formation. In fact, the generating lines through P meeting the sections θ, ϕ in the points A, B' respectively and the remaining generating lines through A, B respectively meeting as before in the point Q, then P and Q will correspond to each other improperly, and in like manner P' and Q will correspond to each other improperly; i.e. P and P' will correspond to each other properly. The relation between P, P' may be expressed by saying that these points are opposite angles of the skew hexagon $PAB'P'A'B$ lying upon the surface, and having the opposite angles A, A' in the section θ, and the opposite angles B, B' in the section ϕ. It is, however, clear from what precedes, that the points P, P' lie in a section passing through the points of intersection of θ, ϕ, and thus the proper transformation so obtained is not the general proper transformation.

128.

DEVELOPMENTS ON THE PORISM OF THE IN-AND-CIRCUMSCRIBED POLYGON.

[From the *Philosophical Magazine*, vol. VII. (1854), pp. 339—345.]

I PROPOSE to develope some particular cases of the theorems given in my paper, "Correction of two Theorems relating to the Porism of the in-and-circumscribed Polygon" (*Phil. Mag.* vol. VI. (1853), [116]). The two theorems are as follows:

THEOREM. The condition that there may be inscribed in the conic $U=0$ an infinity of n-gons circumscribed about the conic $V=0$, depends upon the development in ascending powers of ξ of the square root of the discriminant of $\xi U+V$; viz. if this square root be

$$A+B\xi+C\xi^2+D\xi^3+E\xi^4+F\xi^5+G\xi^6+H\xi^7+\ldots,$$

then for $n=3$, 5, 7, &c. respectively, the conditions are

$$\left| C \right| = 0, \quad \begin{vmatrix} C, & D \\ D, & E \end{vmatrix} = 0, \quad \begin{vmatrix} C, & D, & E \\ D, & E, & F \\ E, & F, & G \end{vmatrix} = 0, \text{ \&c.};$$

and for $n=4$, 6, 8, &c. respectively, the conditions are

$$\left| D \right| = 0, \quad \begin{vmatrix} D, & E \\ E, & F \end{vmatrix} = 0, \quad \begin{vmatrix} D, & E, & F \\ E, & F, & G \\ F, & G, & H \end{vmatrix} = 0, \text{ \&c.}$$

THEOREM. In the case where the conics are replaced by the two circles

$$x^2+y^2-R^2=0,\quad (x-a)^2+y^2-r^2=0,$$

then the discriminant, the square root of which gives the series

$$A+B\xi+C\xi^2+D\xi^3+E\xi^4+\&c.,$$

is

$$(1+\xi)\{r^2+\xi(r^2+R^2-a^2)+\xi^2R^2\}.$$

Write for a moment

$$A+B\xi+C\xi^2+D\xi^3+E\xi^4+\&c.=\sqrt{(1+a\xi)(1+b\xi)(1+c\xi)},$$

then

$$\begin{aligned}
A&=1,\\
2B&=a+b+c,\\
-8C&=a^2+b^2+c^2-2bc-2ca-2ab,\\
16D&=a^3+b^3+c^3-a^2(b+c)-b^2(c+a)-c^2(a+b)+2abc,\\
-128E&=5a^4+5b^4+5c^4-4a^3(b+c)-4b^3(c+a)-4c^3(a+b)\\
&\quad+4a^2bc+4b^2ca+4c^2ab-2b^2c^2-2c^2a^2-2a^2b^2,\\
&\&c.
\end{aligned}$$

To adapt these to the case of the two circles, we have to write

$$r^2(1+a\xi)(1+b\xi)(1+c\xi)=(1+\xi)\{r^2+\xi(r^2+R^2-a^2)+\xi^2R^2\},$$

and therefore

$$\begin{aligned}
c&=1,\\
r^2(a+b)&=r^2+R^2-a^2,\\
r^2ab&=R^2;
\end{aligned}$$

values which after some reductions give

$$\begin{aligned}
A&=1,\\
r^2\cdot 2B&=2r^2+R^2-a^2,\\
-r^4\cdot 8C&=(R^2-a^2)^2-4R^2r^2,\\
r^6\cdot 16D&=(R^2-a^2)\{(R^2-a^2)^2-2r^2(R^2+a^2)\},\\
-r^8\cdot 128E&=5(R^2-a^2)^4-8(R^2-a^2)^2(R^2+2r^2)r^2+16a^4r^4.
\end{aligned}$$

Hence also

$$\begin{aligned}
r^{12}\cdot 1024(CE-D^2)=\{5(R^2-a^2)^4-8(R^2-a^2)^2(R^2+2r^2)r^2+16a^4r^4\}\{(R^2-a^2)^2-4R^2r^2)\}\\
-4\{(R^2-a^2)^3-2(R^2-a^2)(R^2+a^2)r^2\}^2,
\end{aligned}$$

which after all reductions is

$$\begin{aligned}&(R^2-a^2)^6\\ &-12R^2(R^2-a^2)^4r^2\\ &+16R^2(R^2+2a^2)(R^2-a^2)^2r^4\\ &-64R^2a^4r^6.\end{aligned}$$

Hence the condition that there may be, inscribed in the circle $x^2+y^2-R^2=0$ and circumscribed about the circle $(x-a)^2+y^2-r^2=0$, an infinity of n-gons, is for $n=3$, 4, 5, i.e. in the case of a triangle, a quadrangle and a pentagon respectively, as follows.

I. For the triangle, the relation is

$$(R^2-a^2)^2-4R^2r^2=0,$$

which is the completely rationalized form (the simple power of a radius being of course analytically a radical) of the well-known equation

$$a^2=R^2-2Rr,$$

which expresses the relation between the radii R, r of the circumscribed and inscribed circles, and the distance a between their centres.

II. For the quadrangle, the relation is

$$(R^2-a^2)^2-2r^2(R^2+a^2)=0,$$

which may also be written

$$(R+r+a)(R+r-a)(R-r+a)(R-r-a)-r^4=0.$$

(Steiner, *Crelle*, t. II. [1827] p. 289.)

III. For the pentagon, the relation is

$$(R^2-a^2)^6-12R^2(R^2-a^2)^4r^2+16R^2(R^2+2A^2)(R^2-a^2)^2r^4-64R^2a^4r^6=0,$$

which may also be written

$$(R^2-a^2)^2\{(R^2-a^2)^2-4R^2r^2\}^2-4R^2r^2\{(R^2-a^2)^2-4a^2r^2\}^2=0.$$

The equation may therefore be considered as the completely rationalized form of

$$(R^2-a^2)^3+2R(R^2-a^2)^2r-4R^2(R^2-a^2)r^2-8Ra^2r^3=0.$$

This is, in fact, the form given by Fuss in his memoir "De polygonis symmetrice irregularibus circulo simul inscriptis et circumscriptis," *Nova Acta Petrop.* t. XIII. [1802] pp. 166—189 (I quote from Jacobi's memoir, to be presently referred to). Fuss puts $R+a=p$, $R-a=q$, and he finds the equation

$$\frac{p^2q^2-r^2(p^2+q^2)}{r^2q^2-p^2(r^2+q^2)}=\pm\sqrt{\frac{q-r}{q+p}},$$

which, he remarks, is satisfied by $r=-p$ and $r=\dfrac{pq}{p+q}$, and that consequently the rationalized equation will divide by $p+r$ and $pq-r(p+q)$; and he finds, after the division,

$$p^3q^3+p^2q^2(p+q)r-pq(p+q)^2r^2-(p+q)(p-q)^2r^3=0,$$

which, restoring for p, q their values $R+a$, $R-a$, is the very equation above found.

The form given by Steiner (*Crelle*, t. II. p. 289) is

$$r(R-a)=(R+a)\sqrt{(R-r+a)(R-r-a)}+(R+a)\sqrt{(R-r-a)\,2R},$$

which, putting p, q instead of $R+a$, $R-a$, is

$$qr=p\sqrt{(p-r)(q-r)}+p\sqrt{(q-r)(q+p)};$$

and Jacobi has shown in his memoir, "Anwendung der elliptischen Transcendenten u. s. w.," *Crelle*, t. III. [1828] p. 376, that the rationalized equation divides (like that of Fuss) by the factor $pq-(p+q)r$, and becomes by that means identical with the rational equation given by Fuss.

In the case of two concentric circles $a=0$, and putting for greater simplicity $\dfrac{R^2}{r^2}=M$, we have

$$A+B\xi+C\xi^2+D\xi^3+E\xi^4+\&c.=(1+\xi)\sqrt{1+M\xi}.$$

This is, in fact, the very formula which corresponds to the general case of two conics having double contact. For suppose that the polygon is inscribed in the conic $U=0$, and circumscribed about the conic $U+P^2=0$, we have then to find the discriminant of $\xi U+U+P^2$, i.e. of $(1+\xi)U+P^2$. Let K be the discriminant of U, and let F be what the polar reciprocal of U becomes when the variables are replaced by the coefficients of P, or, what is the same thing, let $-F$ be the determinant obtained by bordering K (considered as a matrix) with the coefficients of P. The discriminant of $(1+\xi)U+P^2$ is $(1+\xi)^3K+(1+\xi)^2F$, i.e. it is

$$(1+\xi)^2\{K(1+\xi)+F\},\ =(K+F)(1+\xi)^2(1+M\xi),$$

where $M=\dfrac{K}{K+F}$; or, what is the same thing, M is the discriminant of U divided by the discriminant of $U+P^2$. And M having this meaning, the condition of there being inscribed in the conic $U=0$ an infinity of n-gons circumscribed about the conic $U+P^2=0$, is found by means of the series

$$A+B\xi+C\xi^2+D\xi^3+E\xi^4+\&c.=(1+\xi)\sqrt{1+M\xi}.$$

We have, therefore,

$$A = 1,$$

$$2B = M + 2,$$

$$-8C = M^2 - 4M,$$

$$16D = M^3 - 2M^2,$$

$$-128E = 5M^4 - 8M^3,$$

&c.

$$1024(CE - D^2) = M^4(M^2 - 12M + 16),$$

&c.

Hence for the triangle, quadrangle and pentagon, the conditions are—

I. For the triangle,

$$M + 2 = 0.$$

II. For the quadrangle,

$$M - 4 = 0.$$

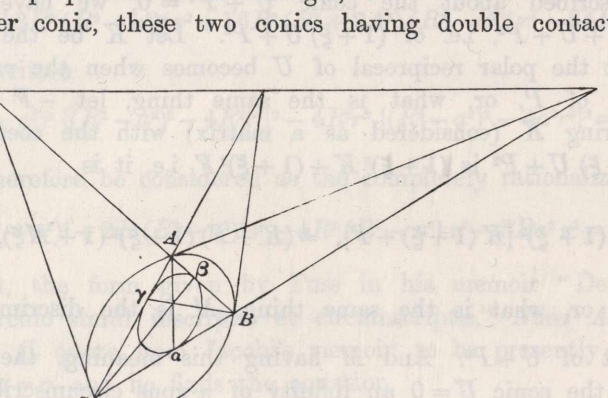
III. For the pentagon,

$$M^2 - 12M + 16 = 0;$$

and so on.

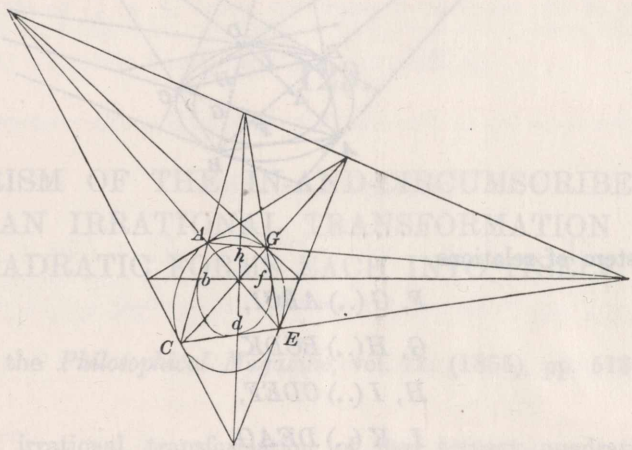
It is worth noticing, that, in the case of two conics having a 4-point contact, we have $F = 0$, and consequently $M = 1$. The discriminant is therefore $(1 + \xi)^3$, and as this does not contain any variable parameter, the conics cannot be determined so that there may be for a given value of n (nor, indeed, for any value whatever of n) an infinity of n-gons inscribed in the one conic, and circumscribed about the other conic.

The geometrical properties of a triangle, &c. inscribed in a conic and circumscribed about another conic, these two conics having double contact with each other,



are at once obtained from those of the system in which the two conics are replaced by concentric circles. Thus, in the case of a triangle, if ABC be the triangle, and α, β, γ be the points of contact of the sides with the inscribed conic, then the tangents to the circumscribed conic at A, B, C meet the opposite sides BC, CA, AB in points lying in the chord of contact, the lines $A\alpha$, $B\beta$, $C\gamma$ meet in the pole of contact, and so on.

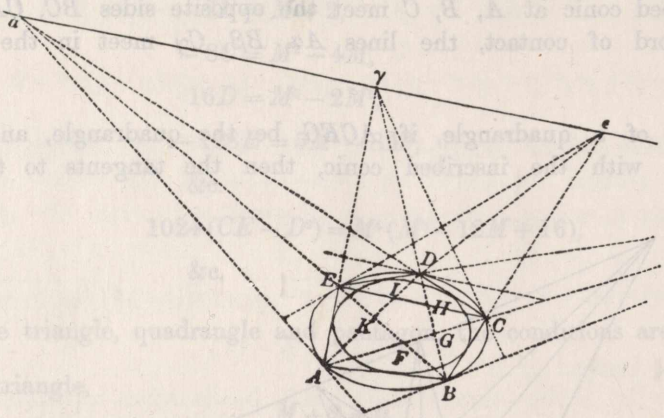
In the case of a quadrangle, if $ACEG$ be the quadrangle, and b, d, f, h the points of contact with the inscribed conic, then the tangents to the circumscribed



conic at the pair of opposite angles A, E and the corresponding diagonal CG, and in like manner the tangents at the pair of opposite angles C, G and the corresponding diagonal AE, meet in the chord of contact. Again, the pairs of opposite sides AC, EG, and the line dh joining the points of contact of the other two sides with the inscribed conic, and the pairs of opposite sides AG, CE, and the line bf joining the points of contact of the other two sides with the inscribed conic, meet in the chord of contact. The diagonals AE, CG, and the lines bf, dh through the points of contact of pairs of opposite sides with the inscribed conic, meet in the pole of contact, &c.

The beautiful systems of 'focal relations' for regular polygons (in particular for the pentagon and the hexagon), given in Sir W. R. Hamilton's Lectures on Quaternions, [Dublin, 1853] Nos. 379—393, belong, it is clear, to polygons which are inscribed in and circumscribed about two conics having double contact with each other. In fact, the focus of a conic is a point such that the lines joining such point with the circular points at infinity (i.e. the points in which a circle is intersected by the line infinity) are tangents to the conic. In the case of two concentric circles, these are to be considered as touching in the circular points at infinity; and consequently, when the concentric circles are replaced by two conics having double contact, the circular points at infinity are replaced by the points of contact of the two conics.

Thus, in the figure (which is simply Sir W. R. Hamilton's figure 81 put into



perspective), the system of relations

$$F,\ G\,(..)\,ABCI,$$
$$G,\ H\,(..)\,BCDK,$$
$$H,\ I\,(..)\,CDEF,$$
$$I,\ K\,(..)\,DEAG,$$
$$K,\ F\,(..)\,EABH,$$

will mean, F, $G\,(..)\,ABCI$, that there is a conic inscribed in the quadrilateral $ABCI$ such that the tangents to this conic through the points F and G pass two and two through the points of contact of the circumscribed and the inscribed conics, and similarly for the other relations of the system. As the figure is drawn, the tangents in question are of course (as the tangents through the foci in the case of the two concentric circles) imaginary.

2 *Stone Buildings, March* 7, 1854.