61.

NOTE ON CERTAIN DIFFERENCE EQUATIONS WHICH POSSESS AN UNIQUE INTEGRAL.

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FOR greater simplicity suppose in what follows that a difference equation is expressed in terms of the arguments

$$u_x,\ u_{x+1},\ \dots\ u_{x+i}.$$

I shall call u_{x+i} the highest and u_x the lowest argument respectively, or collectively the extreme or principal arguments, and the degrees in which they enter into the equation the upper and lower or extreme or principal degrees. It is these partial degrees rather than the total degree of the entire equation which determine the essential character of the solution.

If m is the upper degree and $u_0, u_1, \dots u_{i-1}$ be given it is obvious that for any value of x higher than $(i-1)$, u_x will have m^{x-i+1} values, and consequently in general there will be an infinite number of integrals whether complete or of a given order of deficiency (the deficiency being estimated by the number of relations connecting the initial values $u_0, u_1, \dots u_{i-1}$); but it may be, and is in some cases, possible to assign an integral which shall have m^{x-i+1} values, and in such case there can exist no other; such an integral may be called an unique or exhaustive one, and the equations which possess such integrals may be termed uni-solutional.

As the simplest example of such, suppose

$$u^m{}_{x+1} - u_x{}^n = 0,$$

where m and n are integers.

If we write $$u_x = \alpha^{\left(\frac{n}{m}\right)^x}$$

we have $$u_{x+1} = \left(\alpha^{\left(\frac{n}{m}\right)^x}\right)^{\frac{n}{m}}$$

or $$u^m{}_{x+1} = u_x{}^n.$$

Here $u_x = \alpha^{\left(\frac{n}{m}\right)^x}$ is the one and sole complete integral of the equation; for it possesses m^x values so that there can be no other integrals whatever.

Let us now seek to form difference uni-solutional equations of the 2nd order.

To this end let $u_x = C(\alpha^{2^x} - \beta^{2^x})$, where $\alpha\beta = 1$.

Then calling $\alpha^{2^x} = P$ and $\beta^{2^x} = Q$, $PQ = 1$,

$$u_x = C(P - Q),$$
$$u_{x+1} = C(P^2 - Q^2),$$
$$u_{x+2} = C(P^4 - Q^4).$$

Hence $$\frac{u_{x+1}}{u_x} = P + Q,\quad \frac{u_{x+2}}{u_{x+1}} = P^2 + Q^2 = (P+Q)^2 - 2,$$

and $$\frac{u_{x+2}}{u_{x+1}} = \left(\frac{u_{x+1}}{u_x}\right)^2 - 2.$$

Hence the equation

$$u_x{}^2 u_{x+2} - u^3{}_{x+1} + 2u_x{}^2 u_{x+1} = 0$$

has for its complete integral $u_x = C(\alpha^{2^x} - \alpha^{-2^x})$, and there can be no other because when u_0, u_1 are given u_x is absolutely determined.

But furthermore we may invert the above equation by interchanging u_x and u_{x+2}, which gives the equation

$$(u_x + 2u_{x+1})\, u^2{}_{x+2} - u^3{}_{x+1} = 0,$$

of which the solution will obviously be $u_x = C\left(P - \frac{1}{P}\right)$, where $P = \alpha^{(\frac{1}{2})^x}$.

Suppose u_0, u_1 to be given; then

$$C\left(\alpha - \frac{1}{\alpha}\right) = u_0,\quad C\left(\alpha^{\frac{1}{2}} - \frac{1}{\alpha^{\frac{1}{2}}}\right) = u_1,$$

and calling $$\frac{u_0}{u_1} = 2r,\quad \alpha^{\frac{1}{2}} + \frac{1}{\alpha^{\frac{1}{2}}} = 2r,\quad \alpha^{\frac{1}{2}} - \frac{1}{\alpha^{\frac{1}{2}}} = 2\sqrt{(r^2 - 1)},$$

$$C = \frac{u_0}{4r\sqrt{(r^2-1)}}.$$

Hence

$$u_x = \frac{u_0}{4r\sqrt{(r^2-1)}}\left[\{r + \sqrt{(r^2-1)}\}^{(\frac{1}{2})^{x-1}} - \{r - \sqrt{(r^2-1)}\}^{(\frac{1}{2})^{x-1}}\right],$$

has exactly 2^{x-1} values, for the change of $\sqrt{(r^2-1)}$ into $-\sqrt{(r^2-1)}$ changes simultaneously the signs of the numerator and denominator of this fraction. But by the general principle u_x ought to have 2^{x-1} values in terms of u_0, u_1. Hence the above integral is *exhaustive*.

Suppose now we were to write

$$u_x = C(\alpha^{2^x} + \beta^{2^x}) \text{ with } \alpha\beta = 1;$$

for brevity sake call $u_x = f$, $u_{x+1} = g$, $u_{x+2} = h$, then

$$C(P+Q) = f,$$
$$C(P^2+Q^2) = g,$$
$$C(P^4+Q^4) = h,$$
$$PQ = 1.$$

Hence
$$f^2 = Cg + 2C^2,$$
$$g^2 = Ch + 2C^2,$$
$$C = \frac{f^2-g^2}{g-h},$$
$$f^2 = \frac{f^2-g^2}{g-h} \cdot \frac{2f^2-g^2-gh}{g-h},$$

or
$$f^2g^2 - 2f^2gh + f^2h^2 = 2f^4 - 3f^2g^2 + g^4 - f^2gh + g^3h,$$

or
$$f^2h^2 - (g^3+f^2g)\,h - g^4 + 4f^2g^2 - 2f^4 = 0,$$

or
$$u_x{}^2 u^2_{x+2} - (u^3_{x+1} + u_x{}^2 u_{x+1})\, u_{x+2} - u^4_{x+1} + 4u_x{}^2 u^2_{x+1} - 2u_x{}^4 = 0,$$

of which the correlative equation is

$$-2u^4_{x+2} + (4u^2_{x+1} - u_{x+1}u_x + u_x{}^2)\, u^2_{x+2} - u^3_{x+1}u_x - u^4_{x+1} = 0.$$

A complete solution of the former of these will therefore be

$$u_x = C(\alpha^{2^x} + \beta^{2^x}),$$

and of the latter
$$u_x = C(\alpha^{(\frac{1}{2})^x} + \beta^{(\frac{1}{2})^x}),$$

but neither of these will be an *exhaustive* solution, for in the one the most general value of u_x ought to be a 2^{x-1}-valued function and in the latter a 4^{x-1}-valued function, whereas the actual value is only one-valued in the one case and 2^{x-1}-valued in the other.

Suppose again we write

$$u_x = C(\alpha^{3^x} - \beta^{3^x}), \text{ where } \alpha\beta = 1, \text{ as before,}$$

say
$$u_x = C(P-Q), \text{ where } PQ = 1.$$

Then with the same notation as before

$$C(P-Q) = f,$$
$$C(P^3-Q^3) = g,$$
$$C(P^9-Q^9) = h,$$
$$\frac{g}{f} - 1 = P^2 + Q^2, \quad \frac{h}{g} - 1 = P^6 + Q^6,$$
$$\frac{h}{g} - 1 = \left(\frac{g}{f} - 1\right)^3 - 3\left(\frac{g}{f} - 1\right),$$

or
$$\frac{h}{g} = \left(\frac{g}{f}\right)^3 - 3\left(\frac{g}{f}\right)^2 + 3,$$
$$f^3h - 3f^3g = g^4 - 3g^3f,$$
$$\frac{h-3g}{g-3f} = \frac{g^3}{f^3}.$$

Whence it follows that the integrals of

$$\frac{u_{x+2}-3u_{x+1}}{u_{x+1}-3u_x}-\frac{u^3_{x+1}}{u_x{}^3}=0,$$

and of

$$\frac{u^3_{x+2}}{u^3_{x+1}}-\frac{3u_{x+2}-u_{x+1}}{3u_{x+1}-u_x}=0,$$

are respectively $u_x=C(\alpha^{3^x}-\alpha^{-3^x})$,

and $u_x=C(\alpha^{(\frac{1}{3})^x}-\alpha^{-(\frac{1}{3})^x})$,

with the understanding that $\alpha^{-\frac{1}{3}}.\alpha^{\frac{1}{3}}=1$.

These integrals are evidently *exhaustive.*

By writing $\sqrt{(-1)}\,\alpha$, $-\sqrt{(-1)}\,\alpha^{-1}$ for α, α^{-1} respectively, f, g, h become increased in the ratio of $\sqrt{(-1)}$, $-\sqrt{(-1)}$, $\sqrt{(-1)}$, respectively.

Hence the equations

$$\frac{u_{x+2}+3u_{x+1}}{u_{x+1}+3u_x}-\frac{u^3_{x+1}}{u_x{}^3}=0,$$

and

$$\frac{u^3_{x+2}}{u^3_{x+1}}-\frac{3u_{x+2}+u_{x+1}}{3u_{x+1}+u_x}=0,$$

have for their solutions

$$u_x=C(\alpha^{3^x}+\alpha^{-3^x}) \text{ and } u_x=C(\alpha^{(\frac{1}{3})^x}+\alpha^{-(\frac{1}{3})^x}).$$

Hitherto we have been dealing with *homogeneous* uni-solutional equations. It is easy, however, to form non-homogeneous ones by an obvious process. For, if we write

$$u_x=a_1{}^{m^x}+a_2{}^{m^x}+\ldots+a_i{}^{m^x} \ (m \text{ being an integer}),$$

by eliminating between

$$f_0=\Sigma a,\ f_1=\Sigma a^m,\ f_2=\Sigma a^{m^2},\ \ldots f_i=\Sigma a^{m^i},$$

we shall obtain a relation between the f's of the first degree in f_i and of the degree m^i in f_0, corresponding to which there will be a difference equation of the ith order in which the upper extreme degree is unity and the lower one m^i, of which the integral will be the value of u_x above written, and by interchanging $u_x, u_{x+1}, \ldots u_{x+i}$ respectively with $u_{x+i}, u_{x+i-1}, \ldots u_x$, another in which the lower degree is unity and the upper one m^i, of which the integral will be

$$u_x=a_1{}^{\left(\frac{1}{m}\right)^x}+a_2{}^{\left(\frac{1}{m}\right)^x}+\ldots+a_i{}^{\left(\frac{1}{m}\right)^x},$$

each of which equations will evidently be uni-solutional.

Or, again, if instead of the a's being independent we make their product equal to unity we shall obtain uni-solutional equations of the $(i-1)$th instead of the ith order.

Thus, for example, let

$$u_x=a^{2^x}+b^{2^x}+c^{2^x} \text{ with the condition } abc=1.$$

Then writing $u_x = f$, $u_{x+1} = g$, $u_{x+2} = h$,

$$f = A + B + C,\ g = A^2 + B^2 + C^2,\ h = A^4 + B^4 + C^4,$$

$$f^2 - g = 2(AB + AC + BC),$$

$$2(g^2 - h) = 4(A^2B^2 + A^2C^2 + B^2C^2)$$

$$= (f^2 - g)^2 - 8f.$$

Hence we obtain the uni-solutional equations

$$2u_{x+2} - u^2_{x+1} - 2u_{x+1}u_x^2 + u_x^4 - 8u_x = 0,$$

$$u^4_{x+2} - 2u_{x+1}u^2_{x+2} - 8u_{x+2} - u^2_{x+1} + 2u_x = 0,$$

of which the integrals are known and are exhaustive.

We may in a similar manner obtain uni-solutional *simultaneous* difference equations.

Thus let $\qquad u_x = C(\alpha^{3^x} - \beta^{3^x}),\ v_x = C'(\alpha^{3^x} + \beta^{3^x}),$

and call $\qquad u_x,\ u_{x+1},\ u_{x+2}$ as before f, g, h,

and $\qquad v_x,\ v_{x+1},\ v_{x+2} \qquad l, m, n.$

Then
$$\frac{g}{f} = P^2 + PQ + Q^2,\quad \frac{m}{l} = P^2 - PQ + Q^2,$$

$$\frac{h}{g} = P^6 + P^3Q^3 + Q^6,\ \frac{n}{m} = P^6 - P^3Q^3 + Q^6.$$

Hence
$$\frac{h}{g} - \frac{n}{m} = \tfrac{1}{4}\left(\frac{g}{f} - \frac{m}{l}\right)^3,$$

$$\frac{h}{g} + \frac{n}{m} = 2(P^6 + Q^6)$$

$$= 2(P^2 + Q^2)(P^4 - P^2Q^2 + Q^4)$$

$$= 2(P^2 + Q^2)\{(P^2 + Q^2)^2 - 3P^2Q^2\}$$

$$= \tfrac{1}{4}\left(\frac{g}{f} + \frac{m}{l}\right)\left\{\left(\frac{g}{f} + \frac{m}{l}\right)^2 - 3\left(\frac{g}{f} - \frac{m}{l}\right)^2\right\}$$

$$= -\tfrac{1}{4}\left(\frac{g}{f} + \frac{m}{l}\right)\left(2\frac{g^2}{f^2} - 8\frac{g}{f}\cdot\frac{m}{l} + 2\frac{m^2}{l^2}\right).$$

Hence
$$\frac{h}{g} = \tfrac{1}{8}\left(-\frac{g^3}{f^3} + 3\frac{g^2}{f^2}\cdot\frac{m}{l} + 9\frac{g}{f}\cdot\frac{m^2}{l^2} - 3\frac{m^3}{l^3}\right),$$

$$\frac{n}{m} = \tfrac{1}{8}\left(-3\frac{g^3}{f^3} + 9\frac{g^2}{f^2}\cdot\frac{m}{l} + 3\frac{g}{f}\cdot\frac{m^2}{l^2} - \frac{m^3}{l^3}\right).$$

Obviously, when u_0, u_1; v_0, v_1 are given, each u_x and v_x deduced from the above system of equations has only one value, so that their exhaustive integrals will be

$$u_x = C(\alpha^{3^x} - \beta^{3^x}),\ v_x = C'(\alpha^{3^x} + \beta^{3^x}).$$

The related system found by interchanging f with h and l with n will be

$$\frac{f}{g}=\tfrac{1}{8}\left(-\frac{g^3}{h^3}+3\frac{g^2}{h^2}\cdot\frac{m}{n}+9\frac{g}{h}\cdot\frac{m^2}{n^2}-3\frac{m^3}{n^3}\right),$$

$$\frac{l}{m}=\tfrac{1}{8}\left(-3\frac{g^3}{h^3}+9\frac{g^2}{h^2}\cdot\frac{m}{n}+3\frac{g}{h}\cdot\frac{m^2}{n^2}-\frac{m^3}{n^3}\right).$$

When f, g; l, m are given the system $\frac{1}{h}, \frac{1}{n}$ may be found by solving an equation of the 9th degree. Hence, when u_0, u_1; v_0, v_1 are given, u_2, v_2 will have 9; u_3, v_3, 81, and in general u_x, v_x will have $3^{2(x-1)}$ values which will correspond to the $3^{x-1}.3^{x-1}$ values of u_x, v_x.

The apparent number of values of each of these is $(3^x)^2$, which, however, must be reducible to $3^{x-1}.3^{x-1}$ when expressed in terms of the two initial values of u and of v, similarly to what was noticed at the outset on the reduction of the apparent multiplicity 2^x to a multiplicity 2^{x-1}.

In fact, we write

$$u_x=C\left(\alpha^{(\frac{1}{3})^x}-\beta^{(\frac{1}{3})^x}\right),\quad v_x=C'\left(\alpha^{(\frac{1}{3})^x}+\beta^{(\frac{1}{3})^x}\right),$$

$$u_0=C(\alpha-\beta),\ u_1=C(\alpha^{\frac{1}{3}}-\beta^{\frac{1}{3}});\ v_0=C'(\alpha+\beta),\ v_1=C'(\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}}),$$

$$\alpha^{\frac{2}{3}}+\alpha^{\frac{1}{3}}\beta^{\frac{1}{3}}+\beta^{\frac{2}{3}}=\frac{u_0}{u_1},\quad \alpha^{\frac{2}{3}}-\alpha^{\frac{1}{3}}\beta^{\frac{1}{3}}+\beta^{\frac{2}{3}}=\frac{v_0}{v_1},$$

$$\alpha^{\frac{1}{3}}\beta^{\frac{1}{3}}=\tfrac{1}{2}\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right),$$

$$\alpha^{\frac{2}{3}}+\beta^{\frac{2}{3}}=\tfrac{1}{2}\left(\frac{u_0}{u_1}+\frac{v_0}{v_1}\right),$$

$$\alpha^{\frac{1}{3}}-\beta^{\frac{1}{3}}=\sqrt{\left\{\tfrac{1}{2}\left(\frac{3v_0}{v_1}-\frac{u_0}{u_1}\right)\right\}},$$

$$\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}}=\sqrt{\left\{\tfrac{1}{2}\left(\frac{3u_0}{u_1}-\frac{v_0}{v_1}\right)\right\}},$$

$$C=\frac{u_1}{\sqrt{\left\{\tfrac{1}{2}\left(\frac{3v_0}{v_1}-\frac{u_0}{u_1}\right)\right\}}},\quad C'=\frac{v_1}{\sqrt{\left\{\tfrac{1}{2}\left(\frac{3u_0}{u_1}-\frac{v_0}{v_1}\right)\right\}}},$$

$$\alpha^{\frac{1}{3}}=\sqrt{\left\{\tfrac{1}{8}\left(\frac{3u_0}{u_1}-\frac{v_0}{v_1}\right)\right\}}+\sqrt{\left\{\tfrac{1}{8}\left(\frac{3v_0}{v_1}-\frac{u_0}{u_1}\right)\right\}},$$

$$\beta^{\frac{1}{3}}=\sqrt{\left\{\tfrac{1}{8}\left(\frac{3u_0}{u_1}-\frac{v_0}{v_1}\right)\right\}}-\sqrt{\left\{\tfrac{1}{8}\left(\frac{3v_0}{v_1}-\frac{u_0}{u_1}\right)\right\}},$$

and thus for the final values of u_x and v_x, we find

$$u_x = \frac{u_1}{\sqrt{\left\{\frac{1}{2}\left(\frac{3v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}}} \times \left\{\left[\sqrt{\left\{\frac{1}{8}\left(\frac{3u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}} + \sqrt{\left\{\frac{1}{8}\left(\frac{3v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}}\right]^{(\frac{1}{3})^{x-1}} - \left[\sqrt{\left\{\frac{1}{8}\left(\frac{3u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}} - \sqrt{\left\{\frac{1}{8}\left(\frac{3v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}}\right]^{(\frac{1}{3})^{x-1}}\right\},$$

$$v_x = \frac{v_1}{\sqrt{\left\{\frac{1}{2}\left(\frac{3u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}}} \times \left\{\left[\sqrt{\left\{\frac{1}{8}\left(\frac{3u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}} + \sqrt{\left\{\frac{1}{8}\left(\frac{3v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}}\right]^{(\frac{1}{3})^{x-1}} + \left[\sqrt{\left\{\frac{1}{8}\left(\frac{3u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}} - \sqrt{\left\{\frac{1}{8}\left(\frac{3v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}}\right]^{(\frac{1}{3})^{x-1}}\right\},$$

each of which is unaffected by a change in the signs of the square roots, so that u_x and v_x are seen to be 3^{x-1}-valued functions, and (u_x, v_x) a 9^{x-1}-valued system, as should be the case for an exhaustive solution of the last written difference equations.

Let us tentatively go a step further in the same direction and suppose that we are given

$$u_x = C(\alpha^{5^x} - \beta^{5^x}), \quad v_x = C'(\alpha^{5^x} + \beta^{5^x}),$$

and use $f, g, h; l, m, n$ in the same way as before, and furthermore, write

$$\tfrac{1}{2}\left(\frac{g}{f} + \frac{m}{l}\right) = L, \quad \tfrac{1}{2}\left(\frac{h}{g} + \frac{n}{m}\right) = N,$$

$$\tfrac{1}{2}\left(\frac{g}{f} - \frac{m}{l}\right) = M, \quad \tfrac{1}{2}\left(\frac{h}{g} - \frac{n}{m}\right) = P,$$

we shall find

$$L = A^4 + A^2B^2 + B^4, \quad N = A^{20} + A^{10}B^{10} + B^{20},$$
$$M = AB(A^2 + B^2), \quad P = A^5B^5(A^{10} + B^{10}),$$

(where $A = \alpha^{5^x}$ and $B = \beta^{5^x}$).

Let $$A^2 + B^2 = \lambda, \quad AB = \mu.$$

Then $$L = \lambda^2 - \mu^2, \quad M = \lambda\mu,$$

and it will be seen that

$$N = (\lambda^5 - 5\lambda^3\mu^2 + 5\lambda\mu^4)^2 - \mu^{10},$$
$$P = \lambda^5\mu^5 - 5\lambda\mu^7(\lambda^2 - \mu^2).$$

For $$A^6 + B^6 = \lambda^3 - 3\lambda\mu^2,$$

and consequently

$$\lambda^5 = A^{10} + B^{10} + 5A^2B^2(A^6 + B^6) + 10A^4B^4(A^2 + B^2)$$
$$= A^{10} + B^{10} + 5\mu^2(\lambda^3 - 3\lambda\mu^2) + 10\lambda\mu^4,$$

that is $$A^{10} + B^{10} = \lambda^5 - 5\lambda^3\mu^2 + 5\lambda\mu^4.$$

The above values of N and P (remembering that $AB = \mu$) are found by substituting the expression just obtained for $A^{10} + B^{10}$ in

$$N = A^{20} + A^{10}B^{10} + B^{20},$$
$$P = A^5B^5(A^{10} + B^{10}).$$

From
$$P = \lambda^5\mu^5 - 5\lambda\mu^7(\lambda^2 - \mu^2),$$

(remembering that $\lambda\mu = M$, $\lambda^2 - \mu^2 = L$), we obtain

$$P = M^5 - 5LM\mu^6.$$

Hence
$$\left.\begin{aligned} \frac{M^5 - P}{5LM^4} &= \frac{\mu^3}{\lambda^3} \\ \frac{L}{M} &= \frac{\lambda}{\mu} - \frac{\mu}{\lambda} \end{aligned}\right\}.$$

From these equations we obtain by elimination

$$\left(\frac{P}{M}\right)^2 + (2M^4 + 15L^2M^2 + 5L^4)\frac{P}{M} + M^4(M^4 - 10L^2M^2 + 5L^4) = 0. \quad (1)$$

Similarly by an elimination into the details of which it is unnecessary to enter we obtain

$$3LMP + (L^2 + M^2)N = L(L^2 - 2M^2)(L^2 - M^2)^2, \quad (2)$$

which gives a *linear* relation between N and P.

Equations (1) and (2) form a non-uni-solutional system of which (as also of its inverse) we are in possession of one complete integral, and I have some grounds for suspecting that it may be possible to obtain from this a second (so-called *indirect*) integral, but am unable for the present to pursue the subject further.

The preceding investigation originated in my attention happening to be called to Vieta's well known theorem for approximating to the *Archimedean* constant (π) by means of an indefinite product of cosines of continually bisected angles. The implied connection of ideas will become apparent when one considers that any one of such cosines may be expressed as a sum of two binary exponentials with $\frac{1}{2}$ for the first index, and that thus Vieta's theorem (although presumably obtained by him as a very simple consequence of the method of exhaustions) in its essence depends on the integrability of a uni-solutional difference equation of the 2nd order of the form treated of at the outset of this paper.