thereby the law (the nerve, so to say, of the preceding investigation) that *if all the terms of a natural arithmetical series be increased by the same quantity so as to form a second such series, no prime number can enter in a higher power as a factor of the product of the terms of this latter series, when a suitable term has been* taken away *from it, than the highest power in which it enters as a factor into the product of the terms of the original series.*

In Part II. I shall be able to apply the same method to demonstrate a theorem showing that it is always possible to split up an infinite arithmetical series, if not in the general case, at least for certain values of the common difference, into an infinite number of successive finite and determinable segments such that one or more primes shall be found in each such segment: a theorem which is, so to say, Dirichlet's theorem on arithmetical progressions cut up into slices.

The whole matter is thus made to rest on an elementary fundamental *equality* (Tchebycheff's) which, with the aid of an application of Stirling's theorem, leads (as the former has so admirably shown) *inter alia* to a superior limit to the sum of the logarithms of the primes not exceeding a given number, from which as has been seen in § 2, a superior limit may be deduced to the number of such primes. With the aid of this last limit together with an elementary fundamental *inequality* and a *renewed* application of Stirling's theorem, all my results are made to flow. Thus a theorem of pure form is brought to depend on considerations of greater and less, or as we may express it, Quality is made to stoop its neck to the levelling yoke of Quantity.

Long and vain were my previous efforts to make the desired results hinge upon the properties of transposed Eratosthenes' scales: now we may hope to reverse the process and compel these scales to reveal the secret of their laws under the new light shed upon them by the successful application of the Quantitative method.

† I ought to have stated that the theorem contained in section 2 of Part I. originally appeared in the form of a question (No. 10951) in the *Educational Times* for April of this year.

have the same totient. The simplest case, and the foundation of all that follows, is that in which $k=0$ and $m=2$: this will form the subject of the ensuing chapter which may be regarded as a supplement to Tchebycheff's celebrated memoir of 1850*, and as superseding my article thereon in vol. IV. of the *Amer. Math. Journ.* [Vol. III. of this Reprint, p. 530].

CHAPTER I.

ON THE ASYMPTOTIC LIMITS TO THE NUMBER OF PRIMES INFERIOR TO A GIVEN NUMBER.

§ 1. *Crude determination of the asymptotic limits.*

Call the sum of the logarithms of primes not exceeding x (any real positive quantity) the prime-number-logarithmic sum, or more briefly the prime-log-sum to x, and the sum of such sums to x and all its positive integer roots the prime-log-sum-sum, which in Serret is called $\psi(x)$.

Then it follows from elementary arithmetical principles that the sum of this sum-sum to x and all its aliquot parts, that is

$$\psi(x)+\psi\left(\frac{x}{2}\right)+\psi\left(\frac{x}{3}\right)+\ldots,$$

which we may call the natural series of sum-sums and denote by $T(x)$, is identical with the logarithm of the factorial of the highest integer not exceeding x, and accordingly from Stirling's theorem may be shown to have for its asymptotic limit $x\log x - x$, the superior and inferior limits being this quantity with a residue which, as well for the one as for the other, is a known linear function of $\log x$. Serret, vol. II. p. 226.

If now we take two sets of positive integers,

$$p, p', p'', \ldots; \; q, q', q'', \ldots,$$

together forming what may be termed a *harmonic scheme*, meaning thereby that the sum of the reciprocals of the numbers in the two sets is the same, and extend the T series over x divided by the respective numbers in each set and take the difference between the two sums thus obtained, there will result a new series of the form

$$\sum_{n=1}^{n=\infty} \cdot f(n)\,\psi\left(\frac{x}{n}\right),$$

of which the asymptotic limit will be x multiplied by

$$\Sigma\frac{\log p}{p}-\Sigma\frac{\log q}{q},$$

and the value of $f(n)$ will be

$$\Sigma\frac{n}{p:}-\Sigma\frac{n}{q:},$$

* Published in the *St Petersburg Transactions* for 1854.

where, in general, $\frac{n}{t:}$ means 1 or 0 according as n does or does not contain t, or in other words the "denumerant" of the equation $ty = n$.

I shall call the p's and q's the *stigmata* of the scheme:

$$\Sigma \frac{\log p}{p} - \Sigma \frac{\log q}{q}$$

the stigmatic multiplier, and the new series in $\psi(x)$ a stigmatic series of sum-sums (obtained, it will be noticed, by a four-fold process of summation—namely, two infinite and two finite summations).

It is possible, in general (as will hereafter appear), to deduce from the asymptotic value of a stigmatic series of sum-sums, superior and inferior asymptotic limits to the sum-sum itself. The *asymptotic* limits to the simple sum will then be the same as those last named (Serret, vol. II. p. 236, formulae (8) and (9)*) and will be multiples of x: dividing these respectively by $\log x$, we obtain superior and inferior asymptotic limits to the number of primes not exceeding x (*Messenger*, May 1891, p. 9, footnote [above, p. 694]).

It is obviously simplest always to take unity as one of the stigmata; those employed by Tchebycheff are 1, 30; 2, 3, 5; this *scheme* as I term it leads to the relation

$$\psi\left(\frac{x}{1}\right) - \psi\left(\frac{x}{6}\right) + \psi\left(\frac{x}{7}\right) - \psi\left(\frac{x}{10}\right) + \psi\left(\frac{x}{11}\right) - \psi\left(\frac{x}{12}\right)$$

$$+ \psi\left(\frac{x}{13}\right) - \psi\left(\frac{x}{15}\right) + \psi\left(\frac{x}{17}\right) - \psi\left(\frac{x}{18}\right) + \psi\left(\frac{x}{19}\right) - \psi\left(\frac{x}{20}\right)$$

$$+ \psi\left(\frac{x}{23}\right) - \psi\left(\frac{x}{24}\right) + \psi\left(\frac{x}{29}\right) - \psi\left(\frac{x}{30}\right)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \left(\tfrac{1}{2} \log 2 + \tfrac{1}{3} \log 3 + \tfrac{1}{5} \log 5 - \tfrac{1}{30} \log 30\right) x + \dagger,$$

the series extending to infinity but consisting of repetitions (with a difference) of the above period, obtained by adding for the second period 30, for the third period 60, for the fourth period 90, and so on, to each denominator in the period set out. We may call this a period of 30 terms in which the coefficients are $+1$, 0, or -1. So, in general, whatever the stigmata may be, the stigmatic series will consist of periods of terms in each of which the total number of terms will be the least common multiple of the stigmata.

* The fourth edition, 1879, of Serret's *Cours d'Algèbre Supérieure* is referred to here and throughout the paper.

† The + is used to denote that a quantity is omitted of inferior *order* of magnitude to x. The strict interpretation of the "relation" is that the sum of the stigmatic series less the stigmatic multiplier into x is intermediate to two known linear functions of $\log x$.

Thus, for example, the schemes 1; 2, 2 and 1, 6; 2, 3, 3 would give rise to the relations

$$\psi(x)-\psi\left(\frac{x}{2}\right)+\psi\left(\frac{x}{3}\right)-\psi\left(\frac{x}{4}\right)+\psi\left(\frac{x}{5}\right)-\psi\left(\frac{x}{6}\right)+\ldots$$

$$=(\tfrac{1}{2}\log 2+\tfrac{1}{2}\log 2)\,x+=(\log 2)\,x+\ldots,$$

$$\psi(x)-\psi\left(\frac{x}{3}\right)+\psi\left(\frac{x}{5}\right)-\psi\left(\frac{x}{6}\right)+\psi\left(\frac{x}{7}\right)-\psi\left(\frac{x}{9}\right)+\psi\left(\frac{x}{11}\right)-\psi\left(\frac{x}{12}\right)+\ldots$$

$$=(\tfrac{1}{2}\log 2+\tfrac{2}{3}\log 3-\tfrac{1}{6}\log 6)\,x+=(\tfrac{1}{3}\log 2+\tfrac{1}{2}\log 3)\,x+\ldots$$

of which the periods are 2 and 6 respectively.

The three schemes above given, whose keys, so to say, are 2, 3, 5 respectively (these being the highest prime numbers contained in the stigmata), possess the property that their effective coefficients are alternately plus and minus 1, and, in consequence thereof, we may *immediately* deduce from them asymptotic limits superior and inferior to the logarithmic sum-sum $\psi(x)$.

Thus, calling the stigmatic multipliers in the three cases

$$St_2,\ St_3,\ St_5,$$

we obtain as limits to the coefficient of x in $\psi(x)$,

St_2 and $2St_2$ from the first,

St_3 „ $\frac{3}{2}St_3$ „ „ second,

and St_5 „ $\frac{6}{5}St_5$ „ „ third scheme.

(Compare Serret, pp. 233, 234, where the A is the present St_5.)

The three pairs of limits will thus be

·6931472 : 1·3862944,

·7803552 : 1·1705328,

·9212920 : 1·1055504,

which are in regular order of closer and closer propinquity to unity on each side of it*.

The question then arises can no further schemes be discovered which will enable us to bring the asymptotic coefficients still nearer to this empirical limit†?

* Mr Hammond has noticed that the harmonic scheme 1, 12; 2, 3, 4 will also give rise to a stigmatic series in which the effective terms are alternately positive and negative units, namely,

$$\psi(x)-\psi\left(\frac{x}{4}\right)+\psi\left(\frac{x}{5}\right)-\psi\left(\frac{x}{6}\right)+\psi\left(\frac{x}{7}\right)-\psi\left(\frac{x}{8}\right)+\psi\left(\frac{x}{11}\right)-\psi\left(\frac{x}{12}\right)+\psi\left(\frac{x}{13}\right)-\psi\left(\frac{x}{16}\right)+\ldots,$$

the stigmatic multiplier corresponding to which, say St_{12}, is ·8522758 ..., and therefore will furnish the asymptotic coefficients St_{12} and $\frac{4}{3}St_{12}$, that is, ·8522758 ... and 1·1363687 ....

† The true asymptotic limit to the number of primes below x being according to Legendre's empirical rule $\frac{x}{\log x}$, the asymptotic value of $\psi(x)$ should presumably be x.

It would, I believe, be perfectly futile to seek for stigmatic schemes, involving higher prime numbers than 5, that should give rise to stigmatic series of sum-sums in which the successive coefficients should be alternately positive and negative unity, as in the above instances, but this although a sufficient is not a necessary condition in order that limits to a sum-sum may be capable of being extracted from the known limits to the sum of a series of such sum-sums.

This will be most easily explained by actually exhibiting a new scheme which is effective to the end in view, and showing why it is so.

Such a scheme is 1, 6, 70; 2, 3, 5, 7, 210, which, it will be observed, satisfies the necessary *harmonic* condition: for we have

$$1 + \tfrac{1}{6} + \tfrac{1}{70} = \frac{210 + 35 + 3}{210} = \tfrac{248}{210},$$

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{210} = \frac{105 + 70 + 42 + 30 + 1}{210} = \tfrac{248}{210}.$$

The *stigmatic multiplier* is here

$$\tfrac{1}{2}\log 2 + \tfrac{1}{3}\log 3 + \tfrac{1}{5}\log 5 + \tfrac{1}{7}\log 7 + \tfrac{1}{210}\log 210 - \tfrac{1}{6}\log 6 - \tfrac{1}{70}\log 70 = \cdot 9787955,$$

which I shall call D.

The stigmatic series arranged in sets in two different ways then becomes as a first arrangement

$$\psi(x) - \psi\left(\frac{x}{10}\right):$$

$$+\psi\left(\frac{x}{11}\right) + \psi\left(\frac{x}{13}\right) - \psi\left(\frac{x}{14}\right) - \psi\left(\frac{x}{15}\right);\ +\psi\left(\frac{x}{17}\right) + \psi\left(\frac{x}{19}\right)$$

$$-\psi\left(\frac{x}{20}\right) - \psi\left(\frac{x}{21}\right);\ +\psi\left(\frac{x}{23}\right) - \psi\left(\frac{x}{28}\right);\ +\psi\left(\frac{x}{29}\right) - \psi\left(\frac{x}{30}\right);$$

$$+\psi\left(\frac{x}{31}\right) - \psi\left(\frac{x}{35}\right);\ +\psi\left(\frac{x}{37}\right) - \psi\left(\frac{x}{40}\right);\ +\psi\left(\frac{x}{41}\right) - \psi\left(\frac{x}{42}\right);$$

$$+\psi\left(\frac{x}{43}\right) - \psi\left(\frac{x}{45}\right);\ +\psi\left(\frac{x}{47}\right) - \psi\left(\frac{x}{50}\right);\ +\psi\left(\frac{x}{53}\right) - \psi\left(\frac{x}{56}\right);$$

$$+\psi\left(\frac{x}{59}\right) - \psi\left(\frac{x}{60}\right);\ +\psi\left(\frac{x}{61}\right) - \psi\left(\frac{x}{63}\right);\ +\psi\left(\frac{x}{67}\right) - \psi\left(\frac{x}{70}\right);$$

$$+\psi\left(\frac{x}{71}\right) + \psi\left(\frac{x}{73}\right) - \psi\left(\frac{x}{75}\right) + \psi\left(\frac{x}{79}\right) - \psi\left(\frac{x}{80}\right) + \psi\left(\frac{x}{83}\right)$$

$$-\psi\left(\frac{x}{84}\right) + \psi\left(\frac{x}{89}\right) - \psi\left(\frac{x}{90}\right) + \psi\left(\frac{x}{97}\right) - \psi\left(\frac{x}{98}\right) - \psi\left(\frac{x}{100}\right);$$

$$+\psi\left(\frac{x}{101}\right) + \psi\left(\frac{x}{103}\right) - \psi\left(\frac{x}{105}\right) - \psi\left(\frac{x}{105}\right);\ +\psi\left(\frac{x}{107}\right)$$

$$+\psi\left(\frac{x}{109}\right)-\psi\left(\frac{x}{110}\right)-\psi\left(\frac{x}{112}\right);+\psi\left(\frac{x}{113}\right)-\psi\left(\frac{x}{120}\right);$$

$$+\psi\left(\frac{x}{121}\right)-\psi\left(\frac{x}{126}\right);+\psi\left(\frac{x}{127}\right)-\psi\left(\frac{x}{130}\right);+\psi\left(\frac{x}{131}\right)$$

$$-\psi\left(\frac{x}{135}\right);+\psi\left(\frac{x}{137}\right)+\psi\left(\frac{x}{139}\right)-\psi\left(\frac{x}{140}\right)+\psi\left(\frac{x}{143}\right)$$

$$-\psi\left(\frac{x}{147}\right)+\psi\left(\frac{x}{149}\right)-\psi\left(\frac{x}{150}\right)+\psi\left(\frac{x}{151}\right)-\psi\left(\frac{x}{154}\right)$$

$$+\psi\left(\frac{x}{157}\right)-\psi\left(\frac{x}{160}\right)+\psi\left(\frac{x}{163}\right)-\psi\left(\frac{x}{165}\right)+\psi\left(\frac{x}{167}\right)$$

$$-\psi\left(\frac{x}{168}\right)+\psi\left(\frac{x}{169}\right)-\psi\left(\frac{x}{170}\right)+\psi\left(\frac{x}{173}\right)-\psi\left(\frac{x}{175}\right)$$

$$+\psi\left(\frac{x}{179}\right)-\psi\left(\frac{x}{180}\right)+\psi\left(\frac{x}{181}\right)-\psi\left(\frac{x}{182}\right)+\psi\left(\frac{x}{187}\right)$$

$$-\psi\left(\frac{x}{189}\right)-\psi\left(\frac{x}{190}\right);+\psi\left(\frac{x}{191}\right)+\psi\left(\frac{x}{193}\right)-\psi\left(\frac{x}{195}\right)$$

$$-\psi\left(\frac{x}{196}\right);+\psi\left(\frac{x}{197}\right)+\psi\left(\frac{x}{199}\right)-\psi\left(\frac{x}{200}\right)+\psi\left(\frac{x}{209}\right)$$

$$-\psi\left(\frac{x}{210}\right)-\psi\left(\frac{x}{210}\right);+\psi\left(\frac{x}{211}\right)-\psi\left(\frac{x}{220}\right);$$

....

....

the correlative arrangement being

$$\psi(x)-\psi\left(\frac{x}{10}\right)+\psi\left(\frac{x}{11}\right)+\psi\left(\frac{x}{13}\right):$$

$$-\psi\left(\frac{x}{14}\right)-\psi\left(\frac{x}{15}\right)+\psi\left(\frac{x}{17}\right)+\psi\left(\frac{x}{19}\right);-\psi\left(\frac{x}{20}\right)-\psi\left(\frac{x}{21}\right)$$

$$+\psi\left(\frac{x}{23}\right)-\psi\left(\frac{x}{28}\right)+\psi\left(\frac{x}{29}\right)-\psi\left(\frac{x}{30}\right)+\psi\left(\frac{x}{31}\right)-\psi\left(\frac{x}{35}\right)$$

$$+\psi\left(\frac{x}{37}\right)-\psi\left(\frac{x}{40}\right)+\psi\left(\frac{x}{41}\right)-\psi\left(\frac{x}{42}\right)+\psi\left(\frac{x}{43}\right)-\psi\left(\frac{x}{45}\right)$$

$$+\psi\left(\frac{x}{47}\right)-\psi\left(\frac{x}{50}\right)+\psi\left(\frac{x}{53}\right)-\psi\left(\frac{x}{56}\right)+\psi\left(\frac{x}{59}\right)-\psi\left(\frac{x}{60}\right)$$

$$+\psi\left(\frac{x}{61}\right)-\psi\left(\frac{x}{63}\right)+\psi\left(\frac{x}{67}\right)-\psi\left(\frac{x}{70}\right)+\psi\left(\frac{x}{71}\right)+\psi\left(\frac{x}{73}\right);$$

$$-\psi\left(\frac{x}{75}\right)+\psi\left(\frac{x}{79}\right);-\psi\left(\frac{x}{80}\right)+\psi\left(\frac{x}{83}\right);-\psi\left(\frac{x}{84}\right)+\psi\left(\frac{x}{89}\right);$$

$$-\psi\left(\frac{x}{90}\right)+\psi\left(\frac{x}{97}\right);-\psi\left(\frac{x}{98}\right)-\psi\left(\frac{x}{100}\right)+\psi\left(\frac{x}{101}\right)$$

$$+\psi\left(\frac{x}{103}\right);-\psi\left(\frac{x}{105}\right)-\psi\left(\frac{x}{105}\right)+\psi\left(\frac{x}{107}\right)+\psi\left(\frac{x}{109}\right);$$

$$-\psi\left(\frac{x}{110}\right)-\psi\left(\frac{x}{112}\right)+\psi\left(\frac{x}{113}\right)-\psi\left(\frac{x}{120}\right)+\psi\left(\frac{x}{121}\right)$$

$$-\psi\left(\frac{x}{126}\right)+\psi\left(\frac{x}{127}\right)-\psi\left(\frac{x}{130}\right)+\psi\left(\frac{x}{131}\right)-\psi\left(\frac{x}{135}\right)$$

$$+\psi\left(\frac{x}{137}\right)+\psi\left(\frac{x}{139}\right);-\psi\left(\frac{x}{140}\right)+\psi\left(\frac{x}{143}\right);-\psi\left(\frac{x}{147}\right)$$

$$+\psi\left(\frac{x}{149}\right);-\psi\left(\frac{x}{150}\right)+\psi\left(\frac{x}{151}\right);-\psi\left(\frac{x}{154}\right)+\psi\left(\frac{x}{157}\right);$$

$$-\psi\left(\frac{x}{160}\right)+\psi\left(\frac{x}{163}\right);-\psi\left(\frac{x}{165}\right)+\psi\left(\frac{x}{167}\right);-\psi\left(\frac{x}{168}\right)$$

$$+\psi\left(\frac{x}{169}\right);-\psi\left(\frac{x}{170}\right)+\psi\left(\frac{x}{173}\right);-\psi\left(\frac{x}{175}\right)+\psi\left(\frac{x}{179}\right);$$

$$-\psi\left(\frac{x}{180}\right)+\psi\left(\frac{x}{181}\right);-\psi\left(\frac{x}{182}\right)+\psi\left(\frac{x}{187}\right);-\psi\left(\frac{x}{189}\right)$$

$$-\psi\left(\frac{x}{190}\right)+\psi\left(\frac{x}{191}\right)+\psi\left(\frac{x}{193}\right);-\psi\left(\frac{x}{195}\right)-\psi\left(\frac{x}{196}\right)$$

$$+\psi\left(\frac{x}{197}\right)+\psi\left(\frac{x}{199}\right);-\psi\left(\frac{x}{200}\right)+\psi\left(\frac{x}{209}\right);-\psi\left(\frac{x}{210}\right)$$

$$-\psi\left(\frac{x}{210}\right)+\psi\left(\frac{x}{211}\right)-\psi\left(\frac{x}{220}\right)+\psi\left(\frac{x}{221}\right)+\psi\left(\frac{x}{223}\right):$$

....

....*.

The terms in each arrangement, it will be seen, are separated by marks of punctuation into groups: omitting the first group in either of them, which may be called the outstanding group, in each of the others the sum of the coefficients is zero.

Moreover, the sum of the coefficients from the beginning of each group is always homonymous in sign, that is, will be non-negative in the first and non-positive in the second arrangement: the consequence of this is that all the terms of such groups may be resolved into pairs, whose sum will be necessarily positive in the one and negative in the other.

Thus, for example, in the first arrangement the last but one of the groups may be resolved into the pairs

$$\psi\left(\frac{x}{197}\right)-\psi\left(\frac{x}{200}\right);\ \psi\left(\frac{x}{199}\right)-\psi\left(\frac{x}{210}\right);\ \psi\left(\frac{x}{209}\right)-\psi\left(\frac{x}{210}\right).$$

* Each of these arrangements is to be regarded as made up of the outstanding group and an infinite succession of periodic groups. In the text we have set out the outstanding group and the first period, the other periods will be formed from this one by adding to each denominator in it successive multiples of 210.

each of which is equal to zero or a positive quantity. So the eighth group of the second arrangement is resoluble into the pairs

$$-\psi\left(\frac{x}{98}\right)+\psi\left(\frac{x}{101}\right);\ -\psi\left(\frac{x}{100}\right)+\psi\left(\frac{x}{103}\right),$$

each of which is zero or a negative quantity.

It may be as well to notice in this place that the sum of the coefficients, reckoning from the first term of the outstanding group to the term whose denominator is n, is

$$\sum_{t=0}^{t=n}\Sigma\left(\frac{t}{p:}-\frac{t}{q:}\right),$$

which by virtue of the obvious identity,

$$\sum_{t=0}^{t=n}\left(\frac{t}{i:}\right)=E\left(\frac{n}{i}\right),$$

is equal to
$$\Sigma\left\{E\left(\frac{n}{p}\right)-E\left(\frac{n}{q}\right)\right\}.$$

This formula supplies an easy and valuable test for ascertaining the correctness of the determination of the coefficients up to any given term in the series.

These observations may be extended to any harmonic scheme whatever: for it will be observed that

$$\Sigma\left\{E\left(\frac{n}{p}\right)-E\left(\frac{n}{q}\right)\right\}$$

is a periodic quantity, and therefore possesses both a maximum and a minimum; whence it is easy to see that, by taking the outstanding group of terms sufficiently extensive, all the remaining terms in either kind of arrangement may be separated into groups similar to those above set out; namely, such that the *complete* sum of the coefficients in each group from its first to its end term is zero and up to any intermediate term is *homonymous*, that is, always positive in one and always negative in the other arrangement*.

* For example, from the harmonic scheme 1, 15; 2, 3, 5, 30, we may derive a stigmatic series under the two forms of arrangement

$$\psi(x)-\psi\left(\frac{x}{6}\right):+\psi\left(\frac{x}{7}\right)-\psi\left(\frac{x}{10}\right);\ +\psi\left(\frac{x}{11}\right)-\psi\left(\frac{x}{12}\right);\ +\psi\left(\frac{x}{13}\right)+\psi\left(\frac{x}{17}\right)-\psi\left(\frac{x}{18}\right)$$
$$+\psi\left(\frac{x}{19}\right)-\psi\left(\frac{x}{20}\right)+\psi\left(\frac{x}{23}\right)-\psi\left(\frac{x}{24}\right)+\psi\left(\frac{x}{29}\right)-\psi\left(\frac{x}{30}\right)-\psi\left(\frac{x}{30}\right);\ +\psi\left(\frac{x}{31}\right)-\psi\left(\frac{x}{36}\right):\ \&c.,$$

$$\psi(x)-\psi\left(\frac{x}{6}\right)+\psi\left(\frac{x}{7}\right)-\psi\left(\frac{x}{10}\right)+\psi\left(\frac{x}{11}\right)-\psi\left(\frac{x}{12}\right)+\psi\left(\frac{x}{13}\right)+\psi\left(\frac{x}{17}\right):\ -\psi\left(\frac{x}{18}\right)$$
$$+\psi\left(\frac{x}{19}\right);\ -\psi\left(\frac{x}{20}\right)+\psi\left(\frac{x}{23}\right);\ -\psi\left(\frac{x}{24}\right)+\psi\left(\frac{x}{29}\right);\ -\psi\left(\frac{x}{30}\right)-\psi\left(\frac{x}{30}\right)+\psi\left(\frac{x}{31}\right)$$
$$-\psi\left(\frac{x}{36}\right)+\psi\left(\frac{x}{37}\right)-\psi\left(\frac{x}{40}\right)+\psi\left(\frac{x}{41}\right)-\psi\left(\frac{x}{42}\right)+\psi\left(\frac{x}{43}\right)+\psi\left(\frac{x}{47}\right):\ \&c.$$

In the above arrangements the groups are separated by semicolons and the period is marked out by the colons. In this instance it will be observed that minimum and maximum values of

The consequence of this is that the outstanding group in the first arrangement will always be less, and in the second arrangement always greater, than a function of which the principal, or, as we may call it, the asymptotic term, is the product of x by the stigmatic multiplier, say (St), the complete function being in each case of the form $(St)x$ associated with a known linear function of $\log x$. (Compare Serret, vol. II. p. 232.)

The importance of this observation will become apparent in a subsequent section.

In the case before us (that is, for the scheme in the key of 7) confining our attention to the principal term of either limit, the first arrangement leads immediately (Serret, p. 234) to the superior asymptotic limit $\frac{10}{9}Dx$.

As regards the inferior limit, we have

$$\psi(x) + \psi\left(\frac{x}{13}\right) > Dx,$$

$$\psi(x) > Dx - \frac{1}{13} \cdot \frac{10}{9} Dx > \frac{107}{117} Dx^*.$$

Substituting for D its value ·9787955, we obtain the asymptotic limits 1·0873505 and ·8951370.

The corresponding values got from the Tchebycheffian scheme (1, 30; 2, 3, 5) being 1·1055504 and ·9212920, which are the $\frac{6}{5}A$ and A of Serret.

We know *aliunde* that the true asymptotic values are each of them presumably unity. The superior value above obtained by the new scheme is thus seen to be better, and the inferior value worse than those given by Tchebycheff's scheme. But these values correspond to what may be termed the *crude* determination of the limits which the schemes are capable of affording. The contraction of these asymptotic limits by a method of continual successive approximation will form the subject of the following section†.

$E(n) + E\left(\frac{n}{15}\right) - E\left(\frac{n}{2}\right) - E\left(\frac{n}{3}\right) - E\left(\frac{n}{5}\right) - E\left(\frac{n}{30}\right)$ are 0 and 2, and accordingly in the first arrangement the outstanding group has to be continued until the sum of the coefficients of the terms which it contains is 0, and in the second until such sum is 2.

Writing $Q = \frac{1}{2}\log 2 + \frac{1}{3}\log 3 + \frac{1}{5}\log 5 + \frac{1}{30}\log 30 - \frac{1}{15}\log 15 =$ ·96750 ...,

we may deduce from the above, the asymptotic coefficients $\frac{6}{5}Q$ and $Q - \frac{1}{17} \cdot \frac{6}{5}Q$; that is, 1·1610 ... and ·8992....

* Compare the determination of the limits for the harmonic scheme 1; 2, 3, 6 (*American Journal of Mathematics*, vol. IV. pp. 243, 244 [Vol. III. of this Reprint, p. 542]).

† By the method about to be explained, it should be noticed, we may not merely improve upon the results obtained by the *crude* method from certain harmonic schemes (which form a very restricted class) but may also obtain limits to $\psi(x) \div x$ from harmonic schemes which without its aid would be absolutely sterile (see p. [715]).

§ 2. *On a method of obtaining continually contracting asymptotic limits to* $\frac{\psi(x)}{x}$.

To fix the ideas let us consider the scheme (1, 30; 2, 3, 5) which leads to the stigmatic series

$$(1)-(6)+(7)-(10)+(11)-(12)+(13)-(15)+(17)-(18)+(19)$$
$$-(20)+(23)-(24)+(29)-(30)+(31)\ldots,$$

in which for brevity (n) is used to denote $\psi\left(\frac{x}{n}\right)$.

The sum of this series is, we know, intermediate between

$$Dx+R(\log x) \quad \text{and} \quad D_1x+R_1(\log x),$$

where $$D=\cdot 9212920\ldots, \quad D_1=1\cdot 1055504\ldots=\tfrac{6}{5}D,$$

and R, R_1 signify two known quantities which for uniformity may both be regarded as quadratic functions of $\log x$ (in the first of which the coefficient of $(\log x)^2$ is zero). (Serret, pp. 233, 235.)

Omitting every pair of consecutive terms $-(m)+(\mu)$ in which $\frac{\mu}{m}<\frac{6}{5}$, and using $[\psi(x)]$ to signify the asymptotic value of $\psi(x)$, we find

$$[\psi(x)]>Dx+\left[\psi\left(\frac{x}{24}\right)\right]-\left[\psi\left(\frac{x}{29}\right)\right]>Dx+D\frac{x}{24}-D_1\frac{x}{29},$$

say $$>D'x.$$

Similarly, omitting every consecutive pair of terms $(m)-(\mu)$ in which $\frac{\mu}{m}<\frac{6}{5}$, we find

$$[\psi(x)]<Dx+D_1\frac{x}{6}-D\frac{x}{7}+D_1\frac{x}{10},$$

say $$<D_1'x.$$

If instead of $[\psi(x)]$ we had deduced limits to $\psi(x)$ in the manner indicated above, we should have found

$$\psi(x)>D'x+R'(\log x), \quad \psi(x)<D_1'x+R_1'(\log x);$$

the added terms being each of them quadratic functions of $\log x$.

Repeating this process we shall obtain

$$[\psi(x)]>D''x, \quad [\psi(x)]<D_1''x,$$

where $$D''=D+\tfrac{1}{24}D'-\tfrac{1}{29}D_1', \quad D_1''=D+\tfrac{1}{6}D_1'-\tfrac{1}{7}D'+\tfrac{1}{10}D_1'.$$

Similarly we may write

$$[\psi(x)]>D'''x, \quad [\psi(x)]<D_1'''x,$$

where $$D'''=D+\tfrac{1}{24}D''-\tfrac{1}{29}D_1'', \quad D_1'''=D+\tfrac{1}{6}D_1''-\tfrac{1}{7}D''+\tfrac{1}{10}D_1'',$$

and so on.

If then we write for $D,\ D',\ D'',\ \ldots,\ v_0,\ v_1,\ v_2,\ \ldots,$
and for $D_1,\ D_1',\ D_1'',\ \ldots,\ u_0,\ u_1,\ u_2,\ \ldots,$
we shall find in general

$$[\psi(x)] > v_i x, \quad [\psi(x)] < u_i x;$$

where

$$v_{i+1} = D + \frac{v_i}{24} - \frac{u_i}{29},$$

$$u_{i+1} = D + (\tfrac{1}{6} + \tfrac{1}{10})\, u_i - \tfrac{1}{7} v_i;$$

the complete statement of the inequalities being

$$\psi(x) > v_i x + R^{(i)}(\log x), \quad \psi(x) < u_i x + R_1^{(i)}(\log x),$$

where it is to be noticed that the supplemental terms always remain *quadratic* functions of $\log x$.

(The result thus obtained differs in this particular from that stated by me in the *Amer. Math. Jour.* (vol. IV. p. 241)*; the process therein employed giving as supplemental terms rational integral functions of continually rising degrees of $\log x$. I am indebted to Mr Hammond for drawing my attention to this simple but important circumstance which had strangely escaped my attention previously.) To integrate the equations in u, v we have only to write

$$v_i = V_i + F, \quad u_i = U_i + E,$$

$$F(1 - \tfrac{1}{24}) + \tfrac{1}{29}E = D, \quad V_i = C_1\rho_1^{\,i} + C_2\rho_2^{\,i},$$

$$\tfrac{1}{7}F + (1 - \tfrac{1}{6} - \tfrac{1}{10})\,E = D, \quad U_i = K_1\rho_1^{\,i} + K_2\rho_2^{\,i};$$

and to take for $\rho_1,\ \rho_2$ the two roots of the equation

$$\begin{vmatrix} \rho - \frac{1}{24}, & \frac{1}{29} \\ \frac{1}{7}, & \rho - \frac{1}{6} - \frac{1}{10} \end{vmatrix} = \rho^2 - (\tfrac{1}{6} + \tfrac{1}{10} + \tfrac{1}{24})\,\rho + \tfrac{1}{24}(\tfrac{1}{6} + \tfrac{1}{10}) - \tfrac{1}{203} = 0,$$

that is

$$\rho^2 - \tfrac{37}{120}\rho + \tfrac{113}{18270} = 0.$$

The roots of this equation being each less than 1, on making $i = \infty$ we obtain $v_\infty = F,\ u_\infty = E$, where E, F are deduced from the two algebraic equations

$$\tfrac{23}{24}F + \tfrac{1}{29}E = D,$$

$$\tfrac{1}{7}F + \tfrac{11}{15}E = D.$$

This gives

$$\frac{E}{F} = (\tfrac{23}{24} - \tfrac{1}{7}) \div (\tfrac{11}{15} - \tfrac{1}{29}) = \frac{137 \times 145}{304 \times 56} = \frac{19865}{17024} = q$$

(compare *Amer. Math. Jour.*, vol. IV. p. 242),

$$E = \tfrac{59595}{50999}D = 1{\cdot}0765779\ \ldots,$$

$$F = \tfrac{51072}{50999}D = {\cdot}9226107\ \ldots;$$

whence we may infer that $\psi(x)$ may be made intermediate between two

[* See Vol. III. of this Reprint, p. 539.]

known functions $u_i x + r(\log x)$, $v_i x + s(\log x)$, where u_i, v_i may be brought *indefinitely near* to the numbers

$$1{\cdot}0765779\ldots,\quad {\cdot}9226107\ldots;$$

and the supplemental terms are quadratic functions of $\log x$ depending upon the value of i that may be employed. We may, therefore (subject to an obvious interpretation), treat E and F as asymptotic limits to $\frac{\psi(x)}{x}$.*

If we examine the ratio of the denominators m, μ of any pair of consecutive terms throughout the entire infinite series, whether of the form $(m)-(\mu)$ or $-(m)+(\mu)$, we shall find that $\frac{\mu}{m}$ is always less than q (namely $1{\cdot}16688\ldots$), except in the case of the pairs that have been retained in forming the equations between E and F, from which we may infer that if any of the discarded pairs had been retained we should have obtained values of E and F respectively greater and less than those above set forth.

If, on the other hand, q had turned out to be so much less than $\frac{6}{5}$ as to cause $\frac{\mu}{m}$ in any rejected pair to be greater than q, in such case in order to obtain a value of E the least, and of F the greatest, capable of being extracted from the given scheme, it would have been necessary to take account of every such pair and perform the calculations afresh, thereby obtaining a new value of q (say q') less than the former one; we should then have had to continue the process of examining the rejected pairs and reinstating those (if any) whose denominators furnished a ratio $\frac{\mu}{m}$ greater than q', thereby obtaining a still smaller value q''. Repeating these operations *toties quoties* we should at last arrive at a value of q superior to every ratio $\frac{\mu}{m}$ throughout the entire stigmatic series; the corresponding values of the asymptotic limits would then be the best capable of being deduced from the given scheme.

Per contra had we retained at the start any of the discarded pairs of terms, we should have found for q a value greater than the value of $\frac{\mu}{m}$ in some of the terms retained, which would be a sure indication that the retention of those terms had led to a greater value of q than was necessary; those pairs would then have to be omitted; the q calculated from the reformed equations would be diminished by so doing and the resulting values of E, F

* For the complete analytical determination of the limits to $\psi(x)$ see § 3 of this chapter.

By making i sufficiently great u_i, v_i may be brought indefinitely near to E, F: furthermore, when the superior and inferior limits of $\psi(x) \div x$ are expressed as functions of x and i of the form mentioned in the text, these limits may, by taking x sufficiently great, be brought indefinitely near to u_i, v_i, and therefore to E, F, which I therefore speak of throughout as asymptotic limits to $\psi(x) \div x$. But more strictly the optimistic limits actually arrived at are E' as little as we please greater than E, and F' as little as we please less than F.

would be the best attainable, provided that care was taken at the outset that no rejected pair gave a larger value to $\frac{\mu}{m}$ than any pair that had been retained.

In the case we have considered initial asymptotic limits (namely D and D_1) to $\frac{\psi(x)}{x}$ were obtained from the scheme itself, but it will not always be possible to do this when we are dealing with any harmonic scheme.

Thus, for example, from the fact that the minor arrangement of the stigmatic series corresponding to the scheme [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105] has (1) + (13) for its outstanding group [see p. 718], we may deduce that $\psi(x) + \psi\left(\frac{x}{13}\right)$ has Nx for its inferior asymptotic limit, but are unable from this arrangement to obtain an initial inferior asymptotic limit to $\psi(x)$ itself, and still less shall we be able to obtain an initial superior asymptotic limit to $\psi(x)$ from the major arrangement of the same stigmatic series. It is therefore important to notice that the final asymptotic limits arrived at by the method explained in this section, depend only on the stigmatic multiplier and the coefficients of the stigmatic series, being quite independent of the *initial* values employed, so that in the general case we may start from any given asymptotic limits to $\frac{\psi(x)}{x}$, *however obtained*, without thereby producing any effect in the final result. The limits $u_0 = 2\log 2$ and $v_0 = \log 2$ obtained from the scheme [1; 2, 2] will do as well as any others for our initial asymptotic limits to $\frac{\psi(x)}{x}$, and we may, by substituting these limits in the retained portion of the stigmatic series, arrive at new limits u_1, v_1 which in their turn will give rise to fresh limits u_2, v_2 and so on. We shall in this way obtain a pair of difference equations (connecting u_{i+1}, v_{i+1} with u_i, v_i) which will be of the same form as those previously given [p. 713], and it is to be noticed that in the solutions of these equations, namely

$$u_i = C\rho^i + C_1\rho_1{}^i + E, \quad v_i = K\rho^i + K_1\rho_1{}^i + F,$$

only the values of C, C_1, K, K_1 will depend on the initial values of u, v; so that, provided the roots of the quadratic in ρ (which are always real) are each less than unity, we may, by taking i sufficiently great, make u_i and v_i approach as near as we please to E and F respectively; that is as near as we please to two quantities whose values depend solely on the stigmatic series employed.

The positive and negative divergences from unity of the E and F previously found are respectively

$$\cdot 0765779\ldots, \quad \cdot 0773893\ldots;$$

these divergences as found by Tchebycheff being

$$\cdot 1055504\ldots, \quad \cdot 0787080\ldots,$$

which is already an important gain; but by varying the scheme we shall obtain still better results.

Let us apply the method of indefinite successive approximation to the scheme in the key of 7 treated of in the preceding section, namely [1, 6, 70; 2, 3, 5, 7, 210], for which the stigmatic multiplier (the D of p. [707]), namely

$$\tfrac{1}{2}\log 2 + \tfrac{1}{3}\log 3 + \tfrac{1}{5}\log 5 + \tfrac{1}{7}\log 7 + \tfrac{1}{210}\log 210 - \tfrac{1}{6}\log 6 - \tfrac{1}{70}\log 70$$

is ·9787955....

Preliminary calculations having served to satisfy me that the asymptotic ratio $\frac{E}{F}$ (the q) for this system was not likely to differ much from 1·10, which may be called the *regulator*, I form the corresponding equations for E and F by retaining only those pairs $(m)-(\mu)$ in the stigmatic series for which $\frac{\mu}{m}$ is greater than 1·10.

As previously explained no *error* can result whatever regulator we employ; the worst that can happen will be that the result will not be the best attainable from the scheme, and such imperfection can be ascertained by means of the method previously explained; the result, if the best possible, will prove itself to be so, and, if not the best, will indicate whether the regulator (or criterion of retention) has been taken too small or too great.

Let us examine separately the two arrangements set out in the previous section, the first being employed to obtain by successive approximations the superior, and the second the inferior, limit.

Consider 1° the periodic part of the first arrangement: in the group $(11)+(13)-(14)-(15)$, the pair $(13)-(14)$ being rejected, $(11)-(15)$ remains. Similarly, in the following group $(19)-(20)$ being rejected, $(17)-(21)$ remains; in the third and fourth groups $(23)-(28)$ and $(31)-(35)$ are to be retained. In the following group, all the consecutive pairs from (73) to (98) both inclusive are to be rejected, leaving $(71)-(100)$ available. (The corresponding pair to this in the next period, namely $(281)-(310)$, gives $\frac{310}{281}$, which is less than the assumed regulator.) All the groups in the first period, following $-(100)$, will have to be rejected until we come to the group beginning with (137), which leads to the available pair $(137)-(190)$: in the next period all the ratios will be too small with the exception of $(347)-(400)$ which must be retained, but the term corresponding to this in the third period, namely $(557)-(610)$, will have to be neglected.

Hence, in approximating to the superior limit, we may write

$$u_{i+1} = M + \left(\tfrac{1}{10} + \tfrac{1}{15} + \tfrac{1}{21} + \tfrac{1}{28} + \tfrac{1}{35} + \tfrac{1}{100} + \tfrac{1}{190} + \tfrac{1}{400}\right) u_i$$
$$- \left(\tfrac{1}{11} + \tfrac{1}{17} + \tfrac{1}{23} + \tfrac{1}{31} + \tfrac{1}{71} + \tfrac{1}{137} + \tfrac{1}{347}\right) v_i.$$

2°. In the second arrangement, the first group in the periodic part being $-(14)-(15)+(17)+(19)$, and $\frac{17}{15}$ (and *à fortiori* $\frac{19}{14}$) exceeding the regulator, all these terms are to be preserved.

In addition to these, we shall find in the first period the available couples $-(20)+(73)$ and $-(110)+(139)$, and in the second period $-(230)+(283)$; no other couples will be available, and accordingly, we shall have

$$v_{i+1} = M + (\tfrac{1}{10} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{20} + \tfrac{1}{110} + \tfrac{1}{230})\, v_i$$
$$- (\tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{17} + \tfrac{1}{19} + \tfrac{1}{73} + \tfrac{1}{139} + \tfrac{1}{283})\, u_i.$$

If then we write a, b for the coefficients of u_i, $-v_i$ in the first, and c, d for the coefficients of v_i, $-u_i$ in the second of the above equations, and make $u_i = U_i + E$, $v_i = V_i + F$, we shall obtain

$$u_i = C\rho^i + C_1\rho_1{}^i + E,$$
$$v_i = K\rho^i + K_1\rho_1{}^i + F,$$

where ρ, ρ_1 are the roots of the equation

$$\begin{vmatrix} \rho - a, & b \\ d & , \rho - c \end{vmatrix} = 0,$$

that is $\rho^2 - (a + c)\,\rho + (ac - bd) = 0,$

and E, F are subject to the equations

$$(1-a)\,E + bF = M,$$
$$dE + (1-c)\,F = M,$$

which give

$$E = \frac{1-b-c}{(1-a)(1-c)-bd}\,M, \quad F = \frac{1-a-d}{(1-a)(1-c)-bd}\,M.$$

On performing the calculations, we shall find

$$a = {\cdot}29633\ldots, \quad b = {\cdot}24973\ldots,$$
$$c = {\cdot}30153\ldots, \quad d = {\cdot}30371\ldots,$$
$$1-b-c = {\cdot}44873\ldots, \quad 1-a-d = {\cdot}39995\ldots,$$
$$ac = {\cdot}08935\ldots, \quad bd = {\cdot}07584\ldots,$$
$$a+c = {\cdot}59786\ldots, \quad (1-a)(1-c)-bd = {\cdot}41563\ldots,$$

ρ, ρ_1 will therefore be the roots of

$$\rho^2 - {\cdot}59786\rho + {\cdot}01350 = 0,$$

which are each less than unity.

Also $E = 1{\cdot}0567265\ldots, \quad F = {\cdot}9418543\ldots,$

$$q = \frac{1-b-c}{1-a-d} = 1{\cdot}12196\ldots.$$

This last number being *greater* than the assumed regulator 1·10, and *less* than any of the retained ratios "$\left[\frac{\mu}{m}\right]$", it follows that no better limits than E, F can be extracted from the scheme [1, 6, 70; 2, 3, 5, 7, 210]; or (as we may phrase it) E, F are the optimistic asymptotic limits to that scheme.

Obviously, there is no reason to suppose that these are the closest asymptotic limits that can be obtained from the infinite choice of schemes at our disposal: on the contrary, there is every reason to suppose that these limits may by schemes in higher and higher keys be brought to coincide as nearly as may be desired to each other and to unity.

We shall presently obtain by aid of a new scheme a better result than the E, F of the preceding investigation. But first it should be observed that instead of forming the difference equations in u, v from the two arrangements, say the major and minor, of one and the same stigmatic series (the former meaning the one used to find the superior and the latter the inferior asymptotic limit), we may take these two arrangements, if we please, from two distinct series corresponding to two different schemes.

I have had calculated, from beginning to end, the value of the coefficient of each term in the stigmatic series of sum-sums corresponding to the first natural period, containing 2310 terms of the scheme (1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105), the stigmatic multiplier to which, namely

$$\tfrac{1}{2}\log 2 + \tfrac{1}{3}\log 3 + \tfrac{1}{5}\log 5 + \tfrac{1}{7}\log 7 + \tfrac{1}{11}\log 11 + \tfrac{1}{105}\log 105$$
$$- \tfrac{1}{6}\log 6 - \tfrac{1}{10}\log 10 - \tfrac{1}{210}\log 210 - \tfrac{1}{231}\log 231 - \tfrac{1}{1155}\log 1155,$$

is ·9909532 ... (say N).

This stigmatic series, though too long for printing at full in the restricted space of this Journal, is given later on in a condensed tabular form (see Table A, p. 721). I will proceed to describe its essential features and the use made of it to bring the asymptotic limits closer together. The maximum and minimum sums of its coefficients are 2 and -2: the first terms being $(1)+(13)-(14)-(15)$, the maximum is first reached at the second term; so that the outstanding group in the minor arrangement will be $(1)+(13)$. But the minimum sum, -2, is not reached before the term whose argument is (616). The outstanding group in the major arrangement will therefore contain a very great number of terms, and there might be some trouble in handling the groups, so as to secure the greatest possible advantage. For this reason, I have thought it sufficient for the present to combine the major arrangement of the scheme [1, 6, 70; 2, 3, 5, 7, 210] with the minor one of the scheme [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105].

Maintaining the regulator still at the same value as before, namely 1·10, the major arrangement will remain unaltered from what it was in the preceding case. In the minor arrangement there will be found to exist the following 17 available pairs, all of which, except the last, belong to the first period (the last one belonging to the second period), namely

(14) – (19), (15) – (17), (21) – (31), (33) – (41), (44) – (53), (63) – (73),
(84) – (97), (105) – (241), (110) – (131), (195) – (223), (315) – (481),
(525) – (703), (735) – (943), (945) – (1231), (1484) – (1693),
(1694) – (2323), (4004) – (4633).

We may accordingly write

$$u_{i+1} = M + au_i - bv_i,$$
$$v_{i+1} = N + \gamma v_i - \delta u_i,$$

where

$$a = \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \frac{1}{28} + \frac{1}{35} + \frac{1}{100} + \frac{1}{190} + \frac{1}{400},$$

$$b = \frac{1}{11} + \frac{1}{17} + \frac{1}{23} + \frac{1}{31} + \frac{1}{71} + \frac{1}{137} + \frac{1}{347},$$

$$\gamma = \frac{1}{14} + \frac{1}{15} + \frac{1}{21} + \frac{1}{33} + \frac{1}{44} + \frac{1}{63} + \frac{1}{84} + \frac{1}{105} + \frac{1}{110}$$
$$+ \frac{1}{195} + \frac{1}{315} + \frac{1}{525} + \frac{1}{735} + \frac{1}{945} + \frac{1}{1484} + \frac{1}{1694} + \frac{1}{4004},$$

$$\delta = \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \frac{1}{31} + \frac{1}{41} + \frac{1}{53} + \frac{1}{73} + \frac{1}{97} + \frac{1}{131} + \frac{1}{223}$$
$$+ \frac{1}{241} + \frac{1}{481} + \frac{1}{703} + \frac{1}{943} + \frac{1}{1231} + \frac{1}{1693} + \frac{1}{2323} + \frac{1}{4633},$$

from which, writing
$$(1-a)E + bF = M,$$
$$\delta E + (1-\gamma)F = N,$$
we shall find
$$u_i = C\rho^i + C_1\rho_1{}^i + E,$$
$$v_i = K\rho^i + K_1\rho_1{}^i + F,$$
where ρ, ρ_1 are the roots of

$$\begin{vmatrix} \rho - a, & b \\ \delta\;, & \rho - \gamma \end{vmatrix} = 0,$$

that is
$$\rho^2 - (a+\gamma)\rho + a\gamma - b\delta = 0.$$

The values of a, b; γ, δ are respectively

·2963346 ..., ·2497346 ... ; ·2992774 ..., ·3107808 ...,

from which we see that ρ, ρ_1 being each less than unity the values of u_∞, v_∞ will be E, F, where

$$E = \frac{(1-\gamma)M - bN}{(1-a)(1-\gamma) - b\delta},$$

$$F = \frac{(1-a)N - \delta M}{(1-a)(1-\gamma) - b\delta},$$

and on performing the calculation it will be found that

$$E = 1{\cdot}0551851\ldots, \quad F = {\cdot}9461974.$$

Also
$$q = \frac{E}{F} = 1{\cdot}11518\ldots,$$

which being greater than the assumed regulator, but less than any of the retained ratios $\frac{\mu}{m}$, the results thus obtained are *optimistic*, that is no better can be found without having recourse to some other harmonic scheme.

The advance made upon the determination of the asymptotic limits beyond what was known previously is already remarkable. Tchebycheff's asymptotic numbers stood at

$$1{\cdot}1055504\ldots,$$
$${\cdot}9212920\ldots,$$

corresponding to a divergence from unity

${\cdot}1055504\ldots$ in excess,

and ${\cdot}0787080\ldots$ in defect;

by the combined effect of scheme variation and successive substitution we have succeeded in reducing these divergences to

${\cdot}0551851\ldots$ in excess,

and ${\cdot}0538026\ldots$ in defect;

in which it will be noticed that the divergence for the superior limit is only a little more than half the original one.

The mean of the two limits, it will be seen, is now less than

$$1{\cdot}0007.$$

The annexed table, in which for brevity $\bar{c}$ is written for $-c$, gives in a condensed form the stigmatic series to the scheme [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105].

The coefficients, for all the terms $\psi\left(\frac{x}{m}\right)$ from $m=1$ to $m=1155$ (the half modulus), are written down in regular batches of 10. The coefficients for the ensuing terms up to 2309 can be got from these by the formula $c_{1155+t} = c_{1155-t}$, the term following will have the coefficient zero; the rest of the infinite series is then known from the formula $c_{t+2310i} = c_t$.

Table A.

The coefficients of the first 1155 *terms of the stigmatic series to* [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105]*.

1000000000	$001\bar{1}\bar{1}01010$	$\bar{1}\bar{1}10000\bar{1}10$	$10\bar{1}0\bar{1}01000$
$1\bar{1}\bar{1}\bar{1}01000$	$0010\bar{1}\bar{1}0010$	$10\bar{1}00\bar{1}100\bar{1}$	$1010\bar{1}0\bar{1}010$
$001\bar{1}000\bar{1}10$	$0000001\bar{1}\bar{1}0$	$1010\bar{3}01011\bar{1}$	$0\bar{1}10000000$
$00000\bar{1}1000$	$1\bar{1}00\bar{1}0101\bar{1}$	$000000\bar{1}010$	$100\bar{2}001000$
$0010\bar{2}01\bar{1}10$	$0010\bar{1}\bar{1}0010$	$1\bar{1}000000\bar{1}0$	$1010\bar{1}\bar{1}1\bar{1}10$
$000000000\bar{1}$	$100000000\bar{1}$	$101\bar{1}\bar{1}01010$	$\bar{1}010000\bar{1}10$
$1\bar{1}00\bar{1}01000$	$1\bar{1}00\bar{1}01000$	$001\bar{1}0\bar{1}0010$	$10\bar{1}0\bar{1}0100\bar{1}$
$1010\bar{1}\bar{1}0010$	$001\bar{1}00\bar{1}010$	$0000001\bar{2}00$	$1010\bar{3}01000$
$0\bar{1}1000000\bar{1}$	$10000\bar{1}1000$	$0000\bar{1}0101\bar{1}$	$0\bar{1}1000\bar{1}010$
$10\bar{1}\bar{1}001000$	$001\bar{1}\bar{1}01\bar{1}10$	$0010\bar{2}00010$	$1\bar{1}000\bar{1}10\bar{1}0$
$1010\bar{1}\bar{1}0010$	$0000000\bar{1}1\bar{1}$	$10000000\bar{1}0$	$101\bar{1}\bar{1}0101\bar{1}$
$\bar{1}010000\bar{1}10$	$0000\bar{1}01000$	$1\bar{1}10\bar{1}01000$	$00000\bar{1}0010$
$10\bar{1}\bar{1}00100\bar{1}$	$1010\bar{2}00010$	$001\bar{1}0\bar{1}0010$	$0000000\bar{1}00$
$1010\bar{3}01\bar{1}10$	$0\bar{1}100000\bar{1}0$	$10000\bar{1}100\bar{1}$	$1000\bar{1}0101\bar{1}$
$\bar{1}01000\bar{1}010$	$1\bar{1}0\bar{1}001000$	$0000\bar{1}01\bar{1}10$	$001\bar{1}\bar{1}00010$
$1\bar{1}00\bar{1}010\bar{1}0$	$1010\bar{1}\bar{2}1010$	$000000\bar{1}01\bar{1}$	$1000000\bar{1}00$
$101\bar{1}\bar{1}01000$	$\bar{1}010000\bar{1}1\bar{1}$	$1000\bar{1}01000$	$0\bar{1}10\bar{1}01000$
$0\bar{1}100\bar{1}0010$	$10\bar{1}000100\bar{1}$	$101\bar{1}\bar{1}00010$	$001\bar{1}\bar{1}00010$
$00000\bar{1}1\bar{1}00$	$1010\bar{3}00010$	$0\bar{1}10000\bar{1}00$	$10000\bar{1}10\bar{1}0$
$1000\bar{1}0101\bar{2}$	$001000\bar{1}010$	$000\bar{1}001000$	$0\bar{1}10\bar{1}01\bar{1}10$
$0000\bar{1}00010$	$1\bar{1}0\bar{1}0010\bar{1}0$	$1010\bar{2}\bar{1}1010$	$00000\bar{1}001\bar{1}$
$100000\bar{1}000$	$101\bar{1}\bar{1}01\bar{1}10$	$\bar{1}010000\bar{1}00$	$1000\bar{1}0100\bar{1}$
$1\bar{1}10\bar{1}01000$	$\bar{1}0100\bar{1}0010$	$1\bar{1}\bar{1}000100\bar{1}$	$1000\bar{1}00010$
$001\bar{1}000010$	$0000\bar{1}01\bar{1}00$	$1010\bar{3}\bar{1}1010$	$0\bar{1}1000\bar{1}000$
$10000\bar{1}1\bar{1}00$	$1000\bar{1}0100\bar{1}$	$001000\bar{1}01\bar{1}$	$100\bar{1}001000$
$\bar{1}010\bar{1}01\bar{1}10$	$0\bar{1}10\bar{1}00010$	$1\bar{1}\bar{1}0001\bar{1}0$	$101\bar{1}\bar{1}\bar{1}1010$
$0000\bar{1}0001\bar{1}$	$10000\bar{1}0000$	$101\bar{1}\bar{1}00010$	$\bar{1}010000\bar{2}10$
$1000\bar{1}010\bar{1}0$	$1\bar{1}10\bar{1}0100\bar{1}$	$00100\bar{1}0010$	$00\bar{1}000100\bar{1}$
$1\bar{1}10\bar{1}00010$	$000\bar{1}000010$	$000\bar{1}001\bar{1}00$	$1010\bar{2}$†.

* This table is to be read off in lines. The first three lines set out in full (omitting the *null* terms) will mean

$$\psi(x)+\psi\left(\frac{x}{13}\right)-\psi\left(\frac{x}{14}\right)-\psi\left(\frac{x}{15}\right)+\psi\left(\frac{x}{17}\right)+\psi\left(\frac{x}{19}\right)-\psi\left(\frac{x}{21}\right)-\psi\left(\frac{x}{22}\right)$$
$$+\psi\left(\frac{x}{23}\right)-\psi\left(\frac{x}{28}\right)+\psi\left(\frac{x}{29}\right)+\psi\left(\frac{x}{31}\right)-\psi\left(\frac{x}{33}\right)-\psi\left(\frac{x}{35}\right)+\psi\left(\frac{x}{37}\right)$$
$$+\psi\left(\frac{x}{41}\right)-\psi\left(\frac{x}{42}\right)+\psi\left(\frac{x}{43}\right)-\psi\left(\frac{x}{44}\right)-\psi\left(\frac{x}{45}\right)+\psi\left(\frac{x}{47}\right)+\psi\left(\frac{x}{53}\right)$$
$$-\psi\left(\frac{x}{55}\right)-\psi\left(\frac{x}{56}\right)+\psi\left(\frac{x}{59}\right)+\psi\left(\frac{x}{61}\right)-\psi\left(\frac{x}{63}\right)-\psi\left(\frac{x}{66}\right)+\psi\left(\frac{x}{67}\right)$$
$$-\psi\left(\frac{x}{70}\right)+\psi\left(\frac{x}{71}\right)+\psi\left(\frac{x}{73}\right)-\psi\left(\frac{x}{75}\right)-\psi\left(\frac{x}{77}\right)+\psi\left(\frac{x}{79}\right)$$
$$+\psi\left(\frac{x}{83}\right)-\psi\left(\frac{x}{85}\right)-\psi\left(\frac{x}{88}\right)+\psi\left(\frac{x}{89}\right)+\psi\left(\frac{x}{97}\right)-\psi\left(\frac{x}{98}\right)-\psi\left(\frac{x}{99}\right)+\psi\left(\frac{x}{101}\right)$$
$$+\psi\left(\frac{x}{103}\right)-3\psi\left(\frac{x}{105}\right)+\psi\left(\frac{x}{107}\right)+\psi\left(\frac{x}{109}\right)-\psi\left(\frac{x}{110}\right)-\psi\left(\frac{x}{112}\right)+\psi\left(\frac{x}{113}\right).$$

† By actual summation it will be found as stated above [p. 718] that the sum reckoned from the beginning of the positive and negative integers in this table always lies between 2 and -2 (both inclusive).

If we confine our attention exclusively to the outstanding group of the Major Arrangement, which extends to the 616th term inclusive, *without taking advantage* of any of the other groups, we shall find, on making $E = 1{\cdot}0551851$, $F = {\cdot}9461974$, and N (the stigmatic multiplier) $= {\cdot}9909532$,

$$\frac{[\psi(x)]}{x} < N + \left(\frac{1}{15} + \frac{1}{22} + \frac{1}{28} + \frac{1}{35} + \frac{1}{45} + \frac{1}{56} + \frac{1}{66} + \frac{1}{77} + \frac{1}{88} + \frac{1}{99} + \frac{1}{105} + \frac{1}{126} + \frac{1}{525} + \frac{1}{616}\right) E - \left(\frac{1}{17} + \frac{1}{23} + \frac{1}{29} + \frac{1}{37} + \frac{1}{47} + \frac{1}{59} + \frac{1}{71} + \frac{1}{79} + \frac{1}{89} + \frac{1}{113} + \frac{1}{227}\right) F$$

$< 1{\cdot}0542390 \ldots$ which is inferior in value to E.

This is enough to assure us that a better result than the one last found would be obtained by using the above scheme to furnish the major as well as the minor arrangement, instead of combining it, as we have done, with the scheme [1, 6, 70; 2, 3, 5, 7, 210].

Mr Hammond has been good enough to work out for me in the annexed scholium the *complete* approximation to the limits to $\psi(x)$ given by the original scheme of Tchebycheff [1, 30; 2, 3, 5]: this approximation preserves precisely the same form as that obtained by the crude method, and, although it lies a little out of the track which I had marked out for myself in this paper, will, I think, besides being possibly valuable for future purposes in a more or less remote future, serve as an example to clear up any obscurity that may have pervaded the previous exposition of the purely asymptotic portion of these limits*.

§ 3. *Scholium. Containing an example of the complete ith approximation to the limits to the prime-log-sum-sum to x.*

Using S to denote the stigmatic series

$$\psi(x) - \psi\left(\frac{x}{6}\right) + \psi\left(\frac{x}{7}\right) - \psi\left(\frac{x}{10}\right) + \ldots,$$

we have the inequalities

$$\left.\begin{array}{l} S > Ax - \tfrac{5}{2}\log x - 1 \\ S < Ax + \tfrac{5}{2}\log x \end{array}\right\} \text{(Serret, p. 233),}$$

which, as explained in the preceding section, may be replaced by

$$\psi(x) > Ax - \tfrac{5}{2}\log x - 1 + \psi\left(\frac{x}{24}\right) - \psi\left(\frac{x}{29}\right) \qquad (1),$$

$$\psi(x) < Ax + \tfrac{5}{2}\log x + \psi\left(\frac{x}{6}\right) - \psi\left(\frac{x}{7}\right) + \psi\left(\frac{x}{10}\right) \qquad (2).$$

* In the paragraph [last but one] of p. [709] in the preceding number, a theorem (too simple to require a formal proof) is tacitly assumed which virtually amounts to saying:

If an equal number of black and white beads be strung upon a wire, in such a way that on telling them all, from left to right, more white than black ones are never told off, then the whole number of beads, as they stand, may be sorted into pairs, in each of which a black bead lies to the left of a white one.

If now we assume

$$\psi(x) > p_i Ax + q_i (\log x)^2 + r_i (\log x) + s_i \quad (3),$$

$$\psi(x) < t_i Ax + u_i (\log x)^2 + v_i (\log x) + w_i \quad (4),$$

we obtain, by combining these inequalities with (1),

$$\begin{aligned}\psi(x) > Ax &\qquad - \tfrac{5}{2}\log x \qquad -1\\ &+ \tfrac{1}{24}p_i Ax + q_i(\log x - \log 24)^2 + r_i(\log x - \log 24) + s_i\\ &- \tfrac{1}{29}t_i Ax - u_i(\log x - \log 29)^2 - v_i(\log x - \log 29) - w_i.\end{aligned}$$

Say $\qquad \psi(x) > p_{i+1}Ax + q_{i+1}(\log x)^2 + r_{i+1}(\log x) + s_{i+1},$

where

$$\begin{aligned}p_{i+1} &= \tfrac{1}{24}p_i - \tfrac{1}{29}t_i + 1,\\ q_{i+1} &= q_i - u_i,\\ r_{i+1} &= r_i - v_i + 2u_i \log 29 - 2q_i \log 24 - \tfrac{5}{2},\\ s_{i+i} &= s_i - w_i + q_i(\log 24)^2 - u_i(\log 29)^2 - r_i \log 24 + v_i \log 29 - 1.\end{aligned}$$

Similarly, combining (3) and (4) with (2), we find

$$\begin{aligned}\psi(x) < Ax &\qquad + \tfrac{5}{2}\log x\\ &+ \tfrac{1}{6}t_i Ax + u_i(\log x - \log 6)^2 + v_i(\log x - \log 6) + w_i\\ &- \tfrac{1}{7}p_i Ax - q_i(\log x - \log 7)^2 - r_i(\log x - \log 7) - s_i\\ &+ \tfrac{1}{10}t_i Ax + u_i(\log x - \log 10)^2 + v_i(\log x - \log 10) + w_i.\end{aligned}$$

Say $\qquad \psi(x) < t_{i+1}Ax + u_{i+1}(\log x)^2 + v_{i+1}(\log x) + w_{i+1},$

where

$$\begin{aligned}t_{i+1} &= \tfrac{4}{15}t_i - \tfrac{1}{7}p_i + 1,\\ u_{i+1} &= 2u_i - q_i,\\ v_{i+1} &= 2v_i - r_i + 2q_i \log 7 - 2u_i \log 60 + \tfrac{5}{2},\\ w_{i+1} &= 2w_i - s_i - q_i(\log 7)^2 + u_i\{(\log 6)^2 + (\log 10)^2\} + r_i \log 7 - v_i \log 60.\end{aligned}$$

These, together with the four given above, constitute a set of eight difference equations for the determination of p_i, q_i, r_i, s_i, t_i, u_i, v_i, w_i. Their initial values are furnished by the inequalities

$$\left.\begin{aligned}\psi(x) &> Ax - \tfrac{5}{2}\log x - 1\\ \psi(x) &< \tfrac{6}{5}Ax + \frac{5}{4\log 6}(\log x)^2 + \tfrac{5}{4}\log x + 1\end{aligned}\right\} \text{(Serret, p. 236),}$$

which give $\qquad p_0 = 1,\ q_0 = 0\ ,\ r_0 = -\tfrac{5}{2},\ s_0 = -1,$

$$t_0 = \tfrac{6}{5},\ u_0 = \frac{5}{4\log 6},\ v_0 = \tfrac{5}{4},\ w_0 = 1.$$

The values of p_i, t_i will be found to be

$$p_i = \frac{1}{50999}\left\{51072 - 36\tfrac{1}{2}(\rho^i + \rho_1^i) - 47\tfrac{211}{2320}\left(\frac{\rho^i - \rho_1^i}{\rho - \rho_1}\right)\right\},$$

$$t_i = \frac{1}{50999}\left\{59595 + 801\tfrac{9}{10}(\rho^i + \rho_1^i) + 190\tfrac{2397}{2800}\left(\frac{\rho^i - \rho_1^i}{\rho - \rho_1}\right)\right\},$$

where ρ, ρ_1 are the roots of the equation

$$(\rho - \tfrac{4}{15})(\rho - \tfrac{1}{24}) = \frac{1}{203},$$

and it is easy to verify that these values (which agree with the general ones, involving arbitrary constants, obtained in the preceding section) satisfy the initial conditions

$$p_0 = 1,\ p_1 = \tfrac{1}{24}p_0 - \tfrac{1}{29}t_0 + 1 = 1\tfrac{1}{3480},$$
$$t_0 = \tfrac{6}{5},\ t_1 = \tfrac{4}{15}t_0 - \tfrac{1}{7}p_0 + 1 = 1\tfrac{31}{175}.$$

The values of q_i and u_i, obtained from the equations

$$q_{i+1} = q_i - u_i, \quad u_{i+1} = 2u_i - q_i,$$

with the initial conditions

$$q_0 = 0,\ u_0 = \frac{5}{4 \log 6},$$

are

$$q_i = -\frac{5}{4 \log 6}\left(\frac{\alpha^i - \alpha^{-i}}{\alpha - \alpha^{-1}}\right),$$
$$u_i = \frac{5}{8 \log 6}\left(\alpha^i + \alpha^{-i} + \frac{\alpha^i - \alpha^{-i}}{\alpha - \alpha^{-1}}\right),$$

where α, α^{-1} are the roots of the equation

$$\alpha^2 - 3\alpha + 1 = 0.$$

The values of r_i, s_i, v_i, w_i are linear functions of q_i, u_i whose coefficients are linear functions of i in the case of r_i, v_i and quadratic functions of i in the case of s_i, w_i.

Thus we find, when the constants are properly determined,

$$r_i = -(2 \log 6 + \lambda i)\, u_i + \{\kappa - \lambda - 2 \log 29 + \log 6 - (\kappa + \lambda)\, i\}\, q_i,$$
$$v_i = \quad (3 \log 6 - \kappa i)\, u_i + (2 \log 10 + \lambda - 2\kappa - \lambda i)\, q_i - \tfrac{5}{2},$$

where

$$\kappa = \tfrac{2}{5} \log\left(\frac{24^3 \cdot 60^2}{7 \cdot 29}\right), \quad \lambda = \tfrac{2}{5} \log\left(\frac{24^4 \cdot 60}{7^3 \cdot 29^3}\right).$$

The substitution of these values of r_i and v_i in the equations for determining s_i and w_i, will give a pair of equations of the form

$$s_{i+1} = s_i - w_i \ + (a + bi)\, q_i + (c + di)\, u_i - (1 + \tfrac{5}{2} \log 29),$$
$$w_{i+1} = 2w_i - s_i + (e + fi)\, q_i + (g + hi)\, u_i - \tfrac{5}{2} \log 60,$$

where a, b, c, d, e, f, g, h are known constants, and q_i, u_i are known linear functions of α^i, α^{-i}.

For example, the value of a is

$$(\log 24)^2 - (\kappa - \lambda - 2 \log 29 + \log 6) \log 24 + (2 \log 10 + \lambda - 2\kappa) \log 29.$$

From these equations we should obtain a result of the form

$$s_i = Q_1\alpha^i + R_1\alpha^{-i} + C_1,$$
$$w_i = Q_2\alpha^i + R_2\alpha^{-i} + C_2,$$

in which C_1, C_2 are constants and Q_1, Q_2, R_1, R_2 quadratic functions of i, but the complete determination of these would occupy too much space to be given here.

Sequel to Part II., Chapter I. § 2.

Since § 2 of this chapter was sent to press I have had asymptotic limits to $\psi(x) \div x$ computed by means of a scheme whose stigmata contain simply and in combination all the prime numbers up to 13 inclusive. The numerical results obtained on the one hand and on the other the process employed to determine *à priori* (so as to save the labour of working out the 30030 terms of a complete period) the minimum and maximum values (-1 and 4) of the sum of the coefficients of any number of consecutive terms (the first included) in the stigmatic series proper to the scheme, appear to me too noteworthy to be consigned to oblivion.

This calculation differs from those that precede it in the circumstance that it does not attempt to give the *optimistic* limits which the scheme will afford, notwithstanding which the limits actually obtained will be found to be each of them materially closer to unity than the optimistic limits furnished by any of the preceding schemes.

The scheme I adopt is [1, 6, 10, 14, 105; 2, 3, 5, 7, 11, 13, 385, 1001], which satisfies the necessary condition that the sums of the reciprocals of the numbers on the two sides of the semicolon are equal to one another.

The first thing to be done is to discover the maximum and minimum values of

$$S_n = E\left(\frac{n}{1}\right) + E\left(\frac{n}{6}\right) + E\left(\frac{n}{10}\right) + E\left(\frac{n}{14}\right) + E\left(\frac{n}{105}\right)$$
$$- E\left(\frac{n}{2}\right) - E\left(\frac{n}{3}\right) - E\left(\frac{n}{5}\right) - E\left(\frac{n}{7}\right) - E\left(\frac{n}{11}\right)$$
$$- E\left(\frac{n}{13}\right) - E\left(\frac{n}{385}\right) - E\left(\frac{n}{1001}\right).$$

On taking n equal to 66, it will be found that the value of S_n is -1: I shall proceed to show that this is the minimum, in other words that $-S_n$ cannot be so great as 2.

Denote the fractional part of any quantity x by $F(x)$: if $-S_n$ is not less than 2, then it may be shown that *à fortiori*

$$F\left(\frac{n}{6}\right) + F\left(\frac{n}{10}\right) + F\left(\frac{n}{14}\right) - F\left(\frac{n}{2}\right) - F\left(\frac{n}{3}\right) - F\left(\frac{n}{5}\right) - F\left(\frac{n}{7}\right) + F\left(\frac{n}{105}\right),$$

say $Q(n) + F\left(\frac{n}{105}\right)$ must not be less than 2, and therefore $Q(n)$ must be greater than 1 : now it is not difficult to show that $Q(n)$ is only greater than unity when

$$n = 106 + 210\kappa \quad \text{or} \quad n = 136 + 210\kappa$$

(κ being a positive integer). But corresponding to these two values it will be found that

$$Q(106) + F\left(\frac{106}{105}\right) = \tfrac{1}{3} + \tfrac{2}{5} + \tfrac{3}{7} + \tfrac{1}{105},$$

$$Q(136) + F\left(\frac{136}{105}\right) = \tfrac{1}{3} + \tfrac{2}{5} + \tfrac{2}{7} + \tfrac{31}{105},$$

so that on either supposition $Q(n) + F\left(\frac{n}{105}\right)$ is less than 2.

Hence the minimum value of S_n is -1, and consequently, since the stigmatic excess is here $8 - 5$, the maximum value, as appears from the footnote below, will be $8 - 5 + 1$, that is 4*. (By the stigmatic excess for any scheme I mean the number of stigmata in the right-hand less the number of those in the left-hand set. This excess is obviously equal to the coefficient, with its sign changed, of $\psi\left(\frac{x}{\mu}\right)$ in the stigmatic series, where μ is any common multiple of the stigmata.)

It will be found, on summing up the numbers in Table B, that S_n first attains the value 4 when $n = 1891$, and the value -1 when $n = 66$.

For the inferior limit the outstanding group consists of all the terms up to 1891 inclusive, and for the superior limit all the terms up to 66 inclusive. But in obtaining this limit advantage has been taken of the next three groups, which end with 78, 418, and 2068 respectively. Thus the extreme limit of the following table is 2068, instead of being 30030 (that is 2.3.5.7.11.13) which is the number of terms in a complete period. It contains the coefficients of the first 2068 terms of the stigmatic series for the scheme [1, 6, 10, 14, 105; 2, 3, 5, 7, 11, 13, 385, 1001] written down in horizontal order in regular batches of ten, as was done in Table A for the

* If we call c_n the coefficient of $\psi\left(\frac{x}{n}\right)$ and S_n the sum of such coefficients up to c_n inclusive (regarding c_0 and S_0 as zero), and take μ the least common multiple of the stigmata, we have, obviously,

$$S_\mu = 0, \quad c_n = c_{\mu-n}, \quad \text{and } (S_n + S_{\mu-1-n}) - (S_{n-1} + S_{\mu-n}) = c_n - c_{\mu-n} = 0.$$

Consequently, $\quad S_n + S_{\mu-1-n} = S_0 + S_{\mu-1} = -c_\mu = \eta$ (the stigmatic excess).

This is a valuable formula of verification, and moreover gives a rule for finding either the maximum or minimum coefficient-sum when the other sum is given; for if S_n has the maximum value, $S_{\mu-1-n} = \eta - S_n$; if this is not the minimum let S_n' be less than $\eta - S_n$, then $S'_{\mu-1-n}$ will be greater than S_n, contrary to hypothesis. Hence the minimum value of a coefficient-sum may be found by subtracting the maximum from the stigmatic excess and *vice versâ*.

(I may perhaps be allowed to add that this theorem suggests a generalization of itself, which I think it is safe to anticipate may be formally deduced from it, namely :

If $a_1, a_2, \ldots, a_n$; $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ *be any given positive quantities* (*integer or fractional, rational or irrational*) *such that* $\Sigma a = \Sigma\alpha$, *and if* $-m$, M *be the least and greatest values that* $\Sigma E(ax) - \Sigma E(\alpha x)$ *can assume when* x *is any positive quantity whatever, then* $M - m = \nu - n$.)

scheme [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105] with the unimportant difference that (for typographical convenience) negative coefficients are indicated by dots instead of by bars placed over them.

Table B.

The coefficients of the first 2068 *terms of the stigmatic series to* [1, 6, 10, 14, 105; 2, 3, 5, 7, 11, 13, 385, 1001].

1000000000	00001̇01010	1̇1̇1001̇0010	101̇01̇0101̇0
1011̇1̇01000	01̇101̇00010	101̇01̇1̇1000	10101̇01̇1̇10
00100001̇10	1̇0000010̇1̇0	1011̇1̇01011̇	0010001̇000
0000001001̇	11̇001̇01010	001̇0001̇010	1001̇01̇1000
00102̇01000	00101̇1̇0010	11̇0000001̇0	10102̇011̇10
00000001̇01	1000000001̇	00101̇01010	2̇011̇000010
11̇001̇00000	10001̇01001̇	0011̇000010	102̇01̇01000
10101̇2̇0010	0010001̇000	00000011̇00	11̇101̇01000
00101̇00001̇	10000011̇00	00001̇01010	1̇1̇10001̇010
101̇1̇001000	0011̇1̇00010	00103̇00011̇	100001̇101̇0
10001̇00010	000001̇01̇11	10000000 2̇0	10101̇01011̇
1̇1̇10000010	00002̇01000	11̇101̇011̇00	0000000010
001̇1̇001000	1011̇2̇00010	001001̇1̇010	0000000001̇
10101̇011̇10	000000001̇0	100001̇1001̇	10001̇01000
1̇01̇0001̇010	12̇00001000	00002̇01010	0011̇1̇001̇10
10001̇0101̇0	00101̇1̇1010	0001̇001̇011	1000001̇1̇00
10101̇01001̇	1̇010000011̇	101̇01̇01000	00101̇1̇1000
01̇10000000	102̇0001000	11̇1̇1̇00010	00102̇00010
000001̇1̇100	10101̇00010	1̇0100001̇00	1001̇00101̇0
10001̇00012̇	0010001̇011̇	0000001000	01̇001̇01010
00001̇1̇0010	1001̇00102̇0	10102̇01010	01̇0001̇0011
10001̇01̇000	10101̇012̇10	1̇010000000	00001̇01001̇
1011̇1̇01000	1̇010001̇010	11̇1̇0001001̇	10001̇00010
0001̇000010	00001̇1̇1000	10101̇1̇1000	0010001̇000
11̇00001̇100	10002̇01000	0010001̇1̇11̇	1000001000
3̇0101̇01010	01̇11̇1̇00010	101̇000001̇0	1011̇1̇01011̇
00001̇00011	101̇001̇0000	10101̇1̇0010	1̇0100001̇00
10001̇0101̇0	11̇101̇01001̇	00101̇00010	001̇00011̇00
11̇101̇00010	1̇000000010	0002̇001000	10103̇00010
001001̇0001̇	1000000000	101̇01̇011̇10	001001̇1̇000
10000010 1̇1̇	00101̇01010	1̇1̇101̇00010	11̇001̇0101̇0
10001̇011̇10	0001̇000011	00001̇00000	1011̇1̇1̇1010
1̇010002̇010	10001̇011̇01̇	10101̇0101̇0	0000000011̇
101̇001̇1000	00101̇00000	01̇10000010	01̇1̇0001000
1011̇2̇01010	00101̇001̇00	100001̇1000	00001̇00010
0011̇001̇1̇10	100000001̇0	00101̇01012̇	00101̇00010
00100010̇1̇0	11̇101̇1̇1010	001̇0000001	1001̇000000
11̇102̇01010	1̇0101̇1̇0010	10001̇001̇00	10101̇011̇00
1̇010000000	101̇1̇001002̇	10101̇01̇010	1̇010000011̇
01̇00001000	101̇01̇01010	0011̇01̇0000	10001̇0101̇0

1000ii1010	0i10002̇010	1000i01i00	0010i01i00
0010i0001i	000000101i0	001ii01010	0i0000i011
10i000000i	101ii01010	i000i00010	1000i2̇1000
1010i000i0	001000 0i10	1ii0001 0i0	10102̇0001i
0010000i10	i000001000	0i10i01010	000i000000
100i000000	10002̇0101i	00100ii010	10i0000000
0010ii1i10	0010i000i0	10000010ii	1i10i01010
i000i00011	1i00000i00	1000i01010	2̇01i000010
100i3̇01000	1010ii0000	001000i01i	10i0001i00
1000i00000	00100i001i	0000001 0i0	0010i01010
3̇010000000	10i0i01000	100ii01i10	0010i0i010
00000i1000	001ii00010	0010i0ii	

In Tables I and II below, in addition to pairs of numbers $-(\eta)+(\eta+\theta)$, meaning

$$-\psi\left(\frac{x}{\eta}\right)+\psi\left(\frac{x}{\eta+\theta}\right),$$

and $+(\eta)-(\eta+\theta)$ meaning

$$+\psi\left(\frac{x}{\eta}\right)-\psi\left(\frac{x}{\eta+\theta}\right),$$

there will be found the unpaired numbers (15) and (66) in the one and (19), (229) and (1891) in the other; to understand how these are got, it should be observed that S_n (the sum of the first n numbers in Table B) first becomes 0 when $n=15$, first becomes -1 when $n=66$ and first becomes 2, 3, 4 when $n=19$, 229, 1891 respectively*.

TABLE I.

			+(15)
−	(17)	+ (22)	
−	(19)	+ (21)	
−	(23)	+ (26)	
−	(29)	+ (35)	
−	(41)	+ (45)	
−	(47)	+ (52)	
−	(59)	+ (65)	+(66)
−	(67)	+ (78)	
−	(79)	+ (418)	
−	(107)	+ (135)	
−	(210)	+ (275)	
−	(289)	+ (385)	
−	(419)	+ (2068)	
−	(521)	+ (585)	
−	(629)	+ (795)	
−	(839)	+ (936)	
−	(1049)	+ (1144)	
−	(1717)	+ (1925)	

TABLE II.

+	(15)	− (17)	− (19)
+	(21)	− (31)	
+	(26)	− (29)	
+	(33)	− (43)	
+	(44)	− (61)	
+	(63)	− (73)	
+	(65)	− (71)	
+	(75)	− (103)	− (229)
+	(242)	− (271)	
+	(285)	− (323)	
+	(385)	− (421)	
+	(385)	− (439)	
+	(440)	− (493)	
+	(494)	− (571)	
+	(770)	− (841)	
+	(1155)	− (1273)	− (1891)

* Call Σ the sum of the infinite series given by Table B: it may then easily be verified that

$$\{\psi(x)-\Sigma\}-\left\{\psi\left(\frac{x}{15}\right)+\psi\left(\frac{x}{66}\right)\right\}$$

The reasoning employed in dealing with previous schemes serves to show that superior and inferior asymptotic limits to $\psi(x) \div x$, which we shall call E_1, F_1 in order to distinguish them from the corresponding optimistic limits (E, F), may be found from the equations

$$\left.\begin{aligned} E_1 &= M + aE_1 - bF_1 \\ F_1 &= M + cF_1 - dE_1 \end{aligned}\right\},$$

where a is the sum of the reciprocals of the numbers occurring in Table I with the sign +

b " " " " " –

c " " " in Table II " +

d " " " " " –

and M is the stigmatic multiplier,

namely $$a = \frac{1}{15} + \frac{1}{21} + \frac{1}{22} + \ldots + \frac{1}{2068} = \cdot 33352 \ldots,$$

$$b = \frac{1}{17} + \frac{1}{19} + \frac{1}{23} + \ldots + \frac{1}{1717} = \cdot 30580 \ldots,$$

$$c = \frac{1}{15} + \frac{1}{21} + \frac{1}{26} + \ldots + \frac{1}{1155} = \cdot 26966 \ldots,$$

$$d = \frac{1}{17} + \frac{1}{19} + \frac{1}{29} + \ldots + \frac{1}{1891} = \cdot 27742 \ldots \text{*},$$

---

may be resolved into term-pairs of the form

$$-\psi\left(\frac{x}{\eta}\right) + \psi\left(\frac{x}{\eta+\theta}\right)$$

that shall contain among them all those in Table I, and

$$\{\psi(x) - \Sigma\} + \left\{\psi\left(\frac{x}{19}\right) + \psi\left(\frac{x}{229}\right) + \psi\left(\frac{x}{1891}\right)\right\}$$

into term-pairs of the form $+\psi\left(\frac{x}{\eta}\right) - \psi\left(\frac{x}{\eta+\theta}\right)$ that shall contain among them all those in Table II above.

The maximum value of S_n is here 4: if it had been 2, then instead of 3 unpaired positive terms appended to $\{\psi(x) - \Sigma\}$ there would have been but 1. This is what happens for the scheme [1, 15; 2, 3, 5, 30] given in the footnote on p. [710]: and accordingly, we see that $\{\psi(x) - \Sigma\} + \psi\left(\frac{x}{17}\right)$, for that scheme, is resoluble into paired terms of the form

$$+\psi\left(\frac{x}{\eta}\right) - \psi\left(\frac{x}{\eta+\theta}\right).$$

So again, the minimum being 0 (instead of -1), there will be but 1 unpaired negative term to append to $\{\psi(x) - \Sigma\}$, and accordingly, we see that $\{\psi(x) - \Sigma\} - \psi\left(\frac{x}{6}\right)$ in that scheme is resoluble into term-pairs of the form $-\psi\left(\frac{x}{\eta}\right) + \psi\left(\frac{x}{\eta+\theta}\right)$.

* The above values of a, b, c, d give $a + c = \cdot 603 \ldots$ and $ac - bd = \cdot 005 \ldots$, and consequently the roots of the "characteristic" equation $\rho^2 - (a+c)\rho + (ac - bd) = 0$ satisfy the necessary condition of being each less than unity in absolute value.

and $$M = \frac{1}{2}\log 2 + \frac{1}{3}\log 3 + \frac{1}{5}\log 5 + \frac{1}{7}\log 7 + \frac{1}{11}\log 11 + \frac{1}{13}\log 13 + \frac{1}{385}\log 385 + \frac{1}{1001}\log 1001 - \frac{1}{6}\log 6 - \frac{1}{10}\log 10 - \frac{1}{14}\log 14 - \frac{1}{105}\log 105 = \cdot 98859 \ldots.$$

Hence $$E_1 = \frac{(1-c-b)M}{(1-a)(1-c)-bd} = 1\cdot 04423 \ldots,$$

$$F_1 = \frac{(1-a-d)M}{(1-a)(1-c)-db} = \cdot 95695 \ldots,$$

(so that the mean of E_1 and F_1 is less than ·0006), and $\frac{E_1}{F_1} = 1\cdot 09120 \ldots$ *.

Thus then (see footnote to p. [694]) by taking x sufficiently great, the number of primes not exceeding x, multiplied by $\log x$ and divided by x, may always be made to lie between the numbers

1·04423 ... and ·95695 ...,

the divergences of which from unity are

·04423 ... and ·04304 ... (as against

Tchebycheff's ·10555 ... and ·07807 ...).

These divergences, there is little doubt, would become even more nearly equal than they are, if anyone should feel inclined to undertake the very laborious task of extracting the *optimistic* values (E, F) from the scheme employed.

In order to understand this necessarily abbreviated sketch of a method more easy to think out and apply than to find language to express, I must not conceal that a careful study of the several schemes given, and of the principles embodied in the calculations relating to them, is a *sine quâ non.* It may somewhat lighten the burden thrown upon the reader, if I add a few words concerning one or two points, perhaps inadequately explained in what precedes.

Let μ be the least common multiple of the stigmata of any given harmonic scheme and S_n the sum of the coefficients of

$$\psi(x),\ \psi\left(\frac{x}{2}\right),\ \psi\left(\frac{x}{3}\right), \ldots \psi\left(\frac{x}{n}\right)$$

* In Tables I and II above, the ratio $\frac{\eta+\theta}{\eta}$ is greater than 1·09120 ... for every pair of terms, except $-(1049)+(1144)$ in Table I. In the case of this pair, we have $\frac{1144}{1049} = 1\cdot 0905 \ldots$, which shows that the exclusion of it from that table would have led to asymptotic limits better (but very slightly so) than those arrived at in the text.

in the corresponding stigmatic series. Then from the formula of [p. 710] combined with the equation which connects the stigmata, it follows that

$$S_\mu = 0, \quad S_{n+\mu} = S_n.$$

Hence an infinite number of values of n will give S_n its greatest value; the difference of these values will be of the form $k\mu - \mu'$ where μ' may, and in general will, besides zero have various other values less than μ, thus giving rise to the collections of terms called *groups* (see p. [709]) of which the period of μ terms will be composed. The same will be true when we substitute the word *least* for *greatest*.

If now i be taken *any* number such that S_i has its greatest value it may be shown that the sum of all the terms in the stigmatic series subsequent to the one containing $\psi\left(\frac{x}{i}\right)$ will be *negative* or zero, and similarly when S_i has its least value such sum will be *positive* or zero*; consequently when i is properly determined we can find immediately a superior limit in the one case and an inferior limit in the other, to the sum of the first i terms of the series.

I will conclude this portion of the subject with the remark that from the values of E_1 and F_1 it is easy to infer that if μ is equal to or less than $(\cdot 95695\ldots)k - (1\cdot 04423\ldots)$, and x exceeds a certain ascertainable number whose value depends on k and μ, then between x and kx there will be found more than $\mu \frac{x}{\log x}$ primes†.

* The reason of this is that the sum of all the terms beyond the ith may be separated into partial sums, each containing μ terms, which ultimately vanish. If now

$$\gamma_1 (k\mu + i + 1) + \gamma_2 (k\mu + i + 2) + \ldots + \gamma_\mu (k\mu + i + \mu)$$

be one of them, then $\gamma_1 + \gamma_2 + \ldots + \gamma_t$ will be zero when $t = \mu$, and will have a constant algebraical sign (or else be zero) when $t < \mu$; from which it follows (see footnote p. [722] where, be it observed, a coefficient $+\lambda$ or $-\lambda$ is supposed to be represented by a *sequence* of λ black or λ white beads) that each partial sum may be decomposed into an aggregate of quantities of the form $+(\eta) - (\eta + \theta)$ or $-(\eta) + (\eta + \theta)$ according as the first coefficient in each such sum is positive or negative, and will therefore, if not zero, have the same algebraical sign as that coefficient has, namely $-$ or $+$ according as S_i has its greatest or least value.

† In order that μ may be positive (which ensures the existence of *some* primes between x and kx, when x exceeds a certain limit) it is only necessary to take $k > 1\cdot 09120\ldots$ (which differs very little from $\frac{12}{11}$), whereas if we limited ourselves to the results of the oft-quoted memoir of 1850 [see p. 704, above], we could not prove the existence of prime numbers between x and kx, for a given value of x, however great, unless k exceeds $\frac{6}{5}$.