53.

NOTE ON A PROPOSED ADDITION TO THE VOCABULARY OF ORDINARY ARITHMETIC*.

[*Nature,* XXXVII. (1888), pp. 152, 153.]

THE total number of distinct primes which divide a given number I call its Manifoldness or Multiplicity.

A number whose Manifoldness is n I call an n-fold number. It may also be called an n-ary number, and for $n=1, 2, 3, 4, \ldots$ a unitary (or primary), a binary, a ternary, a quaternary, ... number. Its prime divisors I call the *elements* of a number; the highest powers of these elements which divide a number its *components*; the degrees of these powers its *indices*; so that the indices of a number are the totality of the indices of its several components. Thus, we may say, a prime is a one-fold number whose index is unity.

So, too, we may say that all the components but one of an odd perfect number must have even indices, and that the excepted one must have its base and index each of them congruous to 1 to modulus 4.

Again, a remarkable theorem of Euler, contained in a memoir relating to the Divisors of Numbers (*Opuscula Minora,* II. p. 514), may be expressed by saying that *every even perfect number is a two-fold number, one of whose* components *is a prime, and such that when augmented by unity it becomes a power of* 2, *and double the other component*†.

* Perhaps I may without immodesty lay claim to the appellation of the Mathematical Adam, as I believe that I have given more names (passed into general circulation) to the creatures of the mathematical reason than all the other mathematicians of the age combined.

† It may be well to recall that a perfect number is one which is the half of the sum of its divisors. The converse of the theorem in the text, namely that $2^n(2^{n+1}-1)$, when $2^{n+1}-1$ is a prime, is a perfect number, is enunciated and proved by Euclid in the 36th (the last) proposition of the 9th Book of the "Elements," the second factor being expressed by him as the sum of a geometric series whose first term is unity and the common ratio 2. In Isaac Barrow's English translation, published in 1660, the enunciation is as follows: "If from a unite be taken how many numbers soever 1, A, B, C, D, in double proportion continually, until the whole

Euler's function $\phi(n)$, which means the number of numbers not exceeding n and prime to it, I call the *totient* of n; and in the new nomenclature we may enunciate that *the totient of a number is equal to the product of that number multiplied by the several excesses of unity above the reciprocals of its elements.* The numbers prime to a number and less than it, I call its *totitives.*

Thus we may express Wilson's generalized theorem by saying that any number is contained as a factor in the product of its totitives increased by unity if it is the number 4, or a prime, or the double of a prime, and diminished by unity in every other case.

I am in the habit of representing the totient of n by the symbol τn, τ (taken from the initial of the word it denotes) being a less hackneyed letter than Euler's ϕ, which has no claim to preference over any other letter of the Greek alphabet, but rather the reverse.

It is easy to prove that the half of any perfect number must exceed in magnitude its totient.

Hence, since $\frac{3}{2} \cdot \frac{5}{4}$ is less than 2, it follows that no odd two-fold perfect number exists.

---

added together E be a prime number; and if this whole E multiplying the last produce a number F, that which is produced F shall be a perfect number."

The direct theorem that every even perfect number is of the above form could probably only have been proved with extreme difficulty, if at all, by the resources of Greek Arithmetic. Euler's proof is not very easy to follow in his own words, but is substantially as follows :

Suppose P (an even perfect number) $=2^n A$. Then, using in general $\int X$ to denote the sum of the divisors of X,

$$2=\frac{\int P}{P}=\frac{\int 2^n \cdot \int A}{2^n A}=\frac{2^{n+1}-1}{2^n} \cdot \frac{\int A}{A}.$$

Hence $$\frac{\int A}{A}=\frac{2^{n+1}}{2^{n+1}-1}, \text{ say}=\frac{Q+1}{Q}.$$

Hence $A=\mu Q$, and $\int A=1+\mu+Q+\mu Q+\ldots$ (if μ be supposed >1). Hence unless $\mu=1$ and at the same time Q is a prime

$$\int A>\mu(Q+1),$$

that is $\frac{\int A}{A}$ is greater than itself.

Hence an even number P cannot be a perfect number if it is not of the form $2^n(2^{n+1}-1)$, where $2^{n+1}-1$ is a prime, which of course implies that $n+1$ must itself be a prime.

It is remarkable that Euler makes no reference to Euclid in proving his own theorem. It must always stand to the credit of the Greek geometers that they succeeded in discovering a class of perfect numbers which in all probability are the only numbers which are perfect. Reference is made to *so-called* perfect numbers in Plato's "Republic," H, 546 B, and also by Aristotle, Probl. I E 3 and "Metaph." A 5. Mr Margoliouth has pointed out to me that Muhamad Al-Sharastani, in his *Book of Religious and Philosophical Sects*, Careton, 1856, p. 267 of the Arabic text, assigns reasons for regarding all the numbers up to 10 inclusive as perfect numbers, which he attributes to Pythagoras, but which are purely fanciful and entitled to no more serious consideration than the late Dr Cummings's ingenious speculations on the number of the Beast. My particular attention was called to perfect numbers by a letter from Mr Christie, dated from "Carlton, Selby," containing some inquiries relative to the subject.

Similarly, the fact of $\frac{3}{2} \cdot \frac{7}{6} \cdot \frac{11}{10}$ being less than 2 is sufficient to show that 3, 5 must be the two least elements of any three-fold perfect number; furthermore, $\frac{3}{2} \cdot \frac{5}{4} \cdot \frac{17}{16}$ being less than 2, shows that 11 or 13 must be the third element of any such number if it exists*—each of which hypotheses admits of an easy disproof. But to disprove the existence of a four-fold perfect number by my actual method makes a somewhat long and intricate, but still highly interesting, investigation of a multitude of special cases. I hope, *numine favente,* sooner or later to discover a general principle which may serve as a key to a universal proof of the non-existence of any other than the Euclidean perfect numbers, for a prolonged meditation on the subject has satisfied me that the existence of any one such—its escape, so to say, from the complex web of conditions which hem it in on all sides—would be little short of a miracle. Thus then there seems every reason to believe that Euclid's perfect numbers are the only perfect numbers which exist!

In the higher theory of congruences (see Serret's *Cours d'Algèbre Supérieure*) there is frequent occasion to speak of "a number n which does not contain any prime factor other than those which are contained in another number M."

In the new nomenclature n would be defined as *a number whose elements are all of them elements of M.*

As τN is used to denote the totient of N, so we may use μN to denote its multiplicity, and then a well-known theorem in congruences may be expressed as follows.

The number of solutions of the congruence

$$x^2 - 1 \equiv 0 \pmod P$$

is $2^{\mu P}$ if P is odd,

$2^{\mu P - 1}$ if P is the double of an odd number,

$2^{\mu P}$ if P is the quadruple of an odd number,

and $2^{\mu P + 1}$ in every other case.

In the memoir above referred to, Euler says that no one has demonstrated whether or not any odd perfect numbers exist. I have found a method for determining what (if any) odd perfect numbers exist of any specified order of manifoldness. Thus, for example, I have proved that there exist no perfect odd numbers of the 1st, 2nd, 3rd, or 4th orders of manifold-

* 3, 5, 7 can never co-exist as elements in *any* perfect number as shown by the fact that $\frac{1+3+3^2}{9} \cdot \frac{1+5}{5} \cdot \frac{1+7+49}{49}$, that is $\frac{26}{15}\left(1 + \frac{1}{7} + \frac{1}{49}\right)$, is *greater* than 2. Thus we see that no perfect number can be a multiple of 105. So again the fact that $\frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{18}$ is *less* than 2 is sufficient to prove that any odd perfect number of multiplicity less than 7 must be divisible by 3.

ness, or in other words, no odd primary, binary, ternary, or quaternary number can be a perfect number. Had any such existed, my method must infallibly have dragged each of them to light*.

In connection with the theory of perfect numbers I have found it useful to denote $p^i - 1$ when p and i are left general as the Fermatian function, and when p and i have specific values as the ith Fermatian of p. In such case p may be called the base, and i the index of the Fermatian.

Then we may express Fermat's theorem by saying [cf. p. 625 below] that *either the Fermatian itself whose index is one unit below a given prime or else its base must be divisible by that prime*†.

It is also convenient to speak of a Fermatian divided by the excess of its base above unity as a Reduced Fermatian and of that excess itself as the Reducing Factor.

The spirit of my actual method of disproving the existence of odd perfect numbers consists in showing that an n-fold perfect number must have more than n elements, which is absurd. The chief instruments of the investigation are the two inequalities to which the elements of any perfect number must be subject and the properties of the prime divisors of a Reduced Fermatian with an odd prime index.

* I have, since the above was in print, extended the proof to quinary numbers, and anticipate no difficulty in doing so for numbers of higher degrees of multiplicity, so that it is to be hoped that the way is now paved towards obtaining a general proof of this *palmary* theorem.

† So too we may state the important theorem that if *an element of a Fermatian is its index the component which has that index for its base must be its square.*