NOTES AND REFERENCES.

101. No. V. of this paper gives a correction of a formula (18) in the paper 8, On Lagrange's Theorem.

102. I refer to this paper in my "Note on Riemann's paper 'Versuch einer allgemeinen Auffassung der Integration und Differentiation,' *Werke*, pp. 331—344." *Math. Ann.* t. XVI. (1880), pp. 81—82, for the sake of pointing out the connexion which it has with this paper of Riemann's (contained, as the Editors remark, in a MS. of his student time dated 14 Jan. 1847, and probably never intended for publication): the idea is in fact the same, Riemann considered a function of $x+h$ expanded in a doubly infinite, necessarily divergent, series of integer or fractional powers of h, according to an assigned law: and he thence deduces a theory of fractional differentiation.

114. This Memoir on Steiner's extension of Malfatti's problem is referred to by Clebsch in the paper "Anwendung der elliptischen Functionen auf ein Problem der Geometrie des Raumes," *Crelle*, t. LIII. (1857), pp. 292—308: it is there shown that my fundamental equations, p. 67, are the algebraical integrals of a system of equations

$$\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0,\quad \frac{dz}{\sqrt{Z'}}+\frac{dx}{\sqrt{X'}}=0,\quad \frac{dx}{\sqrt{X''}}+\frac{dy}{\sqrt{Y''}}=0,$$

the integrals of which become comparable when the quartic functions under the square roots differ only by constant factors; and expressing that this is so, he obtains the relations which I assumed to exist between the coefficients α, β, γ, δ, &c., under which the equations admit of solution by quadratics only. And he is thereby led to reduce the problem, not to the foregoing system of fundamental equations, but to other equations connecting themselves with the *usual* form of the Addition-theorem; and with a view thereto to develope a new solution of the Problem.

115, 116. The theory is further developed in my Memoir "On the Porism of the in-and-circumscribed Polygon," *Phil. Trans.* t. CLI., for 1861.

119. I attach some value to the process here explained: the most simple application is that referred to at the end of the paper, for the factorial binomial theorem; to multiply $m+n$ by $m+n-1$, we multiply the m by $(m-1)+n$, and the n by $m+(n-1)$, thus obtaining the result in the form $m(m-1)+2mn+n(n-1)$, and so in other cases.

121. The papers and works relating to the Question are

1. Boole. Proposed Question in the Theory of Probabilities, *Camb. and Dubl. Math. Jour.* t. VI. (1851), p. 286.

2. Cayley. 121, Note on a Question in the Theory of Probabilities, *Phil. Mag.* t. VI. (1853), p. 259.

3. Boole. Solution of a Question in the Theory of Probabilities, *Phil. Mag.* t. VII. (1854), pp. 29—32.

4. Boole. An Investigation of the Laws of Thought, on which are founded the Mathematical Theories of Logic and Probabilities, 8vo. London and Cambridge, 1854 (see in particular pp. 321—326).

5. Wilbraham. On the Theory of Chances developed in Prof. Boole's Laws of Thought, *Phil. Mag.* t. VII. (1854), pp. 465—476.

6. Dedekind. Bemerkungen zu einer Aufgabe der Wahrscheinlichkeitsrechnung, *Crelle*, t. L. (1855), pp. 268—271;

viz. Boole proposed the question in 1, I gave my solution in 2, Boole objected to it in 3, and gave without explanation or demonstration his solution, referring to his then forthcoming work 4, which contains (pp. 321—326) his investigation. Wilbraham in 5 defended my solution, and criticised Boole's: and finally Dedekind in 6 (which does not refer to 4 or 5) completed my solution, by determining the sign of a radical, and establishing between the data, as conditions of a possible experience, the relations $p-\beta q$ and $q-\alpha p$ neither of them negative.

I remark that although Boole in 1, 3, and 4 speaks throughout of "causes," yet it would seem that he rather means "concomitant events": I think that in his point of view the more accurate enunciation of the question would be—The probabilities of two events A and B are α and β respectively; the probability that if the event A present itself the event E will accompany it is p, and the probability that if the event B present itself the event E will accompany it is q; moreover it is assumed that the event E cannot appear in the absence of both the events A and B: required the probability of the event E.

He makes no assumption as to the independence *inter se* of A, and B: and moreover, in thus regarding A and B as events instead of causes, there is no room for regarding E as a consequence of one or the other of A and B, or of both of them.

In my solution I regard A and B as causes: I assume that they are independent causes; and further that either or both of them may act efficiently so as to produce the event E, but that the event E cannot happen unless at least one of them act efficiently, viz. it cannot happen in consequence of the conjoint separately inefficient action of the two causes. On these assumptions it appears to me that my solution, as completed by Dedekind, is correct. This would not preclude the correctness of Boole's solution, if according to what precedes we consider it as the solution of a different question: but I am unable to understand it.

I resume my own solution, completing it according to Dedekind. I write with him u instead of ρ for the required probability of the event E; the equations of the text thus are

$$p=\lambda+(1-\lambda)\,\mu\beta,\ q=\mu+(1-\mu)\,\lambda\alpha;\ u=\lambda\alpha+\mu\beta-\lambda\mu\alpha\beta,$$

and we thence deduce

$$u-\beta q=(1-\beta)\,\lambda\alpha,\ \ u-\alpha p=(1-\alpha)\,\mu\beta;$$

and then eliminating λ, μ, we find

$$u=\frac{u-\beta q}{1-\beta}+\frac{u-\alpha p}{1-\alpha}-\frac{(u-\beta q)(u-\alpha p)}{(1-\beta)(1-\alpha)},$$

or as this equation may be written

$$u^2-u\,(1-\alpha\beta+\alpha p+\beta q)+(1-\beta)\,\alpha p+(1-\alpha)\,\beta q+\alpha\beta pq=0;$$

say we have

$$u=\tfrac{1}{2}\,(1-\alpha\beta+\alpha p+\beta q-\rho),$$

where

$$\begin{aligned}\rho^2&=(1-\alpha\beta+\alpha p+\beta q)^2-4\,(1-\beta)\,\alpha p-4\,(1-\alpha)\,\beta q-4\alpha\beta pq,\\&=(1-2\alpha+\alpha\beta+\alpha p-\beta q)^2+4\alpha\,(1-\alpha)(1-\beta)(1-p),\\&=(1-2\beta+\alpha\beta-\alpha p+\beta q)^2+4\beta\,(1-\beta)(1-\alpha)(1-q),\\&=(1-\alpha\beta+\alpha p-\beta q)^2-4\alpha\,(1-\beta)(p-\beta q),\\&=(1-\alpha\beta-\alpha p+\beta q)^2-4\beta\,(1-\alpha)(q-\alpha p),\end{aligned}$$

and hence also

$$\lambda=\frac{\tfrac{1}{2}\,(1-\alpha\beta+\alpha p-\beta q-\rho)}{(1-\beta)\,\alpha},$$

$$\mu=\frac{\tfrac{1}{2}\,(1-\alpha\beta-\alpha p+\beta q-\rho)}{(1-\alpha)\,\beta}.$$

Here p, q, α, β, as probabilities, are none of them negative or greater than 1; p is the probability that, A acting, E will happen; and βq is the probability that B will act and E happen. But if A act, then even if B does not act, E may happen, or B may act and E happen, that is p is greater than or at least equal to βq, say $p-\beta q$ is not negative. And similarly $q-\alpha p$ is not negative. We thus have as conditions of a possible experience, $p-\beta q$ and $q-\alpha p$ neither of them negative.

The formulæ show that ρ^2 is real; and then further, taking for ρ its positive value, it at once appears that we have u, λ, μ no one of them negative or greater than 1, viz. the values are such as these quantities, as probabilities, ought each of them to have: and we have thus a real solution.

Boole in 1 after remarking that the quadratic equation in u may be written in the form

$$\frac{(1-\alpha p'-u)(1-\beta q'-u)}{1-u}=\alpha'\beta' \quad (p'=1-p, \text{ \&c.})$$

says that this is certainly erroneous; for in the particular case $p=1$, $q=0$ it gives $u=1$ or $u=\alpha(1-\beta)$, whereas the value should be $u=\alpha$. But observe that $p=1$, $q=0$, give $q-\alpha p, =-\alpha$, a negative value, so that the solution does not apply. If we further examine the meaning, A is a cause such that if it act then $(p=1)$ the event is sure to happen; and B is a cause(?) such that if it act then $(q=0)$ the event is sure not to happen; this is self-contradictory unless we make the new assumption that the causes A and B cannot both act. It is remarkable that even in this case my solution gives the plausible result $u=\alpha(1-\beta)$, viz. the probability of the event is the product of the probabilities of A acting, and B not acting.

In further illustration, and at the same time to examine Boole's solution, I write as follows:

	Wilbraham.	Boole.	Cayley.
ABE	ξ	$xyst$	$\alpha\beta(1-\lambda'\mu')$,
ABE'	ξ'	$xys't'$	$\alpha\beta\lambda'\mu'$
$A'BE$	η	$x'ys't$	$\alpha'\beta\mu$
$A'BE'$	η'	$x'ys't'$	$\alpha'\beta\mu'$
$AB'E$	ζ	$xy'st'$	$\alpha\beta'\lambda$
$AB'E'$	ζ'	$xy's't'$	$\alpha\beta'\lambda'$
$A'B'E$	0	0	0
$A'B'E'$	σ'	$x'y's't'$	$\alpha'\beta'$

where in the first column the accent denotes negation: ABE means that the events A, B, E all happen, ABE' that A and B each happen, E' does not happen, and so for the other symbols. And in like manner in the third and fourth columns, where the unaccented letters denote probabilities, an accented letter is the probability of the contrary event, $x'=1-x$, &c.

By hypothesis E cannot happen unless either A or B happen, that is Prob. $A'B'E=0$, or writing $A'B'E$ for the probability (and so in other cases) say $A'B'E=0$. And I then (with Wilbraham) denote the probabilities of the other seven combinations of events by ξ, ξ', η, η', ζ, ζ' and σ'; and (as before) the required probability of the event E by u.

The data of the Problem are $1=1$, $A=\alpha$, $B=\beta$, $AE=\alpha p$, $BE=\beta q$, and we have thence to find $E=u$, where on the left-hand side of the first equation 1 means $ABE+ABE'+\text{\&c.}=\xi+\xi'+\eta+\eta'+\zeta+\zeta'+\sigma'$, and similarly A means

$$ABE+ABE'+AB'E+AB'E', =\xi+\xi'+\zeta+\zeta', \text{ \&c.};$$

we thus have

$$\begin{aligned}
&\xi+\xi'+\eta+\eta'+\zeta+\zeta'+\sigma'=1\ , \\
&\xi+\xi'+\zeta+\zeta' \qquad\qquad\quad =\alpha\ , \\
&\xi+\xi'+\eta+\eta' \qquad\qquad\quad =\beta\ , \\
&\xi+\zeta \qquad\qquad\qquad\qquad\quad =\alpha p, \\
&\xi+\eta \qquad\qquad\qquad\qquad\quad =\beta q, \\
&\xi+\eta+\zeta \qquad\qquad\qquad\quad\ = u,
\end{aligned}$$

six equations for the determination of the eight quantities ξ, ξ', η, η', ζ. ζ', σ', and u.

For the determination of u, it is therefore necessary to find or assume two more equations: in my solution this is in effect done by giving to ξ, ξ', η, η', ζ, ζ', σ' the values in the fourth column, values which satisfy the six equations, and establish the two additional relations

$$\frac{\xi'}{\eta'}=\frac{\zeta'}{\sigma'},\quad \frac{\xi+\xi'}{\eta+\eta'}=\frac{\zeta+\zeta'}{\sigma'},$$

or, as these may be written,

$$\frac{A\,BE'}{A'BE'}=\frac{A\,B'E'}{A'B'E'},\quad \frac{A\,B}{A'B}=\frac{A\,B'}{A'B'};$$

these then are assumptions implicitly made in my solution; they amount to this, that the events A, B are treated as independent, *first* in the case where E does *not* happen; secondly in the case where it is not observed whether E does or does not happen.

Boole in his solution introduces what he calls logical probabilities (but what these mean, I cannot make out): viz. these are Prob. $A=x$, or say simply $A=x$; and similarly, $B=y$, $AE=s$, $BE=t$; then in the case ABE we have A, B, AE, BE, and the logical probability is taken to be $xyst$; and we obtain in like manner the other terms of the third column. And then taking ξ, ξ', η, η', ζ, ζ', σ' to be proportional to the terms of the third column, say $V\xi=xyst$, &c. and substituting in the six equations, we have six equations for the determination of x, y, s, t, V, u, and we thus arrive at the value of the required probability u.

But the assumed values of ξ, ξ', &c. give further

$$\frac{\xi}{\eta}=\frac{\zeta}{\sigma},\quad \frac{\xi'}{\eta'}=\frac{\zeta'}{\sigma'},\ \text{that is}\ \frac{A\,BE}{A'BE}=\frac{A\,B'E}{A'B'E}\ \text{and}\ \frac{A\,BE'}{A'BE'}=\frac{A\,B'E'}{A'B'E'},$$

which are assumptions made in Boole's solution. Wilbraham remarks that the second of these assumed equations, though perfectly arbitrary, is perhaps not unreasonable: it asserts that in those cases where E does not happen, the relation of independence exists between A and B, that is, provided E does not happen, A is as likely to happen whether B happens or does not happen. But that the first of these equations appears to him not only arbitrary, but eminently anomalous: no one (he thinks) can contend that it is either deduced from the data of the problem, or that the mind by the operation of any law of thought recognises it as a necessary or even a reasonable assumption.

To complete Boole's solution: the equations easily give

$$\frac{s'tx'y}{u-\alpha p}=\frac{st'xy'}{u-\beta q}=\frac{s't'}{1-u}=V;$$

and

$$\frac{s't'x'}{1-\alpha p'-u}=\frac{s't'y'}{1-\beta q'-u}=\frac{stxy}{\alpha p+\beta q-u}=V;$$

and multiplying together the first three values, and also the second three values, we have in each case the same numerator $ss'^2tt'^2xx'yy'$, and we thus obtain the equation

$$(u-\alpha p)(u-\beta q)(1-u)-(1-\alpha p'-u)(1-\beta p'-u)(\alpha p+\beta q-u)=0,$$

which, the term in u^2 disappearing, is a quadric equation; it is in fact

$$u^2(-1+\alpha p'+\beta q')+u\{1+\alpha(p-p')+\beta(q-q')-\alpha^2pp'-\beta^2qq'+\alpha\beta(-1+2p'q')\}$$
$$+\{-\alpha p-\beta q+\alpha^2pp'+\beta^2qq'+\alpha\beta(1-p'q')-(\alpha p+\beta q)\alpha\beta p'q'\}=0;$$

or, what is more simple, if we write with Boole $\alpha p=a$, $\beta q=b$, $1-\alpha p'=a'$, $1-\beta q'=b'$, $\alpha p+\beta y=c'$, then the equation is $(u-a)(u-b)(1-u)-(a'-u)(b'-u)(c'-u)=0$, that is

$$(1-a'-b')u^2-\{ab-a'b'+(1-a'-b')c'\}u+(ab-a'b'c')=0,$$

giving

$$u=\frac{ab-a'b'+(1-a'-b')c'+Q}{2(1-a'-b')},$$

where

$$Q^2=\{ab-a'b'+(1-a'-b')c'\}^2-4(1-a'-b')(ab-a'b'c').$$

We have as conditions which must be satisfied by the data, that each of the quantities a', b', c' is greater than each of the quantities a, b; or say, each of the quantities $1-\alpha p'$, $1-\beta q'$, $\alpha p+\beta q$ greater than each of the quantities αp, βq: Q^2 is then real, and taking Q positive, we have u equal to or greater than each of the three quantities and greater than each of the two quantities. The difficulties which I find in regard to this solution have been already referred to.

139. See volume I. Notes and References 13, 14, 15, 16 and 100. I have in the last of these noticed that the terms covariant and invariant were due to Sylvester: and I have referred to papers by Boole, Eisenstein, Hesse, Schläfli and Sylvester. Anterior to the present memoir 139 we have other papers by Boole and Sylvester, one by Hermite (with other papers not directly affecting the theory), a paper by Salmon. and a very important memoir by Aronhold: it will be convenient to give a list as follows:

Boole.

1. Researches on the theory of analytical transformations with a special application to the reduction of the general equation of the second order, *Camb. Math. Jour.* t. II. 1841, pp. 64—73.

2. Exposition of a general theory of linear transformations, Part I. *Camb. Math. Jour.* t. III. 1843, pp. 1—20.

Exposition of a general theory of linear transformations, Part II. *Camb. Math. Jour.* t. III. 1843, pp. 106—119.

3. Notes on linear transformations, *Camb. and Dubl. Math. Jour.* t. IV. 1845, pp. 166—171.

4. On the theory of linear transformations, *Camb. and Dubl. Math. Jour.* t. VI. 1851, pp. 87—106.

5. On the reduction of the general equation of the *n*th degree, *Camb. and Dubl. Math. Jour.* t. VI. 1851, pp. 106—113.

6. Letter to the Editor (reply to Prof. Sylvester), *Camb. and Dubl. Math. Jour.* t. VI. pp. 284, 285.

Sylvester.

1. On the intersections, contacts and other relations of two conics expressed by indeterminate coordinates, *Camb. and Dubl. Math. Jour.* t. V. 1850, pp. 262—282.

2. On a new class of theorems in elimination between quadratic functions, *Phil. Mag.* t. XXXVII. 1850, pp. 213—218.

3. On certain general properties of homogeneous functions, *Camb. and Dubl. Math. Jour.* t. VI. 1851, pp. 1—17.

4. On the intersections of two conics, *Camb. and Dubl. Math. Jour.* t. VI. 1851, pp. 18—20.

5. Reply to Prof. Boole's Observations contained in the November Number of the Journal, *Camb. and Dubl. Math. Jour.* t. VI. 1851, pp. 171—174.

6. Sketch of a memoir on Elimination, Transformation and Canonical forms, *Camb. and Dubl. Math. Jour.* t. VI. 1851, pp. 186—200.

7. On the general theory of Associated Algebraical forms, *Camb. and Dubl. Math. Jour.* t. VI. 1851, pp. 18—20.

8. On Canonical forms, 8vo. London, Bell, 1851.

9. On a remarkable discovery in the theory of Canonical forms and of hyperdeterminants, *Phil. Mag.* t. II. 1851, pp. 391—410.

10. On the Principles of the Calculus of Forms. Part I. Generation of Forms. Sect 1. On Simple Concomitance. 2. On Complex Concomitance. 3. On Commutants. Notes in Appendix (1), (2), (3), (4), (5), (6), (7), (8), *Camb. and Dubl. Math. Jour.* t. VI. 1852, pp. 52—97.

11. On the Principles of the Calculus of Forms. Sect. 4. Reciprocity, also Properties and Analogies of Certain Invariants &c. 5. Applications and Extension of the theory of the Plexus. 6. On the partial differential equations to Concomitants, Orthogonal and Plagional Invariants, &c. Notes in Appendix (9), (10), (11). Postscript, *Camb. and Dubl. Math. Jour.* t. VI. 1852, pp. 179—217.

12. Note on the Calculus of Forms, *Camb. and Dubl. Math. Jour.* t. VIII. 1853 pp. 62—64.

13. On the Calculus of Forms otherwise the theory of Invariants. Sect. 7. On Combinants, *Camb. and Dubl. Math. Jour.* t. VIII. 1853, pp. 256—269.

14. On the Calculus of Forms otherwise the theory of Invariants. Sect. 7. Continued. 8. On the reduction of a sextic function of two variables to its canonical form, *Camb. and Dubl. Math. Jour.* t. IX. 1854, pp. 85, 103.

Salmon. Exercises in the Hyperdeterminant Calculus, *Camb. and Dubl. Math. Jour.* t. IX. 1854, pp. 19—33.

Hermite. Sur la théorie des fonctions homogènes à deux indéterminées, *Camb. and Dubl. Math. Jour.* t. IX. 1854, pp. 172—217.

Aronhold. Zur Theorie der homogenen Functionen von drei Variabeln, *Crelle* t. XXXIX. 1850, pp. 140—159.

In the present Memoir 139, dropping altogether the consideration of linear transformations, I start from the notion of certain operations upon the constants and facients of a quantic, viz. if to fix the ideas we consider the case of a binary quantic $(a, b, \dots b', a' \between x, y)^m$, then there is an operation $\{y\partial_x\}, = a\partial_b + 2b\partial_c \dots + mb'\partial_{a'}$ which performed upon the quantic is tantamount to the operation $y\partial_x$: and similarly an operation $\{x\partial_y\}, = mb\partial_a + (m-1)c\partial_b \dots + a'\partial_{b'}$ which performed upon the quantic is tantamount to the operation $x\partial_y$. Or, what is the same thing, there are two operations $\{y\partial_x\} - y\partial_x$, and $\{x\partial_y\} - x\partial_y$ each of which performed upon the quantic reduces it to zero: to use an expression subsequently introduced, say each of these is an *annihilator* of the quantic. The assumed definition is that any function of the coefficients and variables which is reduced to zero by each of these operators, is a Covariant: and in particular if the function contain the coefficients only (in which case obviously the operators may be reduced to $\{y\partial_x\}$ and $\{x\partial_y\}$ respectively) the function is an Invariant.

I believe I actually arrived at the notion by the simple remark, say that $a\partial_b + 2b\partial_c$ operating upon $ac - b^2$ reduced it to zero, and that the same operation performed upon $ax^2 + 2bxy + cy^2$ reduced it to $2axy + 2by^2$ which is $= y\partial_x \{ax^2 + 2bxy + cy^2\}$. But the earliest published mention of the notion is in the year 1852 in Note 7 of Sylvester's paper on the Principles of the Calculus of Forms (Sylvester 10). Here, connecting it with the theory of linear transformations, he writes "There is one principle of *paramount* importance which has not been touched upon in the preceding pages,... The principle now in question consists in introducing the idea of *continuous* or *infinitesimal* variation into the theory. To fix the ideas suppose C to be a function of the coefficients of $\phi(x, y, z)$ such that it remains unaltered when x, y, z become respectively fx, gy, hz, where $fgh = 1$. Next suppose that C does not alter when x becomes $x + ey + \epsilon z$, where e, ϵ are indefinitely small; it is easily and obviously demonstrable that if this be true for e, ϵ indefinitely small, it must be true for all values of e, ϵ. Again suppose that C alters neither when x receives such infinitesimal increment, y and z remaining constant, nor when y and z separately receive corresponding increments z, x and x, y in the respective cases remaining constant. ...C will remain constant for any concurrent linear transformations of x, y, z when the modulus is unity. This all-important principle...also *instantaneously* gives the necessary and sufficient conditions to which an invariant of any given order of any homogeneous function whatever is subject, and thereby reduces the problem of discovering invariants to a definite form." And in section 6 of the same paper (Sylvester 11) referring to the Note, he writes "This method may also be extended to concomitants generally. M. Aronhold as I collect from private information was the first to think of the application of this method to the subject: but it was Mr Cayley who communicated to me the equations which define the invariants of functions of two variables. The method by which I obtain these equations and prove their sufficiency is my own, but I believe has been adopted by Mr Cayley in a Memoir about to be published in *Crelle's Journal* [? 100]. I have also recently been informed of a paper about to appear in *Liouville's Journal* from the pen of M. Eisenstein, where it appears that the same idea and mode of treatment have been made use of. Mr Cayley's communication to me was made in the early part of December last [1851] and my method (the result of a remark made long before) of obtaining these and the more general equations and of demonstrating their sufficiency imparted a few weeks subsequently—I believe between January and February of the present year [1852]," and then applying the principle to the binary quadric, he proceeds to consider the theory of the operator $a\dfrac{d}{db}+2b\dfrac{d}{dc}+3c\dfrac{d}{dd}+\ldots$, and the other operator with the coefficients in the reverse order, as applied to an invariant ϕ of the quantic. The theory of these operators was thus familiar to Sylvester in 1852, but it was in nowise made the foundation of the structure.

I notice as contained in the paper Boole (4), what is probably the first statement of the "provectant" process of forming an invariant; for example, from the quartic function $(a, b, c, d, e \between x, y)^4$ he derives

$$\tfrac{1}{48}(a, b, c, d, e \between \partial_y, -\partial_x)^4 \,.\, (a, b, c, d, e \between x, y)^4 = ae - 4bd + 3c^2, \text{ the quadrinvariant};$$

and similarly from the Hessian $(ac-b^2,\ 2(ad-bc),\ ae+2bd-3c^2,\ 2(be-cd),\ ce-d^2 \between x, y)^4$ is derived the cubinvariant $ace - ad^2 - b^2e + 2bcd - 3c^2$. Mention is also made of the function $A(\beta\delta-\gamma^2)+B(\beta\gamma-\alpha\delta)+C(\alpha\gamma-\beta^2)$, ($A$, B, C given quadric functions, α, β, γ, δ given cubic functions of (a, b, c, d, e, f)), which is the octinvariant Q of the binary quintic.

The papers of Sylvester contain a great number of important results which will some of them be referred to in connexion with the later Memoirs on Quantics.

Hermite's discovery of the invariant of the degree 18 of the quintic, and the demonstration of his law of reciprocity are both given in the Memoir by him which is above referred to.

147. Upon looking at any one of the Tables, for instance VIII (*a*), it will be noticed (1) that the partition symbols in the outside top line and left-hand column respectively are differently arranged, (2) that the numbers of each pair of equal numbers (see the Memoir) are not symmetrically situate, and (3) that the table is what may be called a half-square; viz. the squares above (or, in the case of a (*b*) table, those below) the sinister diagonal are all vacant; the squares in the sinister diagonal itself are all occupied by units ($+1$ or -1 as the case may be). It is possible (and that in many ways) to give the same arrangement to the partition-symbols in the outside line and column respectively, and at the same time to retain the half-square form of the table: or (what is far more important) we may with Faà di Bruno, give the same arrangement to the partition-symbols, and at the same time make the table symmetrical, viz. cause the two numbers of each pair of equal numbers to be symmetrically situate in regard to the dexter diagonal of the square—but we cannot at the same time retain accurately the half-square form of the table. The general principle is that in the outside column (or line) the partition-symbols which are conjugate to each other have symmetrical positions, while the self-conjugate symbols are collected at the middle of the column (or line); there is then in regard to these self-conjugate symbols a sort of dislocation of the sinister diagonal, the units which belong to them being transferred to the dexter diagonal, and in the sinister diagonal replaced by zeros, for instance at the crossing of the two diagonals we may have $\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline\end{array}$ instead of $\begin{array}{|c|c|}\hline & 1 \\ \hline 1 & \\ \hline\end{array}$. A Table thus arranged may be called Symmetric.

Again as remarked by Fiedler, the two corresponding tables (*a*) and (*b*) may be united into a single table; the sinister diagonal is the same for each of them, and if we then insert into the (*b*) table below the sinister diagonal the numbers of the (*a*) table, we have a table which is to be read according to the lines for the numbers above and in the sinister diagonal; and according to the columns for the numbers in and below the same diagonal. This may be called a United table: it may be unsymmetric, or be rearranged so as to be made symmetric.

The tables have been rearranged as above, and extended to the order 14: I give the following references.

Fiedler. Elemente der Neueren Geometrie &c. (1862), pp. 73 et seq. (II. to X, (*a*) and (*b*) united, unsymmetric).

Faà di Bruno. Sur les Fonctions Symétriques, *Comptes Rendus*, t. 76 (1873), pp. 163—168 (II to VIII, (*b*), symmetric, there is some error in VIII, inasmuch as it is presented without the dislocation of the sinister diagonal).

——— Théorie des Fonctions Binaires, 8vo. Turin &c. 1876. II to XI (*b*) symmetric.

Durfee. Tables of the Symmetric Functions of the Twelfthic, *Amer. Math. Jour.* t. v. (1882), pp. 45—60. XII (*a*) and (*b*) unsymmetric.

Rehorovsky. Tafeln der symmetrischen Functionen der Wurzeln und der Coefficienten-Combinationen vom Gewichte eilf und zwölf. Wien, Denks. t. 26 (1883), pp.

53—60. XI (*a*) and (*b*), XII (*a*) and (*b*): unsymmetric, united. Is referred to in the next mentioned paper.

Durfee. The Tabulation of Symmetric Functions, *Amer. Math. Jour.* t. v. (1882), pp. 348, 349. XII (*a*) and (*b*); symmetric, united.

MacMahon. Symmetric Functions of the 13[ic], *Amer. Math. Jour.* t. vi. (1884), pp. 289—300. XIII (*b*); symmetric.

Cayley. Symmetric Functions of the roots for the degree 10 for the Form $1 + bx + \frac{cx^2}{1.2} + \ldots = (1-\alpha x)(1-\beta x)(1-\gamma x)\ldots$ *Amer. Math. Jour.* t. vii. (1885), pp. 47—56. II to X (*b*), unsymmetric. The calculation of the tables for this new form (MacMahon's) of the coefficients afforded a complete verification of the (*b*) tables, showing that there was not a single error in these tables as published in the *Philosophical Transactions.*

Durfee. Symmetric Functions of the 14[ic], *Amer. Math. Jour.* t. ix. (1887), pp. 278—292. XIV (*b*) symmetric, the arrangement is different from and seemingly better than that in the tables XII (*b*) and XIII (*b*).

MacMahon. Properties of a Complete Table of Symmetric Functions, *Amer. Math. Jour.* t. x. (1888), pp. 42—46.

——— Memoir on a New Theory of Symmetric Functions, *Amer. Math. Jour.* t. xi. (1889), pp. 1—36. (*a*) and (*b*) Tables for the weights 1 to 6 and their several partitions. To explain this, observe that the general idea is to ignore the coefficients altogether, regarding them as merely particular symmetric functions of the roots: thus the (*b*) table for the weight 4 (partition 1^4) is in fact the table IV (*b*) giving the symmetric functions (4), (31), (2^2), (21^2), (1^4) in terms of (1^4), $(1^3)(1)$, $(1^2)^2$, $(1^2)(1)^2$, $(1)^4$, that is in terms of the combinations e, bd, c^2, b^2c, b^4 of the coefficients, but that the other tables weight 4 to a different partition, give the values of symmetric functions (combinations of the foregoing) which are expressible in terms of other symmetric functions of the roots: for instance weight 4 (partition 21^2) gives (4), (31), (2^2), and (21^2) in terms of (21^2), $(21)(1)$, $(2)(1^2)$ and $(2)(1)^2$. A leading idea in this valuable memoir is that of the "Separations" of a Partition.

150. The theory is developed in an incomplete form. If to fix the ideas we consider a quintic equation $(a, b, c, d, e, f \between x, 1)^5 = 0$, then a single equality $\alpha = \beta$ between the roots implies a onefold relation between the coefficients (a, b, c, d, e, f): this is completely and precisely expressed by means of a single equation ($\nabla = 0$, where ∇ is the discriminant, $= a^4f^4 + \&c.$). Similarly a system of two equalities $\alpha = \beta = \gamma$, or $\alpha = \beta$, $\gamma = \delta$ as the case may be, implies a twofold relation between the coefficients (a, b, c, d, e, f) and the question arises, to determine the order of this twofold relation, and to find how it can be completely and precisely expressed, whether by two equations $A = 0$, $B = 0$, or if need be by a larger number of equations $A = 0$, $B = 0$, $C = 0$, &c. between the coefficients; this is not done in the memoir,

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but what is done is only to find two or more equations satisfied in virtue of the system of the two equalities between the roots. And similarly in the case of a system of more than two equalities. See my paper 77, where this notion of the order of a system of equations was established.

152. The next later memoir on the theory of Matrices, so far as I am aware is that by Laguerre, "Sur le Cacul des Systèmes Lineaires," *Jour. Ec. Polyt.* t. XXV. (1867), pp. 215—264. A "système lineaire" is what I called a matrix, and the mode of treatment is throughout very similar to that of my memoir; in particular we have in it my theorem of the equation satisfied by a matrix of any order. The memoir contains a theorem relating to the integral functions of *two* matrices A, B of the same order, viz. this is expressible in the form $m + pA + qB + rAB$. For later developments see the papers by Sylvester in the *American Mathematical Journal*.

158. The notion of the "Absolute" was I believe first introduced in the present memoir. In reference to the theory of distance founded upon it and here developed, I refer to the papers

Klein, Ueber die sogenannte Nicht-Euklidische Geometrie, *Math. Ann.* t. IV. (1871), pp. 573—625.

Cayley, On the Non-Euclidian Geometry, *Math. Ann.* t. V. (1872), pp. 630—634.

Klein, Ueber die sogenannte Nicht-Euklidische Geometrie, *Math. Ann.* t. VI. (1873), pp. 112—145.

In his first paper Klein substitutes, for my $\cos^{-1}$ expression for the distance between two points, a logarithmic one; viz. in linear geometry if the two fixed points are A, B then the assumed definition for the distance of any two points P, Q is

$$\text{dist.}(PQ) = c\log\frac{AP\,.\,BQ}{AQ\,.\,BP};$$

this is a great improvement, for we at once see that the fundamental relation, $\text{dist.}(PQ) + \text{dist.}(QR) = \text{dist.}(PR)$, is satisfied: in fact we have

$$\text{dist.}(QR) = c\log\frac{AQ\,.\,BR}{AR\,.\,BQ},$$

and thence

$$\text{dist.}(PQ) + \text{dist.}(QR) = c\log\frac{AP\,.\,BR}{AR\,.\,BQ}, = \text{dist.}\,PR.$$

But in my Sixth Memoir, the question arises, what is meant by "coordinates": if in linear geometry (x, y) are the coordinates of a point P, does this mean that $x : y$ is the ratio of the distances in the ordinary sense of the word of the point P from two fixed points A, B: and if so, does the notion of distance in the new sense ultimately depend on that of distance in the ordinary sense? And similarly in Klein's definition, do AP, BQ, AQ, BP denote distances in the ordinary sense of the word, and if so does the notion of distance in the new sense ultimately depend on that of distance in the ordinary sense?

As to my memoir, the point of view was that I regarded "coordinates" not as distances or ratios of distances, but as an assumed fundamental notion, not requiring or admitting of explanation. It recently occurred to me that they might be regarded as mere numerical values, attached arbitrarily to the point, in such wise that for any given point the ratio $x : y$ has a determinate numerical value, and that to any given numerical value of $x : y$ there corresponds a single point. And I was led to interpret Klein's formulæ in like manner; viz. considering A, B, P, Q as points arbitrarily connected with determinate numerical values a, b, p, q, then the logarithm of the formula would be that of $(a-p)(b-q) \div (a-q)(b-p)$. But Prof. Klein called my attention to a reference (p. 132 of his second paper) to the theory developed in Staudt's Geometrie der Lage, 1847 (more fully in the Beiträge zur Geometrie der Lage, Zweites Heft, 1857). The logarithm of the formula is $\log(A, B, P, Q)$, and, according to Staudt's theory (A, B, P, Q), the anharmonic ratio of any four points, has independently of any notion of distance the fundamental properties of a numerical magnitude, viz. any two such ratios have a sum and also a product, such sum and product being each of them a like ratio of four points determinable by purely descriptive constructions. The proof is easiest for the product: say the ratios are (A, B, P, Q) and (A', B', P', Q'): then considering these as given points we can construct R, such that $(A', B', P', Q') = (A, B, Q, R)$: the two ratios are thus (A, B, P, Q) and (A, B, Q, R), and we say that their product is (A, B, P, R) {observe as to this that introducing the notion of distance, the two factors are $\dfrac{AP.BQ}{AQ.BP}$ and $\dfrac{AQ.BR}{AR.BQ}$ and thus their product $= \dfrac{AP.BR}{AR.BP}$, which is (A, B, P, R), which is the foundation of the definition}. Next for the sum, we construct $Q_{,}$ such that $(A', B', P', Q') = (A, B, P, Q_{,})$; the sum then is $(A, B, P, Q) + (A, B, P, Q_{,})$; and if we then construct S such that (A, A), $(Q, Q_{,})$, (B, S) are an involution, we say that $(A, B, P, Q) + (A, B, P, Q_{,}) = (A, B, P, S)$. {Observe as to this that again introducing the notion of distance the last mentioned equation is $\dfrac{AP.BQ}{AQ.BP} + \dfrac{AP.BQ_{,}}{AQ_1.BP} = \dfrac{AP.BS}{AS.BP}$, that is $\dfrac{BQ}{AQ} + \dfrac{BQ_{,}}{AQ_{,}} = \dfrac{BS}{AS}$, which expresses that S is determined as above; in fact the equation $\dfrac{b-q}{a-q} + \dfrac{b-q_{,}}{a-q_{,}} = \dfrac{b-s}{a-s}$ is readily seen to be equivalent to

$$\begin{vmatrix} 1, & b+s, & bs \\ 1, & 2a, & a^2 \\ 1, & q+q_1, & qq_1 \end{vmatrix} = 0\}.$$

It must however be admitted that, in applying this theory of Staudt's to the theory of distance, there is at least the appearance of arguing in a circle, since the construction for the product of the two ratios, is in effect the assumption of the relation,

$$\text{dist. } PQ + \text{dist. } QR = \text{dist. } PR.$$

I may refer also to the Memoir, Sir R. S. Ball "On the theory of the Content," *Trans. R. Irish Acad.* vol. XXIX. (1889), pp. 123—182, where the same difficulty is discussed. The opening sentences are—"In that theory [Non-Euclidian Geometry] it seems as if we try to replace our ordinary notion of distance between two points by the logarithm of a certain anharmonic ratio. But this ratio itself involves the notion of distance measured in the ordinary way. How then can we supersede the old notion of distance by the Non-Euclidian notion, inasmuch as the very definition of the latter involves the former?"

An extensive list of papers is given, Halsted, Bibliography of Hyper-Space and of Non-Euclidean Geometry, *Amer. Math. Jour.* t. I. (1878), pp. 261—276 and 384—385, also t. II. (1879), pp. 65—70.

END OF VOL. II.