41.

INAUGURAL LECTURE AT OXFORD
12 December 1885.

ON THE METHOD OF RECIPROCANTS AS CONTAINING AN EXHAUSTIVE THEORY OF THE SINGULARITIES OF CURVES*.

[*Nature*, XXXIII. (1886), pp. 222—231.]

It is now two years and seven days since a message by the Atlantic cable containing the single word "Elected" reached me in Baltimore informing me that I had been appointed Savilian Professor of Geometry in Oxford, so that for three weeks I was in the unique position of filling the post and drawing the pay of Professor of Mathematics in each of two Universities: one, the oldest and most renowned, the other—an infant Hercules—the most active and prolific in the world, and which realises what only existed as a dream in the mind of Bacon—the House of Solomon in the New Atlantis.

To Johns Hopkins, who endowed the latter, and in conjunction with it a great Hospital and Medical School, between which he divided a vast fortune accumulated during a lifetime of integrity and public usefulness, I might address the words familiarly applied to one dear to all Wykehamists:—

"Qui condis lævâ, condis collegia dextrâ,
Nemo tuarum unam vicit utraque manú."

The chair which I have the honour to occupy in this University is made illustrious by the names and labours of its munificent and enlightened founder, Sir Henry Saville; of Thomas Briggs, the second inventor of logarithms; of Dr Wallis, who, like Leibnitz, drove three abreast to the temple of fame—being eminent as a theologian, and as a philologer, in addition to being illustrious as the discoverer of the theorem connected with the quadrature of the circle named after him, with which every schoolboy is supposed to be familiar, and as the author of the *Arithmetica Infinitorum*, the precursor of Newton's *Fluxions*; of Edmund Halley, the trusted friend and counsellor of Newton, whose work marks an epoch in the history of astronomy, the reviver of the study of Greek geometry and discoverer of the proper motions of the so-

[* The tables referred to in the text are given pp. 301, 302 below.]

called fixed stars; and by one in later times not unworthy to be mentioned in connection with these great names, my immediate predecessor, the mere allusion to whom will, I know, send a sympathetic thrill through the hearts of all here present, to whom he was no less endeared by his lovable nature than an object of admiration for his vast and varied intellectual acquirements, whose untimely removal, at the very moment when his fame was beginning to culminate, cannot but be regarded as a loss, not only to his friends and to the University for which he laboured so strenuously, but to science and the whole world of letters.

As I have mentioned, the first to occupy this chair was that remarkable man Thomas Briggs, concerning whose relation to the great Napier of Merchiston, the fertile nursery of heroes of the pen and the sword, an anecdote, taken from the *Life* of Lilly, the astrologer, has lately fallen under my eyes, which, with your permission, I will venture to repeat:—

"I will acquaint you (says Lilly) with one memorable story related unto me by John Marr, an excellent mathematician and geometrician, whom I conceive you remember. He was servant to King James and Charles the First. At first, when the lord Napier, or Marchiston, made public his logarithms, Mr Briggs, then reader of the astronomy lectures at Gresham College, in London, was so surprised with admiration of them, that he could have no quietness in himself until he had seen that noble person the lord Marchiston, whose only invention they were: he acquaints John Marr herewith, who went into Scotland before Mr Briggs, purposely to be there when those two so learned persons should meet. Mr Briggs appoints a certain day when to meet at Edinburgh; but failing thereof, the lord Napier was doubtful he would not come. It happened one day as John Marr and the lord Napier were speaking of Mr Briggs: 'Ah John (said Marchiston), Mr Briggs will not now come.' At the very moment one knocks at the gate; John Marr hastens down, and it proved Mr Briggs to his great contentment. He brings Mr Briggs up into my lord's chamber, where almost *one quarter of an hour was spent*, each beholding other almost with admiration *before one word was spoke*. At last Mr Briggs began: 'My lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help unto astronomy, namely, the logarithms; but, my lord, being by you found out, *I wonder nobody else found it out before*, when now known it is so easy.' He was nobly entertained by the lord Napier; and every summer after that, during the lord's being alive, this venerable man Mr Briggs went purposely into Scotland to visit him*."

* A very similar story is told of the meeting of Leopardi and Niebuhr in Rome. What Briggs said of logarithms may be said almost in the same words of the subject of this lecture:—"This most excellent help to geometry which, being found out, one wonders nobody else found it out

Some apology may be needed, and many valid reasons might be assigned, for the departure, in my case, from the usual course, which is that every professor on his appointment should deliver an inaugural lecture before commencing his regular work of teaching in the University. I hope that my remissness, in this respect, may be condoned if it shall eventually be recognised that I have waited, before addressing a public audience, until I felt prompted to do so by the spirit within me craving to find utterance, and by the consciousness of having something of real and more than ordinary weight to impart, so that those who are qualified by a moderate amount of mathematical culture to comprehend the drift of my discourse, may go away with the satisfactory feeling that their mental vision has been extended and their eyes opened, like my own, to the perception of a world of intellectual beauty, of whose existence they were previously unaware.

This is not the first occasion on which I have appeared before a general mathematical audience, as the messenger of good tidings, to announce some important discovery. In the year 1859 I gave a course of seven or eight lectures at King's College, London, at each of which I was honoured by the attendance of my lamented predecessor, on the subject of "The Partitions of Numbers and the Solution of Simultaneous Equations in Integers," in which it fell to my lot to show how the difficulties might be overcome which had previously baffled the efforts of mathematicians, and especially of one bearing no less venerable a name than that of Leonard Euler, and also laid the basis of a method which has since been carried out to a much greater extent in my "Constructive Theory of Partitions," published in the *American Journal of Mathematics*, in writing which I received much valuable co-operation and material contributions from many of my own pupils in the Johns Hopkins University*. Several years later, in the same place, I delivered a lecture on the well-known theorem of Newton, which fills a chapter in the *Arithmetica Universalis*, where it was stated without proof, and of which many celebrated mathematicians, including again the name of Euler, had sought for a proof in vain. In that lecture I supplied the missing demonstration, and owed my success, I believe, chiefly to merging the theorem to be proved, in one of

before; when, now known, it is so easy." I quite entered into Briggs's feelings at his interview with Napier when I recently paid a visit to Poincaré in his airy perch in the Rue Gay-Lussac in Paris (will our grandchildren live to see an Alexander Williamson Street in the north-west quarter of London, or an Arthur Cayley Court in Lincoln's Inn, where he once abode?). In the presence of that mighty reservoir of pent-up intellectual force my tongue at first refused its office, my eyes wandered, and it was not until I had taken some time (it may be two or three minutes) to peruse and absorb as it were the idea of his external youthful lineaments that I found myself in a condition to speak.

* In one of those lectures, two hundred copies of the notes for which were printed off and distributed among my auditors, I founded and developed to a considerable extent the subject since rediscovered by M. Halphen under the name of the Theory of Aspects.

greater scope and generality. In mathematical research, reversing the axiom of Euclid, and converting the proposition of Hesiod, it is a continual matter of experience, as I have found myself over and over again, that the whole is less than its part. On a later occasion, taking my stand on the wonderful discovery of Peaucellier, in which he had realised that exact parallel motion which James Watt had believed to be impossible, and exhausted himself in contrivances to find an imperfect substitute for, in the steam-engine, I think I may venture to say that I brought into being a new branch of mechanico-geometrical science, which has been, since then, carried to a much higher point by the brilliant inventions of Messrs Kempe and Hart. I remember that my late lamented friend, the Lord Almoner's Reader of Arabic in this University, subsequently editor of the *Times*, Mr Chenery, who was present on that occasion in an unofficial capacity, remarked to me after the lecture, which was delivered before a crowded auditory at the Royal Institution, that when they saw two suspended opposite Peaucellier cells, coupled toe-and-toe together, swing into motion, which would have been impossible had not the two connected moving points each described an accurate straight line, "the house rose at you." (The lecture merely illustrated experimentally two or three simple propositions of Euclid, Book III.)

The matter that I have to bring before your notice this afternoon is one far bigger and greater, and of infinitely more importance to the progress of mathematical science, than any of those to which I have just referred. No subject during the last thirty years has more occupied the minds of mathematicians, or lent itself to a greater variety of applications, than the great theory of Invariants. The theory I am about to expound, or whose birth I am about to announce, stands to this in the relation not of a younger sister, but of a brother, who, though of later birth, on the principle that the masculine is more worthy than the feminine, or at all events, according to the regulations of the Salic law, is entitled to take precedence over his elder sister, and exercise supreme sway over their united realms. Metaphor apart, I do not hesitate to say that this theory, *minor natu potestate major*, infinitely transcends in the extent of its subject-matter, and in the range of its applications, the allied theory to which it stands in so close a relation. The very same letters of the alphabet which may be employed in the two theories, in the one may be compared to the dried seeds in a botanical cabinet, in the other to buds on the living branch ready to burst out into blossom, flower and fruit, and in their turn supply fresh seed for the maintenance of a continually self-perpetuating cycle of living forms. In order that I may not be considered to have lost myself in the clouds in making such a statement, let me so far anticipate what I shall have to say on the meaning of Reciprocants and their relation to the ordinary Invariantive or Covariantive forms by taking an instance which happens to be common (or at least, by a slight geometrical adjustment, may be made so) to the two theories. I ask you to compare the form

$$a^2d - 3abc + 2b^3$$

as it is read in the light of the one and in that of the other. In the one case the a, b, c, d stand for the coefficients of a so-called Binary Quantic, and its evanescence serves to express some particular relation between three points lying in a right line. In the other case the letters are interpreted* to mean the successive differential derivatives of the 2nd, 3rd, 4th, 5th orders of one Cartesian co-ordinate of a curve in respect to the other. The equation expressing this evanescence is capable of being integrated, and this integral will serve to denote a relation between the two co-ordinates which furnishes the necessary and sufficient condition in order that the point of the curve of any or no specified order (for it may be transcendental) to which the co-ordinates may refer, may admit of having, at the point where the condition is satisfied, a contact with a conic of a higher order than the common. In the one case the letters employed are dead and inert atoms; in the other they are germs instinct with motion, life, and energy.

A curious history is attached to the form which I have just cited, one of the simplest in the theory, of which the narrative may not be without interest to many of my hearers, even to those whose mathematical ambition is limited to taking a high place in the schools.

At pp. 19 and 20 of Boole's *Differential Equations* (edition of 1859) the author cites this form as the left-hand side of an equation which he calls the "Differential Equation of lines of the second order," and attributes it to Monge, adding the words, "But here our powers of geometrical interpretation fail, and results such as this can scarcely be otherwise useful than as a registry of integrable forms." In this vaticination, which was quite uncalled for, the eminent author, now unfortunately deceased, proved himself a false prophet, for the form referred to is among the first that attracts notice in crossing the threshold of the subject of Reciprocants, and is but one of a crowd of similar and much more complicated expressions, no less than it susceptible of geometrical interpretation and of taking their place on the register of integrable forms. A friend, with whom I was in communication on the subject, and whom I see by my side, remarked to me, in reference to this passage:—"I cannot help comparing a certain passage in Boole to Ezekiel's valley of the dry bones: 'The valley was full of bones, and lo, they were very dry.' The answer to the question, 'Can these bones live?' is supplied by the advent of the glorious idea of the Reciprocants; and the grand invocation, 'Come from the four winds, O breath, and breathe upon these slain, that they may live,' may well be used here. That they will

[* $a=\frac{1}{2!}\frac{d^2y}{dx^2}$, $b=\frac{1}{3!}\frac{d^3y}{dx^3}$, $c=\frac{1}{4!}\frac{d^4y}{dx^4}$, $d=\frac{1}{5!}\frac{d^5y}{dx^5}$.]

'live and stand up upon their feet an exceeding great army' is what we may expect to happen." This, as you will presently see, is just what actually has happened.

Not knowing where to look in Monge for the implied reference, I wrote to an eminent geometer in Paris to give me the desired information; he replied that the thing could not be in Monge, for that M. Halphen, who had written more than one memoir on the subject of the differential equation of a conic, had made nowhere any allusion to Monge in connection with the subject. Hereupon, as I felt sure that a reference contained in repeated editions of a book in such general use as Boole's *Differential Equations* was not likely to be erroneous, I addressed myself to M. Halphen himself, and received from him a reply, from which I will read an extract:—

"En premier lieu, c'est une chose nouvelle pour moi que l'équation différentielle des coniques se trouve dans Boole, dont je ne connais pas l'ouvrage. Je vais, bien entendu, le consulter avec curiosité. Ce fait a échappé à tout le monde ici, et l'on a cru généralement que j'avais le premier donné cette équation. *Nil sub sole novi!* Il m'est naturellement impossible de vous dire où la même équation est enfouie parmi les œuvres de Monge. Pour moi, c'est dans *Le Journal de Math.* (1876), p. 375, que j'ai eu, je crois, la première occasion de développer cette équation sous la forme même que vous citez; et c'est quand je l'ai employée, l'année suivante, pour le problème *sur les lois de Kepler* (*Comptes rendus*, 1877, t. LXXXIV. p. 939), que M. Bertrand l'a remarquée comme neuve. Ce qui vous intéresse plus, c'est de connaître la forme simplifiée sous laquelle j'ai donné plus tard cette équation dans le *Bulletin* de la Société Mathématique. C'est sous cette dernière forme que M. Jordan la donne dans son cours de l'École Polytechnique (t. I. p. 53)."

All my researches to obtain the passage in Monge referred to by Boole have been in vain*.

I will now proceed to endeavour to make clear to you what a Reciprocant means: the above form, which may be called the *Mongian*, would afford an example by which to illustrate the term; but I think it desirable to begin with a much easier one. Consider then the simple case of a single term, the second derivative of one variable, y, in respect to another, x. Every tyro in algebraical geometry knows that this, or rather the fact of its evanescence, serves to characterise one or more points in a curve which possess, so to say,

* Search has been made in the collected works of Monge and in manuscripts of his own or Prony in the library of the Institute, but without effect. I have also made application to the Universal Information Society, who undertake to answer "every conceivable question," but nothing has so far come of it. Perhaps until the citation from Monge is verified it will be safer in future to refer to the so-called Mongian as the Boole-Mongian. It may be regarded as the starting-point of the Differential Invariant Theory, as the Schwarzian is of the deeper-lying and more comprehensive Reciprocant Theory.

a certain indelible and intrinsic character, or what is technically called a singularity; in this case an inflexion such as exists in a capital letter S, or Hogarth's line of beauty.

If we invert the two variables, exchanging, that is to say, one with the other, the fact of this indelibility draws with it the consequence that in general these two reciprocal functions must vanish together, and as a fact each is the same as the other multiplied or divided by the third power of the first derivative of the one variable with respect to the other taken negatively. In this case we are dealing with a single derivative and its reciprocal. The question immediately presents itself whether there may not be a combination of derivatives possessing a similar property. We know that no *single* derivative except the second does.

Such a combination actually presents itself in a form which occurs in the solution of Differential Equations of the second order, the form

$$\frac{dy}{dx}\cdot\frac{d^3y}{dx^3}-\frac{3}{2}\left(\frac{d^2y}{dx^2}\right)^2,$$

which, after the name of its discoverer, Schwarz, we may agree to call a Schwarzian (Cayley's "Schwarzian Derivative*"). If in this expression the x and y be interchanged, its value, barring a factor consisting of a power of the first derivative, remains unaltered, or, to speak more strictly, merely undergoes a change of algebraical sign. We may now arrive at the generalised conception of an algebraical function of the derivatives of one variable in respect to another, which, if we agree to pay no regard to the algebraical sign, or to any power of the first derivative that may appear as a factor, will remain unaltered when the dependent and independent variables are interchanged one with another; and we may agree to call any such function a Reciprocant.

But here an important distinction arises—there are Reciprocants such as the one I first mentioned, $\frac{d^2y}{dx^2}$, or such as the Mongian to which allusion has

* More strictly speaking this is Cayley's Schwarzian derivative cleared of fractions—it may well be called the Schwarzian (see my note on it in the *Mathematical Messenger* for September or October past). Prof. Greenhill in regard to the Schwarzian derivative proper writes me as follows:—

"I found the reference in a footnote to p. 74 of Klein's *Vorlesungen über das Ikosaëder*, &c., in which Klein thanks Schwarz for sending him the reference to a paper by Lagrange, 'Sur la construction des cartes géographiques' in the *Nouveaux Mémoires de l'Académie de Berlin*, 1779. Compare also Schwarz's paper in Bd. 75 of *Borchardt's Journal*, where further literary notices are collected together. Klein says further that in the 'Sächsischen Gesellschaft von Januar 1883,' he has considered the inner meaning (*innere Bedeutung*) of the differential equation

$$\frac{\eta'''}{\eta}-\frac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2=f(\eta),\text{ where }\eta'=\frac{d\eta}{dz}\ldots."$$

There are two papers by Lagrange, one immediately following the other, "Sur la construction des cartes géographiques," but I have not been able to discover the Schwarzian derivative in either of them.

been made in the letter from M. Halphen, in which the second and higher differential derivatives alone appear, the first differential derivative not figuring in the expression. These may be termed Pure Reciprocants. Thus I repeat $\frac{d^2y}{dx^2}$, and

$$9\left(\frac{d^2y}{dx^2}\right)^2 \cdot \frac{d^5y}{dx^5} - 45\,\frac{d^2y}{dx^2}\cdot\frac{d^3y}{dx^3}\cdot\frac{d^4y}{dx^4} + 40\left(\frac{d^3y}{dx^3}\right)^3$$

are pure reciprocants. Those from which the first derivative $\frac{dy}{dx}$ is not excluded may be called Mixed Reciprocants. An example of such kind of Reciprocants is afforded by the Schwarzian above referred to. This distinction is one of great moment, for a little attention will serve to make it clear that every pure reciprocant expressed in terms of x and y marks an intrinsic feature or singularity in the curve, whatever its nature may be, of which x and y are the co-ordinates; for if in place of the variables (x, y) any two linear functions of these variables be substituted, a pure reciprocant, by virtue of its reciprocantive character, must remain unaltered save as to the immaterial fact of its acquiring a factor containing merely the constants of substitution*.

The consequence is that every pure reciprocant corresponds to, and indicates, some singularity or characteristic feature of a curve, and *vice versâ* every such singularity of a general nature and of a descriptive (although not necessarily of a projective) kind, points to a pure reciprocant.

Such is not the case with mixed reciprocants. They will not in general remain unaltered when linear substitutions are impressed upon the variables. Is it then necessary, it may be asked, to pay any attention to mixed reciprocants; or may they not be formally excluded at the very threshold of the inquiry? Were I disposed to put the answer to this question on mere personal grounds, I feel that I should be guilty of the blackest ingratitude, that I should be kicking down the ladder by which I have risen to my present commanding point of view, if I were to turn my back on these humble mixed reciprocants, to which I have reason to feel so deeply indebted; for it was the putting together of the two facts of the substantial permanence under linear substitutions impressed upon the variables of the Schwarzian form and the simpler one which marks the inflexions of a curve—it was, if I may so say, the collision in my mind of these two facts—that kindled the spark and fired the train which set my imagination in a blaze by the light of which the whole horizon of Reciprocants is now illumined.

* The form as it stands shows that for y a linear function of x and y may be substituted; and the form *reciprocated* (by the interchange of x and y) shows that a similar substitution may be made for x. Hence arbitrary linear substitutions may be simultaneously impressed on x and y without inducing any change of form.

But it is not necessary for me to defend the retention of mixed reciprocants on any such narrow ground of personal predilection. The whole body of Reciprocants, pure and mixed, form one complete system, a single garment without rent or seam, a complex whole in which all the parts are inextricably interwoven with each other. It is a living organism, the action of no part of which can be thoroughly understood if dissevered from connection with the rest.

It was in fact by combining and interweaving mixed reciprocants that I was led to the discovery of the pure binomial reciprocant, which comes immediately after the trivial monomial one,—the earliest with which I became acquainted, and of the existence of compeers to which I was for some time in doubt, and only became convinced of the fact after the discovery of the Partial Differential Equation, the master-key to this portion of the subject, which gives the means of producing them *ad libitum* and ascertaining all that exist of any prescribed type. Of this partial differential equation I shall have occasion hereafter to speak; but this is not all, for, as we shall presently see, mixed reciprocants are well worthy of study on their own account, and lead to conclusions of the highest moment, whether as regards their applications to geometry or to the theory of transcendental functions and of ordinary differential equations.

The singularities of curves, taking the word in its widest acceptation, may be divided into three classes: those which are independent of homographic deformation and which remain unaltered in any perspective picture of the curve; those which, having an express or tacit reference to the line at infinity, are not indelible under perspective projection, but using the word descriptive with some little latitude may, in so far as they only involve a reference to the line at infinity as a line, be said to be of a purely descriptive character; and, lastly, those which are neither projective nor purely descriptive, having relation to the points termed, in ordinary parlance, "circular points at infinity"—for which the proper name is "centres of infinitely distant pencils of rays," that is, pencils, every ray of which is infinitely distant from every point external to it. Such, for instance, would be the character of points of maximum or minimum curvative, which, as we shall see, indicate, or are indicated by, that particular class of Mixed to which I give the name of "Orthogonal Reciprocants." All purely descriptive singularities alike, whether projective or non-projective, are indicated by pure reciprocants, and are subject to the same Partial Differential Equation; just as, in the Theory of Binary Quantics, Invariants, although under one aspect they may be regarded as a self-contained special class, admit of being and are most advantageously studied in connection with, and as forming a part of, the whole family of forms commonly known by the name of "semi-, or subinvariants," but which I find it conduces to much greater clearness of expression and avoidance of ambiguity or periphrasis to designate as Binariants.

The question may here be asked, How, then, are projective and non-projective pure reciprocants to be discriminated by their external characters?

I believe that I know the answer to this question, which is, that the former are subject to satisfy a second partial differential equation of a certain simple and familiar type, but this is a matter upon which it is not necessary for me to enter on the present occasion*. It is enough for our present purpose to remark that every projective pure reciprocant must, so to say, be in essence a masked ternary covariant. For instance, if we take the simplest of all such, namely, a, that is $\frac{d^2y}{dx^2}$, we have, if $\phi(x, y) = 0$,

$$\frac{d^2y}{dx^2} \cdot \left(\frac{d\phi}{dy}\right)^3 = \begin{vmatrix} \frac{d^2\phi}{dx^2} & \frac{d^2\phi}{dxdy} & \frac{d\phi}{dx} \\ \frac{d^2\phi}{dxdy} & \frac{d^2\phi}{dy^2} & \frac{d\phi}{dy} \\ \frac{d\phi}{dx} & \frac{d\phi}{dy} & \cdot \end{vmatrix}$$

which, for facility of reference, let me call M. Obviously we might instead of $a = 0$ substitute $M = 0$ to mark an inflexion. And now if we write Φ as the completed form of ϕ, when made homogeneous by the substitution of z for unity; and if we suppose it to be of n dimensions in x, y, z, and call its Hessian H, we shall obtain the syzygy

$$(n-1)^2 \left(\frac{d\phi}{dy}\right)^3 a + H + \left\{\frac{d^2\Phi}{dx^2} \cdot \frac{d^2\Phi}{dy^2} - \left(\frac{d^2\Phi}{dxdy}\right)^2\right\} \Phi = 0.$$

Hence the system $\Phi = 0$, $a = 0$, will be in effect the same as the system $\Phi = 0$, $H = 0$, and in this sense a may be said to carry H as it were in its bosom. And so in general every pure projective reciprocant may, in the language of insect transformation, be regarded as passing, so to say, first from the grub to the pupa or chrysalis, and from this again, divested of all superfluous integuments, to the butterfly or imago state.

Non-projective pure reciprocants undergo only one such change. There is no possibility of their ever emerging into the imago—their development being finally arrested at the chrysalis stage.

It would, I think, be an interesting and instructive task to obtain the imago or Hessianised transformation of the Mongian, but I am not aware

* In Paris, from which I correct the proofs, I have succeeded in reducing this conjecture to a certainty and in establishing the marvellous fact that every Projective Reciprocant, or, which is the same thing, every Differential Invariant, is, at the same time, an Ordinary Subinvariant. Thus a differential invariant (or projective reciprocant) may be regarded as *a single personality clothed with two distinct natures*—that of a reciprocant and that of a subinvariant.

that anyone has yet done, or thought of doing, this*. It seems to me that by substituting Reciprocants in lieu of Ternary Covariants we are as it were stealing a dimension from space, inasmuch as Reciprocants, that is, Ternary Covariants in their undeveloped state, are closely allied to, and march *pari passu* with, the familiar forms which appertain to merely binary quantics.

I will now proceed to bring before your notice the general partial differential equation which supplies the necessary and sufficient condition to which all pure reciprocants are subject.

It is highly convenient to denote the successive derivatives

$$\frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \frac{d^4y}{dx^4},\ \ldots$$

by the simple letters a, b, c, ...

The first derivative $\dfrac{dy}{dx}$ plays so peculiar a part in this theory that it is necessary to denote it by a letter standing aloof from the rest, and I call it t. This last letter, I need not say, does not make its appearance in any pure reciprocant. This being premised, I invite your attention to the equation in question, in which you will perceive the symbols of operation are separated from the object to be operated upon.

Writing $V = 3a^2\delta_b + 10ab\delta_c + (15ac + 10b^2)\,\delta_d + \ldots.$ and calling any pure reciprocant R,

$$VR = 0$$

is the equation referred to.

I cannot undertake, within the brief limits of time allotted to this lecture, to explain how this operation, or, as it may be termed, this annihilator V is arrived at. The table of binomial coefficients, or rather half series of binomial coefficients, shown† in Chart 4, will enable you to see what is the law of the numerical coefficients of its several terms. Let the words *weight, degree, extent* (extent, you will remember, means the number of places by which the most remote letter in the form is separated from the first letter in the alphabet) of a pure reciprocant signify the same things as they would do if the letters $a, b, c, \ldots$ referred, according to the ordinary notation, to Binariants instead of to Reciprocants. The number of binariants linearly independent of each other whose weight, degree and extent or order are w, i, j is given by the partition formula $(w\,; i, j) - (w-1\,; i, j)$ where in general $(w; i, j)$ means the number of ways of partitioning w into i or fewer parts none greater than j.

* M. Halphen informs me that this has been done by Cayley in the *Phil. Trans.* for 1865, and subsequently in a somewhat simplified form by Painvin, *Comptes Rendus*, 1874. But neither of these authors seems to have had the Boole-Mongian objectively before him, so that a slight supplemental computation is wanting to establish the equation between it and the function which either of them finds to vanish at a *sextactic* point.

[† p. 302 below.]

It follows immediately from the mere form of V that the corresponding formula in the case of Reciprocants of a given type $w.i.j$ will be

$$(w; i, j) - (w-1; i+1, j)$$

the augmentation of i in the second term of the formula being due to the fact that, whereas in the partial differential equation for Binariants it is the letters themselves which appear as coefficients, it is quadratic functions of these in the case of Reciprocants. From the form of V we may also deduce a rigorous demonstration of the existence of Reciprocants strictly analogous to those with which you are familiar in the Binariant Theory, which are pictured in Chart 2, and are now usually designated as Protomorphs, as being the forms by the interweaving of which with one another (or rather by a sort of combined process of mixture and precipitation), all others, even the irreducible ones, are capable of being produced. The corresponding forms for Reciprocants you will see exhibited in the same table. Each series of Protomorphs may of course be indefinitely extended as more and more letters are introduced. In the table I have not thought it necessary to go beyond the letter g. You also know that besides Protomorphs there are other irreducible forms, the organic radicals, so to say, into which every compound form may be resolved, always limited in number, whatever the number of letters or primal elements we may be dealing with. The same thing happens to Reciprocants as you will notice in the comparative table in Chart 2. Without going into particulars, I will ask you to take from me upon faith the assurance that there is no single feature in the old familiar theory, whether it relates to Protomorphs, to Ground-forms, to Perpetuants, to Factorial constitution, to Generating Functions, or whatever else sets its stamp upon the one, which is not counterfeited by and reproduced in the parallel theory.

So much—for time will not admit of more—concerning pure reciprocants.

Let me now say a few words *en passant* on Mixed Reciprocants.

Pure Reciprocants, we have seen, are the analogues of Invariants, or else of the leading terms, for that is what are Semi- or Subinvariants, of Covariantive expansions; each is subject to its own proper linear partial differential equation. Mixed Reciprocants are the exact analogues of the coefficients in such expansions other than those of the leading terms. Starting from the leading terms as the unit point, the coefficients of rank ω are subject to a partial differential equation of order ω; and just so, mixed reciprocants, if involving t up to the power ω, are subject to a partial differential equation of that same order.

I have alluded to a peculiar class of mixed under the name of "Orthogonal Reciprocants." They are distinguished, as I have proved, by the beautiful property that, if differentiated with respect to t, the result must be itself a Reciprocant. In Chart 1 you will see this illustrated in the case of a mixed reciprocant $(1+t^2)b-3ta^2$, which serves to indicate the existence of points of maximum and minimum curvature. Its differential coefficient with respect to t is the oft-alluded-to Schwarzian, transliterated into the simpler notation. Proceeding in the inverse order—of Integration instead of Differentiation—I call your attention to a mixed reciprocant, of a very simple character, one which presents itself at the very outset of the theory, namely

$$tc-5ab,$$

which, integrated in respect to t between proper limits, yields the elegant orthogonal reciprocant

$$(t^2+1)c-10abt+15a^3.$$

Expressed in the ordinary notation, this, equated to zero, takes the form

$$\left\{\left(\frac{dy}{dx}\right)^2+1\right\}\frac{d^4y}{dx^4}-10\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}\cdot\frac{d^3y}{dx^3}+15\left(\frac{d^2y}{dx^2}\right)^3=0.$$

Mr Hammond has integrated this, treated as an ordinary differential equation, and has obtained the complete primitive expressed through the medium of two related Hyper-Elliptic Functions connecting the variables x and y (see* Chart 3). It may possibly turn out to be the case that every mixed reciprocant is either itself an Orthogonal Reciprocant, or by integration, in respect to t, leads to one.

It will of course be understood that, in interpreting equations obtained by equating to zero an Orthogonal Reciprocant, the variables must be regarded as representing not general but rectangular Cartesian co-ordinates.

Here seems to me to be the proper place for pointing out to what extent I have been anticipated by M. Halphen in the discovery of this new world of Algebraical Forms. When the subject first dawned upon my mind, about the end of October or the beginning of November last, I was not aware that it had been approached on any side by any one before me, and believed that I was digging into absolutely virgin soil. It was only when I received M. Halphen's letter, dated November 25, in relation to the Mongian business already referred to, accompanied by a presentation of his memoirs on Differential Invariants, that I became aware of there existing any link of connection between his work and my own. A Differential Invariant, in the sense in which the term is used by M. Halphen, is not what at first blush I supposed it to be, and as in my haste to repair what seemed to me an omission to be without loss of time supplied, I wrote to M. Hermite it was, in a letter which has been or is about to be inserted in the *Comptes Rendus* of the Institute of France; it is not, I say, identical with what I have termed a general pure reciprocant, but only with that peculiar species of Pure Reciprocants to which I have in a preceding part of this lecture referred as corresponding and pointing to Projective Singularities. In his

[* p. 302 below.]

splendid labours in this field Halphen has had no occasion to construct or concern himself with that new universe of forms viewed as a whole, whether of Pure or Mixed Reciprocants, which it has been the avowed and principal object of this lecture to bring under your notice.

I anticipate deriving much valuable assistance in the vast explorations remaining to be made in my own subject from the new and luminous views of M. Halphen, and possibly he may derive some advantage in his turn from the larger outlook brought within the field of vision by my allied investigations.

Let me return for a moment to that simplest class of pure reciprocants which I have called protomorphs. Each of these will be found (as may be shown either by a direct process of elimination, or by integrating the equations obtained by equating them severally to zero, regarded as ordinary differential equations between x and y) each of these, I say, will be found to represent some simple kind of singularity at the point (x, y) of the curve to which these co-ordinates are supposed to refer. Thus, for instance, No. 1 marks a single point of inflexion; No. 2, points of closest contact with a common parabola; No. 3, what our Cayley has called sextactic points, referring to a general conic; No. 4, points of closest contact with a common cubical parabola; and so on. The first and third, it will be noticed, represent projective singularities, and as such, in M. Halphen's language, would take the name of Differential Invariants. The second and fourth, having reference to the line at infinity in the plane of the curve, are of a non-projective character, and as such would not appear in M. Halphen's system of Differential Invariants. It is an interesting fact that every simple parabola, meaning one whose equation can be brought under the form $y = x^{\frac{m}{n}}$, corresponds to a linear function of a square of the third, and the cube of the second protomorph, and consequently will in general be of the sixth degree. In the particular case of the cubical parabola, the numerical parameter of this equation is such that the highest powers of b cancel each other so that the form sinks one degree, and becomes represented by the *Quasi-Discriminant*, No. 4.

This simple instance will serve to illustrate the intimate connection which exists between the projective and non-projective reciprocants, and the advantage, not to say necessity, of regarding them as parts of one organic whole.

It would take me too far to do more than make the most cursory allusion to an extension of this theory similar to that which happens when in the ordinary theory of invariants we pass from the consideration of a single Quantic to that of two or more. There is no difficulty in finding the partial differential equation to double reciprocants which, as far as I have as yet pursued the investigation, appear to be functions of $a, b, c, \ldots$; $a', b', c', \ldots$; and of $(t - t')$.

The theory of double reciprocants will then include as a particular case the question of determining the singularities of paired points of two curves at which their tangents are parallel, and consequently the theory of common tangents to two curves and of bi-tangents to a single one.

I think I may venture to say that a general pure multiple reciprocant which marks off relative singularities, whether projective or non-projective, of a group of curves, is a function of the second and higher differential derivatives appertaining to the several curves of the group, and of the differences of the first derivatives, whereas in a mixed multiple reciprocant these last-named differences are replaced by the first derivatives themselves. As a particular case, when the group dwindles to an individual and there is only one t, this letter disappears altogether from the form, for there are no differences of a single quantity.

In the chart (marked No. 2) you will see the table of Protomorphs carried on as far as the letter g inclusive, and will not fail to notice what may be termed the higher organisation of Reciprocantive as compared with ordinary Invariantive Protomorphs; the degrees of the latter oscillate or librate between the numbers 2 and 3, whereas in the former the degree is variable according to a certain transcendental law dependent on the solution of a problem in the Partition of Numbers. Another interesting difference between general Invariants and general Pure Reciprocants consists in the fact that, whilst the number of the former ultimately (that is, when the extent is indefinitely increased) becomes indefinitely great, that of the latter is determinate for any given degree even for an infinite number of letters.

In carrying on the table of protomorphs up to the letter h (see Chart 6) a new phenomenon presents itself, to which, however, there is a perfect parallel in the allied theory. An arbitrary constant enters into the form, its general value being a linear function of U and W (for which see Chart 6). But this is not all. If you examine the terms in both U and W (there are in all twelve such) you will find that these twelve do not comprise all of the same type to which they belong. There is a Thirteenth (a banished Judas), equally *à priori* entitled to admission to the group, but which does not make its appearance among them, namely, b^4d. I rather believe that a similar phenomenon of one or more terms, whose presence might be expected, but which do not appear, presents itself in the allied invariantive theory, but cannot speak with certainty as to this point, as the circumstance has not received, and possibly does not merit, any very particular attention.

Still, in the case before us, this unexpected absence of a member of the family, whose appearance might have been looked for, made an impression on my mind, and even went to the extent of acting on my emotions. I began to think of it as a sort of lost Pleiad in an Algebraical Constellation, and in the end, brooding over the subject, my feelings found vent, or sought relief, in a rhymed effusion, a *jeu de sottise*, which, not without some apprehension of appearing singular or extravagant, I will venture to rehearse. It will at least serve as an interlude, and give some relief to the strain upon your attention before I proceed to make my final remarks on the general theory.

TO A MISSING MEMBER

Of a Family Group of Terms in an Algebraical Formula.

Lone and discarded one! divorced by fate,
Far from thy wished-for fellows—whither art flown?
Where lingerest thou in thy bereaved estate,
Like some lost star, or buried meteor stone?
Thou mindst me much of that presumptuous one
Who loth, aught less than greatest, to be great,
From Heaven's immensity fell headlong down
To live forlorn, self-centred, desolate:
Or who, new Heraklid, hard exile bore,
Now buoyed by hope, now stretched on rack of fear,
Till throned Astræa, wafting to his ear
Words of dim portent through the Atlantic roar,
Bade him "the sanctuary of the Muse revere
And strew with flame the dust of Isis' shore."

Having now refreshed ourselves and bathed the tips of our fingers in the Pierian spring, let us turn back for a few brief moments to a light banquet of the reason, and entertain ourselves as a sort of after-course with some general reflections arising naturally out of the previous matter of my discourse. It seems to me that the discovery of Reciprocants must awaken a feeling of surprise akin to that which was felt when the galvanic current astonished the world previously accustomed only to the phenomena of machine or frictional electricity. The new theory is a ganglionic one: it stands in immediate and central relation to almost every branch of pure mathematics—to Invariants, to Differential Equations, ordinary and partial, to Elliptic and Transcendental Functions, to Partitions of Numbers, to the Calculus of Variations, and above all to Geometry (alike of figures and of complexes), upon whose inmost recesses it throws a new and wholly unexpected light. The geometrical singularities which the present portion of the theory professes to discuss are in fact the distinguishing *features* of curves; their *technical* name, if applied to the human countenance, would lead us to call a man's eyes, ears, nose, lips, and chin his singularities; but these singularities make up the character and expression, and serve to distinguish one individual from another. And so it is with the so-called singularities of curves.

Comparing the system of ground-forms which it supplies with those of the allied theory, it seems to me clear that some common method, some yet undiscovered, deep-lying, Algebraical principle remains to be discovered, which shall in each case alike serve to demonstrate the finite number of these forms (these organic radicals) for any specified number of letters. The road to it, I believe, lies in the Algebraical Deduction of ground-forms from the Protomorphs*. Gordan's method of demonstration, so difficult and so complicated, requiring the devotion of a whole University semester to master, is inapplicable to reciprocants, which, as far as we can at present see, do not lend themselves to symbolic treatment.

How greatly must we feel indebted to our Cayley, who while he was, to say the least, the joint founder of the symbolic method, set the first, and out of England little if at all followed, example of using as an engine that mightiest instrument of research ever yet invented by the mind of man—a Partial Differential Equation, to define and generate invariantive forms.

With the growth of our knowledge, and higher views now taken of invariantive forms, the old nomenclature has not altogether kept pace, and is in one or two points in need of a reform not difficult to indicate. I think that we ought to give a general name—I propose that of Binariants—to every rational integral form which is nullified by the general operator

$$\lambda a\delta_b + \mu b\delta_c + \nu c\delta_d + \dots,$$

where λ, μ, ν, ... are arbitrary numbers.

This operator, I think, having regard to the way in which its segments link on to one another, may be called the Vermicular.

Binariants corresponding to unit values of λ, μ, ν, ... may be termed standard binariants. Those for which these numbers are the terms of the natural arithmetical series 1, 2, 3, ... Invariantive binariants, which may be either complete or incomplete invariants; these latter are what are usually termed semi- or sub-invariants. I may presently have to speak of a third class of binariants for which the arbitrary multipliers are the numbers 3, 8, 15, 24 ... (the squares of the natural numbers each diminished by unity) which, if the theorem I have in view is supported by the event, will have to be termed Reciprocantive Binariants. But first let me call attention to what seems a breach of the asserted parallelism between the Invariantive and the

* See the section on the Algebraical Deduction of the Ground-forms of the Quintic in my memoir on Subinvariants in the *American Journal of Mathematics*. [Vol. III. of this Reprint, p. 580.]

Reciprocantive theories. In the former we have complete and incomplete invariants, but we have drawn no such distinction between one set of pure reciprocants and another. A parallel distinction does however exist.

If we use w, i, j to signify the weight, degree, and extent of an invariantive form, w is never less than the half product of ij; when equal to it the form is complete. In the case of reciprocants certain observed facts seem to indicate that there exists an analogous but less simple inequality. If this conjecture is verified it is not merely $\frac{ij}{2} - w$, but $\frac{ij}{2} - (j-2) - w$, which is never negative: and when this is zero, the form may be said to be complete*. There would then be thus complete forms in each of the two theories; in the earlier one they take a special name: this is the only difference.

We have spoken of Pure Reciprocants as being either projective or non-projective, but so far have abstained from particularising the external characters by which the former may be distinguished from the latter. I have good reason to suspect that the former are distinguished from the latter by being Binariants; that, in addition to being subject to annihilation by the operator V, they are also subject to annihilation by the Vermicular operator when made special by the use of the numerical multipliers 3, 8, 15 ... above alluded to, or in other words (as previously mentioned incidentally) are subject to satisfy two simultaneous partial differential equations instead of only one†.

* If this should turn out to be true, the "crude generating function" for reciprocants would be almost identical with that of in- and co-variants of the same extent j. The denominators would be absolutely identical; as regards the numerators, while that for invariantive forms is $1-a^{-1}x^{-2}$ the numerator for reciprocants would be $1-a^{-2}x^{-2j}$. As I write abroad and from memory there is just a chance that the index of a here given may be erroneous.

† As already stated in a previous footnote this conjecture is fully confirmed, my own proof having been corroborated (if it needed corroboration) by another entirely different one invented by M. Halphen, who fully shares my own astonishment at the fact of there being forms (half-horse, half-alligator) at once reciprocants and sub-invariants, and as such satisfying two simultaneous partial differential equations.

If instead of denoting the successive differential derivatives (starting from the second) $a, b, c, \ldots$ we call them $1.2.a$, $1.2.3.b$, $1.2.3.4.c$, ... the two Annihilators will be

$$a\delta_b + 2b\delta_c + 3c\delta_d + 4d\delta_e + \ldots$$

and

$$4\,\frac{a^2}{2}\,\delta_b + 5ab\delta_c + 6\,(ac + \tfrac{1}{2}b^2)\,\delta_d + 7\,(ad + bc)\,\delta_e + \ldots$$

the latter being my new operator, the Reciprocator V, accommodated to the above-stated change of notation for the successive differential derivatives.

Hardly necessary is it for me to point out in explanation of the semi-sums $\frac{1}{2}b^2$, ... that we may write the MacMahonised V under the form

$$4a^2\delta_b + 5\,(ab + ba)\,\delta_c + 6\,(ac + b^2 + ca)\,\delta_d + 7\,(ad + bc + cb + da)\,\delta_e + \ldots.$$

It is to be presumed that in addition to mixed reciprocants (the ocean into which flows the sea of pure reciprocants, as into that again empties itself the river of projective reciprocants) there may exist a theory of forms in which y as well as $\frac{dy}{dx}$ will appear, or, so to say, doubly mixed reciprocants, the most general of all, in which case we must speak of the content of these as the

Projective Reciprocants we have seen are disguised or masked Ternary Covariants—Covariants in the grub, the first undeveloped state. Now ternary covariants are capable, it may or may not be generally known, of satisfying 6 reducible to 2 simultaneous Partial Differential Equations, and at first sight it might be surmised that nothing would be gained by the substitution of the two new for the two old simultaneous partial differential equations. But the fact is not so, for the old partial differential equations are perfectly unmanageable, or at least have never, as far as I know, been handled by any one, for they have to do with a *triangular heap*, whereas the new ones are solely concerned with a *linear series* of coefficients.

I have alluded to there being a particular form common to the two theories. In the one theory it is the Mongian alluded to in the correspondence, which has been read, with M. Halphen. In the other it is the source of the skew covariant to the cubic. If the latter be subjected to a sort of MacMahonic numerical adjustment, it becomes absolutely identical with the former. Let us imagine that before the invention of Reciprocants an Algebraist happened to have had both forms present to his mind, and had thought of some contrivance for lowering the coefficients of the Mongian written out with the larger coefficients, and had thus stumbled upon this striking fact. It could not have failed to vehemently arouse his curiosity, and he would have set to work to discover, if possible, the cause of this coincidence. He would in all probability have addressed himself to the form which precedes the source alluded to in the natural order of genesis, and have applied a similar adjustment to the much simpler form, $ac-b^2$: having done so he would have tried to discover to what singularity it pointed—but his efforts to do so we know must have been fruitless, and he would have felt disposed to throw down his work in despair, for the intermediate ideas necessary to make out the parallelism would not have been present to his mind. So long as we confine ourselves to Differential Invariants, that is, to projective pure reciprocants, we are like men walking on those elevated ridges, those more than Alpine summits, such as I am told* exist in Thibet, where it may be the labour of days for two men who can see and speak to each other to come together. Reciprocants supply the bridge to span the yawning ravine and to bring allied forms into direct proximity.

---

ocean and of the others as sea, river, and brook. Curious is it to reflect that in the theory which as it exists comprises Invariantives, Reciprocants, and Invariantive Reciprocants or Reciprocant Invariantives, the order of discovery was (1) Invariantives (Eisenstein, Boole, &c.); (2) Invariantive Reciprocants (Monge and Halphen); (3) Reciprocants (Schwarz, the author of this lecture).

* I think my informant was my friend Dr Inglis, of the Athenæum Club, who some time ago undertook a journey in the Himalayas in the hopes of coming upon the traces of a lost religion which he thought he had reason to believe existed among mankind in the pre-Glacial period of the earth's history.

I have spoken of mixed reciprocants as being subject to satisfy not a linear partial differential equation, but one of a higher order dependent on the intensity, so to say, of its mixedness—the highest power, that is to say, of the first differential derivative which it contains, and it might therefore be supposed that these forms are much more difficult to be obtained than pure reciprocants. But the fact is just the reverse, for as I discovered in the very infancy of the inquiry, and have put on record in the September or October number* of the *Mathematical Messenger*, mixed reciprocants may be evolved in unlimited profusion by the application of simple and explicit processes of multiplication and differentiation. From any reciprocant whatever, be it mixed or pure, new mixed ones may be educed infinitely infinite in number, inasmuch as at each stage of the process, arbitrary functions of the first differential derivative may be introduced.

The wonderful fertility of this method of generation excited warm interest on the part of one of the greatest of living mathematicians, the expression of which acted as a powerful incentive to me to continue the inquiry. They may be compared with the shower of December meteors shooting out in all directions and covering the heavens with their brilliant trains, all diverging from one or more fixed radiant-points, the radiant-point in the theory before us being the particular form selected to be operated upon.

The new doctrine which I have endeavoured thus imperfectly to adumbrate has taken its local rise in this University, where it has already attracted some votaries to its side, and will, I hope, eventually obtain the cooperation of many more. I have ventured with this view to announce it as the subject of a course of lectures during the ensuing term.

When I lately had the pleasure of attending the new Slade Professor's inaugural discourse, I heard him promise to make his pupils participators in his work, by painting pictures in the presence of his class. I aspire to do more than this—not only to paint before the members of my class, but to induce them to take the palette and brush and contribute with their own hands to the work to be done upon the canvas. Such was the plan I followed at the Johns Hopkins University, during my connection with which I may have published scores of Mathematical articles and memoirs in the journals of America, England, France, and Germany, of which probably there was scarcely one which did not originate in the business of the class-room; in the composition of many or most of them I derived inestimable advantage from the suggestions or contributions of my auditors. It was frequently a chase, in which I started the fox, in which we all took a common interest, and in which it was a matter of eager emulation between my hearers and myself to try which could be first in at the death.

[* p. 255 above.]

During the past period of my professorship here, imperfectly acquainted with the usages and needs of the University, I do not think that my labours have been directed so profitably as they might have been either as regards the prosecution of my own work or the good of my hearers: my attention has been distracted between theories waiting to be ushered into existence and providing for the daily bread of class-teaching. I hope that in future I may be able to bring these two objects into closer harmony and correlation, and I think I shall best discharge my duty to the University by selecting for the material of my work in the class-room any subject on which my thoughts may, for the time being, happen to be concentrated, not too alien to, or remote from, that which I am appointed to teach; and thus, by example, give lessons in the difficult art of mathematical thinking and reasoning—how to follow out familiar suggestions of analogy till they broaden and deepen into a fertilising stream of thought—how to discover errors and to repair them, guided by faith in the existence and unity of that intellectual world which exists within us, and is at least as real as that with which we are environed.

The *American Mathematical Journal*, conducted under the auspices of the Johns Hopkins University, which has gained and retains a leading position among the most important of its class, whether measured by the value of its contents or the estimation in which it is held by the Mathematical world, bears as its motto—

πραγμάτων ἔλεγχος οὐ βλεπομένων.

I have the pleasure of seeing among my audience this day the most distinguished geometer of Holland, Prof. Schoute, who has done me the signal honour of coming over to England to be present at this lecture, who hospitably entertained me at Groningen (in a vacation visit which I recently paid to his country, the classic soil which has given birth to an Erasmus, a Grotius, a Boerhaave, a Spinoza, a Huyghens, and a Rembrandt), and who was kind enough, in proposing my health at a party where many of his colleagues were present, to say that he felt sure "that I should return to England cheered and invigorated, and would, ere long, light on some discovery which would excite the wonder of the Mathematical world."

I do not venture to affirm, nor to think, that this vaticination has been fulfilled in the terms in which it was uttered, but can most truly say that the discovery, which it has been my good fortune to be made the medium of revealing, has excited my own deepest feelings of ever-increasing wonder rising almost to awe, such as must have come over the revellers who saw the handwriting start out more and more plainly on the wall, or the *scienziati* crowding round the blurred palimpsest as they began to be able to decipher characters and piece together the sentences of the long lost and supposed irrecoverable *De Republicâ*.

When I was at Utrecht, on my way to Groningen, Mr Grinwis, the Professor of Mathematics at that University, showed me an English book on "Differential Equations," which had just appeared, of which he spoke in high terms of praise, and said it contained over 800 examples. I wrote at once for the book to England, and on seeing it on my arrival, forgetting that it had been ordered, mistook it for a present from the author or publisher, and, what is unusual with me, read regularly into it, until I came to the section on Hyper-geometrical series, where the Schwarzian Derivative (so named by Cayley after Prof. Schwarz) is spoken of.

Perhaps I ought to blush to own that it was new to me, and my attention was riveted by the property it possesses, in common with the more simple form which points to inflexions on curves, of remaining substantially unaltered, of persisting as a factor at least of its altered self, when the variables which enter it are interchanged. Following out this indication, I at once asked myself the question, "ought there not to exist combinations of derivatives of *all* orders possessing this property of reciprocation?" That question was soon answered, and the universe of mixed reciprocants stood revealed before me. These mixed reciprocants, by simple processes of combination, led me to the discovery of the first pure reciprocant, $3b^2 - 5ac$: whereupon I again put the question to myself, "are there, or are there not, others of this form, and if so, what are they?"

In an unexpected manner the question was answered, and my curiosity gratified to the utmost by the discovery of the partial differential equation which is the central point of the theory, and at once discloses the parallelism between it and the familiar doctrine of Invariants. Two principal exponents of that doctrine, who have infused new blood into it, and given it a fresh point of departure—Capt. MacMahon and Mr Hammond—I have the pleasure of seeing before me. Mr Kempe, who is also present, has lately entered into and signally distinguished himself in the same field, availing himself in so doing of his profound insight into the subject of linkages, his interest in which I believe I may say received its first impulse from the lecture which he heard me deliver upon it at the Royal Institution in January 1874, on the very night when the Prime Minister for the time being sent round letters to his supporters announcing his intention to dissolve Parliament. Of the two events I have ever regarded the lecture as by far the more important to the permanent interests of society. He has lately applied ideas founded upon linkages to produce a most original and remarkable scheme for explaining the nature of the whole pure body of Mathematical truth, under whatever different forms it may be clothed, in a memoir which has been recommended to be printed in the *Transactions* of the Royal Society, and which, I think, cannot fail when published to excite the deepest interest alike in the Mathematical and the Philosophical worlds*.

I also feel greatly honoured by the presence of Prof. Greenhill, who will be known to many in this room from his remarkable contributions to the theory of Hydrodynamics and Vortex Motion, and who has sufficient candour and largeness of mind to be able to appreciate researches of a different character from those in which he has himself gained distinction.

I should not do justice to my feelings if I did not acknowledge my deep obligations to Mr Hammond for the assistance which he has rendered me, not only in preparing this lecture which you have listened to with such exemplary patience, but in developing the theory; I am indebted to him for many valuable suggestions tending to enlarge its bounds, and believe have been saved, by my conversations with him, from falling into some serious errors of omission or oversight. Saving only our Cayley (who, though younger than myself, is my spiritual progenitor—who first opened my eyes and purged them of dross so that they could see and accept the higher mysteries of our common Mathematical faith), there is no one I can think of with whom I ever have conversed, from my intercourse with whom I have derived more benefit. It would be an immense gain to Science, and to the best interests of the University, if something could be done to bring such men as Mr Hammond (and, let me add, Mr Buchheim, who ought never to have been allowed to leave it) to come and live among us. I am sure that with their endeavours added to my own and those of that most able body of teachers and researchers with whom I have the good fortune to be associated—my brother Professors and the Tutorial Staff of the University—we could create such a School of Mathematics as might go some way at least to revive the old scientific renown of Oxford, and to light such a candle in England as, with God's grace, should never be put out†.

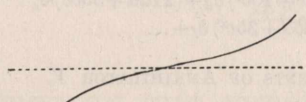
* In his memoir for the *Phil. Trans.* Mr Kempe contends that any whatever mathematical proposition or research is capable of being represented by some sort of simple or compound linkage. One would like to know by what sort of linkage he would represent the substance of the memoir itself.

† I have purposely confined myself in my lecture to reciprocants, indicative of properties of plane curves, but had in view to extend the theory to the case of higher dimensions in space leading to reciprocants involving the differential derivatives of any number of variables y, z, ... M. Halphen, with whom I have had the great advantage of being in communication during my stay in Paris, has anticipated me in this part of my plan, and has found that the same method which I have used to obtain the Annihilator V applied to a system of variables leads to an Annihilator of a very similar form to V, and at my request will publish his results in a forthcoming number of the *Comptes Rendus.* Thus the dominion of reciprocants is already extended over the whole range of forms unlimited in their own number as well as in that of the variables which they contain.

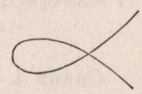
TABLES OF SINGULARITIES AND FORMULÆ REFERRED TO IN THE PRECEDING LECTURE.

CHART 1.

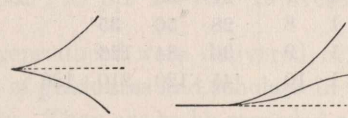
Inflexion



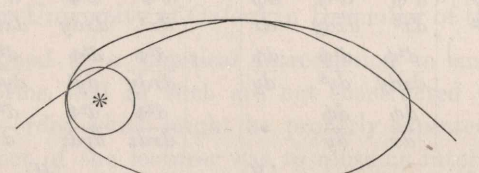
Node



Cusp



Points of maximum and minimum curvature



Bitangent

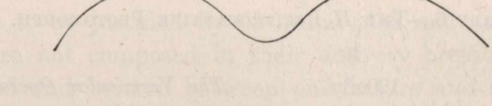


CHART 2.—PROTOMORPHS.

Binariants.	*Reciprocants.*
a	a
$ac-b^2$	$3ac-5b^2$
$a^2d-3abc+2b^3$	$9a^2d-45abc+40b^3$
$ae-4bd+3c^2$	$5a^2e-35abd+7ac^2+35b^2c$
$a^2f+5abe+2acd+8b^2d-6bc^2$	$45a^3f-420a^2be-42a^2cd+1120ab^2d-315abc^2$ $-1120b^3c$
$ag-6bf+15ce-10d^2$	$a^2g-12abf-450ace+792b^2e+588ade^2$ $-2772bcd+1925c^3$

Chart 3.

No. 1. a

No. 2. $3ac-5b^2$

No. 3. $9a^2d-45abc+40b^3$

No. 4. $45a^3d^2-450a^2bc+192a^2c^3+400ab^3d+165ab^2c^2-400b^4c$

$$x=\int\frac{dt}{\sqrt{\{\kappa\,(1-15t^2+15t^4-t^{o})+\lambda\,(3t-10t^3+3t^5)\}}}+\mu$$

$$y=\int\frac{tdt}{\sqrt{\{\kappa\,(1-15t^2+15t^4-t^6)+\lambda\,(3t-10t^3+3t^5)\}}}+\nu$$

$$V=3a^2\delta_b+10ab\delta_c+(15ac+10b^2)\,\delta_d+(21ad+35bc)\,\delta_e$$
$$+(28ae+56bd+35c^2)\,\delta_f+\ldots$$

Chart 4.—Coefficients of Annihilator V.

1	4	3			
1	5	10			
1	6	15	10		
1	7	21	35		
1	8	28	56	35	
1	9	36	84	126	
1	10	45	120	210	126

Chart 5.—Reciprocant Transformations.

Grub	Chrysalis			Imago		
$\frac{d^2y}{dx^2}$	$\frac{d^2\phi}{dx^2}$	$\frac{d^2\phi}{dxdy}$	$\frac{d\phi}{dx}$	$\frac{d^2\Phi}{dx^2}$	$\frac{d^2\Phi}{dxdy}$	$\frac{d^2\Phi}{dxdz}$
	$\frac{d^2\phi}{dxdy}$	$\frac{d^2\phi}{dy^2}$	$\frac{d\phi}{dy}$	$\frac{d^2\Phi}{dxdy}$	$\frac{d^2\Phi}{dy^2}$	$\frac{d^2\Phi}{dydz}$
	$\frac{d\phi}{dx}$	$\frac{d\phi}{dy}$	$\bullet$	$\frac{d^2\Phi}{dxdz}$	$\frac{d^2\Phi}{dydz}$	$\frac{d^2\Phi}{dz^2}$
(a)		(M)			(H)	

$$(n-1)^2\left(\frac{d\phi}{dy}\right)^3a+H+\left\{\frac{d^2\Phi}{dx^2}\cdot\frac{d^2\Phi}{dy^2}-\left(\frac{d^2\Phi}{dxdy}\right)^2\right\}\Phi=0.$$

$\dfrac{dy}{dx}\dfrac{d^3y}{dx^3}-\dfrac{3}{2}\left(\dfrac{d^2y}{dx^2}\right)^2$ is the Schwarzian, otherwise written $tb-\dfrac{3a^2}{2}$.

Chart 6.—The H Reciprocantive Protomorph.

U	W	
$65a^4h$	$120a^3cf$	*The Vermicular Operator.*
$-975a^3bg$	$-200a^2b^2f$	$\lambda a\delta_b+\mu b\delta_c+\nu c\delta_d+\pi d\delta_e+\ldots$
$-990a^3cf$	$-195a^3de$	
$+6200a^2b^2f$	$-145a^2bce$	*Examples.*
$+4690a^2bce$	$+1000ab^3e$	
$-1540ab^3e$	$+1365a^2bd^2$	$a\delta_b+b\delta_c+c\delta_d+d\delta_e+\ldots$
$-2730a^2bd^2$	$-777a^2c^2d$	$a\delta_b+2b\delta_c+3c\delta_d+4d\delta_e+\ldots$
$+7161a^2c^2d$	$-22260ab^2cd$	$3a\delta_b+8b\delta_c+15c\delta_d+24d\delta_e+\ldots$
$+3080ab^2cd$	$+2485abc^3$	
$-24255abc^3$	$+105b^3c^2$	☞ b^4d does not appear in either U or W.
$+25410b^3c^2$		
	$H+\Lambda U+MW$	

Λ and M are arbitrary numbers.