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ON THE PORISM OF THE IN-AND-CIRCUMSCRIBED TRIANGLE, AND ON AN IRRATIONAL TRANSFORMATION OF TWO TERNARY QUADRATIC FORMS EACH INTO ITSELF.

[From the *Philosophical Magazine*, vol. IX. (1855), pp. 513—517.]

THERE is an irrational transformation of two ternary quadratic forms each into itself, based upon the solution of the following geometrical problem,

Given that the line

$$lx+my+nz=0$$

meets the conic

$$(a, b, c, f, g, h \between x, y, z)^2=0$$

in the point (x_1, y_1, z_1); to find the other point of intersection.

The solution is exceedingly simple. Take (x_2, y_2, z_2) for the coordinates of the other point of intersection, we must have identically with respect to x, y, z,

$$(a, \ldots \between x, y, z)^2 . (\mathfrak{A}, \ldots \between l, m, n)^2 - k(lx+my+nz)^2$$
$$=(a, \ldots \between x_1, y_1, z_1 \between x, y, z) . (a, \ldots \between x_2, y_2, z_2 \between x, y, z)$$

to a constant factor *près*.

Assume successively $x, y, z=\mathfrak{A}, \mathfrak{H}, \mathfrak{G}$; $\mathfrak{H}, \mathfrak{B}, \mathfrak{F}$; $\mathfrak{G}, \mathfrak{F}, \mathfrak{C}$; it follows that

$$\begin{aligned} x_2 : y_2 : z_2 = {} & y_1 z_1 \{\mathfrak{A}(\mathfrak{A}, \ldots \between l, m, n)^2 - (\mathfrak{A}l+\mathfrak{H}m+\mathfrak{G}n)^2\} \\ & : z_1 x_1 \{\mathfrak{B}(\mathfrak{A}, \ldots \between l, m, n)^2 - (\mathfrak{H}l+\mathfrak{B}m+\mathfrak{F}n)^2\} \\ & : x_1 y_1 \{\mathfrak{C}(\mathfrak{A}, \ldots \between l, m, n)^2 - (\mathfrak{G}l+\mathfrak{F}m+\mathfrak{C}n)^2\}; \end{aligned}$$

or, what is the same thing,

$$x_2 : y_2 : z_2 = y_1z_1\,(bn^2 + cm^2 - 2fmn) \\ : z_1x_1\,(cl^2 + an^2 - 2gnl) \\ : x_1y_1\,(am^2 + bn^2 - 2hlm).$$

It is not necessary for the present purpose, but it may be as well to give the corresponding solution of the problem:

Given that one of the tangents through the point (ξ, η, ζ) to the conic

$$(a, b, c, f, g, h \between x, y, z)^2 = 0$$

is the line $l_1x + m_1y + n_1z = 0$; to find the equation to the other tangent.

Let $l_2x + m_2y + n_2z = 0$ be the other tangent, then

$$(a, \ldots \between \xi, \eta, \zeta)^2 \cdot (a, \ldots \between x, y, z)^2 - \{(a \ldots \between \xi, \eta, \zeta \between x, y, z)\}^2 \\ = (l_1x + m_1y + n_1z)\,(l_2x + m_2y + n_2z)$$

to a constant factor *près*. Assume successively $y = 0, z = 0$; $z = 0, x = 0$; $x = 0, y = 0$; then we have

$$l_2 : m_2 : n_2 = m_1n_1\,\{a\,(a, \ldots \between \xi, \eta, \zeta)^2 - (a\xi + h\eta + g\zeta)^2\} \\ : n_1\,l_1\,\{b\,(a, \ldots \between \xi, \eta, \zeta)^2 - (h\xi + b\eta + f\zeta)^2\} \\ : l_1\,m_1\,\{c\,(a, \ldots \between \xi, \eta, \zeta)^2 - (g\xi + f\eta + c\zeta)^2\};$$

or, as they may be more simply written,

$$l_2 : m_2 : n_2 = m_1n_1\,(\mathfrak{B}\zeta^2 + \mathfrak{C}\eta^2 + 2\mathfrak{F}\eta\zeta) \\ : n_1\,l_1\,(\mathfrak{C}\xi^2 + \mathfrak{A}\zeta^2 - 2\mathfrak{G}\zeta\xi) \\ : l_1\,m_1\,(\mathfrak{A}\eta^2 + \mathfrak{B}\xi^2 - 2\mathfrak{H}\xi\eta).$$

Returning now to the solution of the first problem, I shall for the sake of simplicity consider the formulæ obtained by taking for the equation of the conic,

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 0.$$

We see, therefore, that if this conic be intersected by the line $lx + my + nz = 0$ in the points (x_1, y_1, z_1) and (x_2, y_2, z_2), then

$$x_2 : y_2 : z_2 = y_1z_1\,(\gamma m^2 + \alpha n^2) \\ : z_1x_1\,(\alpha n^2 + \beta l^2) \\ : x_1y_1\,(\beta l^2 + \alpha m^2).$$

We have, in fact, *identically*

$$ly_1z_1(\beta n^2+\gamma m^2)+mz_1x_1(\gamma l^2+\alpha n^2)+nx_1y_1(\alpha m^2+\beta l^2)$$
$$=(\alpha mnx_1+\beta nly_1+\gamma lmz_1)(lx_1+my_1+nz_1)-lmn(\alpha x_1^2+\beta y_1^2+\gamma z_1^2).$$

$$\alpha y_1^2z_1^2(\beta n^2+\gamma m^2)^2+\beta z_1^2x_1^2(\gamma l^2+\alpha n^2)^2+\gamma x_1^2y_1^2(\alpha m^2+\beta l^2)^2$$
$$=\alpha\beta\gamma\{-l^3x_1^3-m^3y_1^3-n^3z_1^3$$
$$+(my_1+nz_1)l^2x_1^2+(nz_1+lx_1)m^2y_1^2+(lx_1+my_1)n^2y_1^2-2lmnx_1y_1z_1\}(lx_1+my_1+nz_1)$$
$$-(l^4\beta\gamma x_1^2+m^4\gamma\alpha y_1^2+n^4\alpha\beta z_1^2)(\alpha x_1^2+\beta y_1^2+\gamma z_1^2);$$

which show that if $lx_1+my_1+nz_1=0$ and $\alpha x_1^2+\beta y_1^2+\gamma_1 z^2=0$, then also $lx_2+my_2+nz_2=0$ and $\alpha x_2^2+\beta y_2^2+\gamma z_2^2=0$: this is, of course, as it should be.

I shall now consider l, m, n as *given functions* of x_1, y_1, z_1 satisfying identically the equations

$$lx_1+my_1+nz_1=0,$$
$$l^2bc+m^2ca+n^2ab=0,$$

equations which express that $lx+my+nz=0$ is the tangent from the point (x_1, y_1, z_1) to the conic $ax^2+by^2+cz^2=0$. And I shall take for α, β, γ the following values, viz.

$$\alpha=ax_1^2+by_1^2+cz_1^2-a(x_1^2+y_1^2+z_1^2),$$
$$\beta=ax_1^2+by_1^2+cz_1^2-b(x_1^2+y_1^2+z_1^2),$$
$$\gamma=ax_1^2+by_1^2+cz_1^2-c(x_1^2+y_1^2+z_1^2);$$

so that x_1, y_1, z_1 continuing absolutely indeterminate, we have *identically* $\alpha x_1^2+\beta y_1^2+\gamma z_1^2=0$. Also taking Θ as a function of x_1, y_1, z_1, the value of which will be subsequently given, I write

$$x_2=\Theta y_1z_1(\beta n^2+\gamma m^2),$$
$$y_2=\Theta z_1x_1(\gamma l^2+\alpha n^2),$$
$$z_2=\Theta x_1y_1(\alpha m^2+\beta l^2);$$

so that x_1, y_1, z_1 are arbitrary, and x_2, y_2, z_2 are taken to be determinate functions of x_1, y_1, z_1. The point (x_2, y_2, z_2) is geometrically connected with the point (x_1, y_1, z_1) as follows, viz. (x_2, y_2, z_2) is the point in which the tangent through (x_1, y_1, z_1) to the conic $ax^2+by^2+cz^2=0$ meets the conic passing through the point (x_1, y_1, z_1) and the points of intersection of the conics $ax^2+by^2+cz^2=0$ and $x^2+y^2+z^2=0$. Consequently, in the particular case in which (x_1, y_1, z_1) is a point on the conic $x^2+y^2+z^2=0$, the point (x_2, y_2, z_2) is the point in which this conic is met by the tangent through (x_1, y_1, z_1) to the conic $ax^2+by^2+cz^2=0$.

It has already been seen that $lx_1+my_1+nz_1=0$ and $\alpha x_1^2+\beta y_1^2+\gamma z_1^2=0$ identically; consequently we have identically $lx_2+my_2+nz_2=0$ and $\alpha x_2^2+\beta y_2^2+\gamma z_2^2=0$. The latter equation, written under the form

$$(ax_1^2+by_1^2+cz_1^2)(x_2^2+y_2^2+z_2^2)-(x_1^2+y_1^2+z_1^2)(ax_2^2+by_2^2+cz_2^2)=0,$$

shows that if x_2, y_2, z_2 are such that $x_2^2+y_2^2+z_2^2=x_1^2+y_1^2+z_1^2$, then that also $ax_2^2+by_2^2+cz_2^2=ax_1^2+by_1^2+cz_1^2$. I proceed to determine Θ so that we may have $x_2^2+y_2^2+z_2^2=x_1^2+y_1^2+z_1^2$. We obtain immediately

$$\frac{1}{\Theta^2}(x_2^2+y_2^2+z_2^2)=(l^4x_1^2+m^4y_1^2+n^4z_1^2)(\alpha^2x_1^2+\beta^2y_1^2+\gamma^2z_1^2)$$
$$-(\alpha^2l^4x_1^4+\beta^2m^4y_1^4+\gamma^2n^4z_1^4-2\beta\gamma m^2n^2y_1^2z_1^2-2\gamma\alpha n^2l^2z_1^2x_1^2-2\alpha\beta l^2m^2x_1^2y_1^2);$$

write for a moment

$$ax_1^2+by_1^2+cz_1^2=p,\ x_1^2+y_1^2+z_1^2=q, \text{ so that } \alpha=p-aq,\ \beta=p-bq,\ \gamma=p-cq,$$

then

$$\alpha^2x_1^2+\beta^2y_1^2+\gamma^2z_1^2=qp^2-2p\,.\,pq+(a^2x_1^2+b^2y_1^2+c^2z_1^2)\,q^2, = q\,\{(a^2x_1^2+b^2y_1^2+c^2z_1^2)\,q-p^2\},$$
$$=q\,\{(b-c)^2y_1^2z_1^2+(c-a)^2z_1^2x_1^2+(a-b)^2x_1^2y_1^2\},$$

$$\alpha^2l^4x_1^4+\beta^2m^4y_1^4+\gamma^4n^4z_1^4-2\beta\gamma m^2n^2y_1^2z_1^2-2\gamma\alpha n^2l^2z_1^2x_1^2-2\alpha\beta l^2m^2x_1^2y_1^2$$
$$=\quad p^2\,\{l^4x_1^4+m^4y_1^4+n^4z_1^4-2m^2n^2y_1^2z_1^2-2n^2l^2z_1^2x_1^2-2l^2m^2x_1^2y_1^2\}$$
$$-2pq\,\{al^4x_1^4+bm^4y_1^4+cn^4z_1^4-(b+c)\,m^2n^2y_1^2z_1^2-(c+a)\,n^2l^2z_1^2x_1^2-(a+b)\,l^2m^2x_1^2y_1^2\}$$
$$+\quad q^2\,\{a^2l^4x_1^4+b^2m^4y_1^4+c^2n^4z_1^4-2bcm^2n^2y_1^2z_1^2-2can^2l^2z_1^2x_1^2-2abl^2m^2x_1^2y_1^2\},$$

the first line of which vanishes in virtue of the equation $lx_1+my_1+nz_1=0$; we have therefore

$$\frac{1}{\Theta^2}(x_2^2+y_2^2+z_2^2)\div(x_1^2+y_1^2+z_1^2)$$
$$=(l^4x_1^2+m^4y_1^2+n^4z_1^2)\,\{(b-c)^2y_1^2z_1^2+(c-a)^2z_1^2x_1^2+(a-b)^2x_1^2y_1^2\}$$
$$+2\,(ax_1^2+by_1^2+cz_1^2)\,\{al^4x_1^4+bm^4y_1^4+cn^4z_1^4-(b+c)\,m^2n^2y_1^2z_1^2-(c+a)\,n^2l^2z_1^2x_1^2-(a+b)\,l^2m^2x_1^2y_1^2\}$$
$$-(x_1^2+y_1^2+z_1^2)\,\{a^2l^4x_1^4+b^2m^4y_1^4+c^2n^4z_1^4-2bcm^2n^2y_1^2z_1^2-2can^2l^2z_1^2x_1^2-2abl^2m^2x_1^2y_1^2\}.$$

Hence reducing the function on the right-hand side, and putting

$$(x_2^2+y_2^2+z_2^2)\div(x_1^2+y_1^2+z_1^2)=1,$$

we have

$$\frac{1}{\Theta^2}=a^2l^4x_1^6+b^2m^4y_1^6+c^2n^4z_1^6$$
$$+(c^2m^4-2b^2m^2n^2)\,y_1^4z_1^2+(a^2n^4-2c^2n^2l^2)\,z_1^4x_1^2+(b^2l^4-2a^2l^2m^2)\,x_1^4y_1^2$$
$$+(b^2n^4-2c^2m^2n^2)\,y_1^2z_1^4+(c^2l^4-2a^2n^2l^2)\,z_1^2x_1^4+(a^2m^4-2b^2l^2m^2)\,x_1^2y_1^4$$
$$+\{l^4(b-c)^2+m^4(c-a)^2+n^4(a-b)^2$$
$$+2m^2n^2(bc-ca-ab)+2n^2l^2(-bc+ca-ab)+2l^2m^2(-bc-ca+ab)\}\,x_1^2y_1^2z_1^2.$$

The value of Θ might probably be expressed in a more simple form by means of the equations $lx_1+my_1+nz_1=0$ and $l^2bc+m^2ca+n^2ab=0$, even without solving these equations; but this I shall not at present inquire into.

Recapitulating, l, m, n are considered as functions of x_1, y_1, z_1 determined (to a common factor *près*) by the equations

$$lx_1 + my_1 + nz_1 = 0,$$
$$l^2bc + m^2ca + n^2ab = 0;$$

Θ is determined as above, and then writing

$$\alpha = ax_1^2 + by_1^2 + cz_1^2 - a(x_1^2 + y_1^2 + z_1^2),$$
$$\beta = ax_1^2 + by_1^2 + cz_1^2 - b(x_1^2 + y_1^2 + z_1^2),$$
$$\gamma = ax_1^2 + by_1^2 + cz_1^2 - c(x_1^2 + y_1^2 + z_1^2),$$

we have

$$x_2 = \Theta y_1 z_1 (\beta n^2 + \gamma m^2),$$
$$y_2 = \Theta z_1 x_1 (\gamma l^2 + \alpha n^2),$$
$$z_2 = \Theta x_1 y_1 (\alpha m^2 + \beta l^2);$$

and these values give

$$lx_2 + my_2 + nz_2 = 0,$$
$$x_2^2 + y_2^2 + z_2^2 = x_1^2 + y_1^2 + z_1^2,$$
$$ax_2^2 + by_2^2 + cz_2^2 = ax_1^2 + by_1^2 + cz_1^2.$$

In connexion with the subject I may add the following transformation, viz. if

$$3\sqrt{\alpha}\, x' = \sqrt{3\beta}\,(y - z) + \sqrt{(3\alpha - 2\beta)(x^2 + y^2 + z^2) + 2\beta(yz + zx + xy)},$$

then reciprocally

$$3\sqrt{\beta}\, x = -\sqrt{3\alpha}\,(y' - z') + \sqrt{(3\beta - 2\alpha)(x'^2 + y'^2 + z'^2) + 2\alpha(y'z' + z'x' + x'y')}.$$

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2,$$
$$\beta(x^2 + y^2 + z^2 - yz - zx - xy) = \alpha(x'^2 + y'^2 + z'^2 - y'z' - z'x' - x'y').$$

Suppose $1 + \rho + \rho^2 = 0$, then

$$x^2 + y^2 + z^2 - yz - zx - xy = (x + \rho y + \rho^2 z)(x + \rho^2 + \rho z);$$

and in fact

$$3\sqrt{\alpha}\,(x' + \rho y' + \rho^2 z') = -\sqrt{3\beta}\,(1 + 2\rho)(x + \rho y + \rho^2 z),$$
$$3\sqrt{\alpha}\,(x' + \rho^2 y' + \rho z') = \sqrt{3\beta}\,(1 + 2\rho)(x + \rho^2 y + \rho z).$$

The preceding investigations have been in my possession for about eighteen months.

2 *Stone Buildings, April* 18, 1855.