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ON THE HOMOGRAPHIC TRANSFORMATION OF A SURFACE OF THE SECOND ORDER INTO ITSELF.

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I PASS to the improper transformation. Sir W. R. Hamilton has given (in the note, p. 723 of his Lectures on Quaternions [Dublin, 1853)] the following theorem:—If there be a polygon of $2m$ sides inscribed in a surface of the second order, and $(2m-1)$ of the sides pass through given points, then will the $2m$-th side constantly touch two cones circumscribed about the surface of the second order. The relation between the extremities of the $2m$-th side is that of two points connected by the general improper transformation; in other words, if there be on a surface of the second order two points such that the line joining them touches two cones circumscribed about the surface of the second order, then the two points are as regards the transformation in question a pair of corresponding points, or simply a pair. But the relation between the two points of a pair may be expressed in a different and much more simple form. For greater clearness call the surface of the second order U, and the sections along which it is touched by the two cones, θ, ϕ; the cones themselves may, it is clear, be spoken of as the cones θ, ϕ. And let the two points be P, Q. The line PQ touches the two cones, it is therefore the line of intersection of the tangent plane through P to the cone θ, and the tangent plane through P to the cone ϕ. Let one of the generating lines through P meet the section θ in the point A, and the other of the generating lines through P meet the section ϕ in the point B. The tangent planes through P to the cones θ, ϕ respectively are nothing else than the tangent planes to the surface U at the points A, B respectively. We have therefore at these points two generating lines meeting in the point P; the other two generating lines at the points A, B meet in like manner in the point Q. Thus P, Q are opposite angles of a skew quadrangle formed by four generating lines (or, what is the same thing, lying upon the surface of the second order), and having its other two angles, one of them on the section θ and the other on the section ϕ; and if we consider the side PA as belonging determinately to one or the other of the two systems of generating lines, then when P is given, the corresponding point Q is, it is clear, completely determined. What precedes may be recapitulated in the statement, that in the improper transformation of a surface of the second order into itself, we have, as corresponding points, the opposite angles of a skew quadrangle lying upon the surface, and having the other two opposite angles upon given plane sections of the surface. I may add, that attending only to the sections through the points of intersection of θ, ϕ, if the point P be situate anywhere in one of these sections, the point Q will be always situate in the other of these sections, i.e. the sections correspond to each other in pairs; in particular, the sections θ, ϕ are corresponding sections, so also are the sections Θ, Φ (each of them two generating lines) made by tangent planes of the surface. Any three pairs of sections form an involution; the two sections which are the sibiconjugates of the involution are of course such, that, if the point P be situate in either of these sections, the corresponding point Q will be situate in the same section. It may be noticed that when the two sections θ, ϕ coincide, the line joining the corresponding points passes through a fixed point, viz. the pole of the plane of the coincident sections; in fact the lines PQ and AB are in every case reciprocal polars, and in the present case the line AB lies in a fixed plane, viz. the plane of the coincident sections, the line PQ passes therefore through the pole of this plane. This agrees with the remarks made in the first part of the present paper.

The analytical investigation in the case where the surface of the second order is represented under the form $xy - zw = 0$ is so simple, that it is, I think, worth while to reproduce it here, although for several reasons I prefer exhibiting the final result in relation to the form $x^2 + y^2 + z^2 + w^2 = 0$ of the equation of the surface of the second order. I consider then the surface $xy - zw = 0$, and I take $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$ for the coordinates of the poles of the two sections θ, ϕ, and also (x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2) as the coordinates of the points P, Q. We have of course $x_1y_1 - z_1w_1 = 0$, $x_2y_2 - z_2w_2 = 0$. The generating lines through P are obtained by combining the equation $xy - zw = 0$ of the surface with the equation $xy_1 + yx_1 - zw_1 - wz_1 = 0$ of the tangent plane at P. Eliminating x from these equations, and replacing in the result x_1 by its value $\frac{z_1w_1}{y_1}$, we have the equation

$$(yz_1 - zy_1)(yw_1 - wy_1) = 0.$$

We may if we please take $yz_1 - zy_1 = 0$, $xy_2 + yx_1 - zw_1 - wz_1 = 0$ as the equations of the line PA; this leads to

$$\left.\begin{array}{l} yz_1 - zy_1 = 0, \\ xy_1 + yx_1 - zw_1 - wz_1 = 0, \end{array}\right\} \quad \left.\begin{array}{l} yw_2 - wy_2 = 0, \\ xy_2 + yx_2 - zw_2 - wz_2 = 0, \end{array}\right\}$$

for the equations of the lines PA, QA respectively; and we have therefore the coordinates of the point A, coordinates which must satisfy the equation

$$\beta x + \alpha y - \delta z - \gamma w = 0$$

of the plane θ. This gives rise to the equation

$$y_2(\alpha y_1 - \delta z_1) - w_2(\gamma y_1 - \beta z_1) = 0.$$

We have in like manner

$$\left.\begin{array}{r} yw_1 - y_1 w = 0, \\ xy_1 + yx_1 - zw_1 - wz_1 = 0, \end{array}\right\} \quad \left.\begin{array}{r} yz_2 - zy_2 = 0, \\ xy_2 + yx_2 - zw_2 - wz_2 = 0, \end{array}\right\}$$

for the equations of the lines PB, QB respectively; and we may thence find the coordinates of the point B, coordinates which must satisfy the equation

$$\beta' x + \alpha' y - \delta' z - \gamma' w = 0$$

of the plane ϕ. This gives rise to the equation

$$y_2(\alpha' y_1 - \gamma' w_1) - z_2(\delta' y_1 - \beta' w_1).$$

It is easy, by means of these two equations and the equation $x_2 y_2 - z_2 w_2 = 0$, to form the system

$$\begin{aligned} x_2 &= (\alpha y_1 - \delta z_1)(\alpha' y_1 - \gamma' w_1), \\ y_2 &= (\gamma y_1 - \beta z_1)(\delta' y_1 - \beta' w_1), \\ z_2 &= (\gamma y_1 - \beta z_1)(\alpha' y_1 - \gamma' w_1), \\ w_2 &= (\alpha y_1 - \delta z_1)(\delta' y_1 - \beta' w_1); \end{aligned}$$

or, effecting the multiplications and replacing $z_1 w_1$ by $x_1 y_1$, the values of x_2, y_2, z_2, w_2 contain the common factor y_1, which may be rejected. Also introducing on the left-hand sides the common factor MM', where $M^2 = \alpha\beta - \gamma\delta$, $M'^2 = \alpha'\beta' - \gamma'\delta'$, the equations become

$$\begin{aligned} MM'x_2 &= \gamma'\delta x_1 + \alpha\alpha' y_1 - \alpha'\delta z_1 - \alpha\gamma' w_1, \\ MM'y_2 &= \beta\beta' x_1 + \gamma\delta' y_1 - \beta\delta' z_1 - \beta'\gamma w_1, \\ MM'z_2 &= \beta\gamma' x_1 + \gamma\alpha' y_1 - \beta\alpha' z_1 - \gamma\gamma' w_1, \\ MM'w_2 &= \beta'\delta x_1 + \alpha\delta' y_1 - \delta\delta' z_1 - \alpha\beta' w_1, \end{aligned}$$

values which give identically $x_2 y_2 - z_2 w_2 = x_1 y_1 - z_1 w_1$. Moreover, by forming the value of the determinant, it is easy to verify that the transformation is in fact an improper one. We have thus obtained the equations for the improper transformation of the surface $xy - zw = 0$ into itself. By writing $x_1 + iy_1$, $x_1 - iy_1$ for x_1, y_1, &c., we have the following system of equations, in which (a, b, c, d), (a', b', c', d') represent, as before, the coordinates of the poles of the plane sections, and $M^2 = a^2 + b^2 + c^2 + d^2$, $M'^2 = a'^2 + b'^2 + c'^2 + d'^2$, viz. the system[1]

[1] The system is very similar in form to, but is *essentially* different from, that which could be obtained from the theory of quaternions by writing

$$MM'(w_2 + ix_2 + jy_2 + kz_2) = (d + ia + jb + kc)(w + ix + jy + kz)(d' + ia' + jb' + kc');$$

the last-mentioned transformation is, in fact, *proper*, and not *improper*.

$$MM'x_2 = (aa' - bb' - cc' - dd')\,x_1 + (\ \ ab' + a'b + cd' - c'd)\,y_1$$
$$+ (\ \ ac' + a'c + db' - d'b)\,z_1 + (\ ad' + a'd + bc' - b'c)\,w_1,$$
$$MM'y_2 = (ab' + a'b - cd' + c'd)\,x_1 + (-aa' + bb' - cc' - dd')\,y_1$$
$$+ (\ \ bc' + b'c - da' + d'a)\,z_1 + (\ bd' + b'd - ac' + a'c)\,w,$$
$$MM'z_2 = (ac' + a'c - db' + d'b)\,x_1 + (\ \ bc' + b'c - ad' + a'd)\,y_1$$
$$+ (-aa' - bb' + cc' - dd')\,z_1 + (\ cd' + c'd - ba' + b'a)\,w_1,$$
$$MM'w_2 = (ad' + a'd - bc' + b'c)\,x_1 + (\ \ bd' + b'd - ca' + c'a)\,y_1$$
$$+ (\ \ cd' + c'd - ab' + a'b)\,z_1 + (-aa' - bb' - cc' + dd')\,w_1,$$

values which of course satisfy identically $x_2^2 + y_2^2 + z_2^2 + w_2^2 = x_1^2 + y_1^2 + z_1^2 + w_1^2$, and which belong to an improper transformation. We have thus obtained the improper transformation of the surface of the second order $x^2 + y^2 + z^2 + w^2 = 0$ into itself.

Returning for a moment to the equations which belong to the surface $xy - zw = 0$, it is easy to see that we may without loss of generality write $\alpha = \beta = \alpha' = \beta' = 0$; the equations take then the very simple form

$$MM'x_2 = \gamma'\delta x_1,\quad MM'y_2 = \gamma\delta' y_1,\quad MM'z_2 = -\gamma\gamma' w_1,\quad MM'w_2 = -\delta\delta' z_1,$$

where $MM' = \sqrt{-\gamma\delta}\,\sqrt{-\gamma'\delta'}$; and it thus becomes very easy to verify the geometrical interpretation of the formulæ.

It is necessary to remark, that, whenever the coordinates of the points Q are connected with the coordinates of the points B by means of the equations which belong to an improper transformation, the points P, Q have to each other the geometrical relation above mentioned, viz. there exist two plane sections θ, ϕ such that P, Q are the opposite angles of a skew quadrangle upon the surface, and having the other two opposite angles in the sections θ, ϕ respectively. Hence combining the theory with that of the proper transformation, we see that if A and B, B and C, ..., M and N are points corresponding to each other properly or improperly, then will N and A be points corresponding to each other, viz. properly or improperly, according as the number of the improper pairs in the series A and B, B and C, ..., M and N is even or odd; i.e. if all the sides but one of a polygon satisfy the geometrical conditions in virtue of which their extremities are pairs of corresponding points, the remaining side will satisfy the geometrical condition in virtue of which its extremities will be a pair of corresponding points, the pair being proper or improper according to the rule just explained.

I conclude with the remark, that we may by means of two plane sections of a surface of the second order obtain a proper transformation. For, if the generating lines through P meet the sections θ, ϕ in the points A, B respectively, and the remaining generating lines through A, B respectively meet the sections ϕ, θ respectively in B', A', and the remaining generating lines through B', A' respectively meet in a point P'; then will P, P' be a pair of corresponding points in a proper transformation. In fact, the generating lines through P meeting the sections θ, ϕ in the points A, B respectively, and the remaining generating lines through A, B respectively meeting as before in the point Q, then P and Q will correspond to each other improperly, and in like manner P' and Q will correspond to each other improperly; i.e. P and P' will correspond to each other properly. The relation between P, P' may be expressed by saying that these points are opposite angles of the skew hexagon $PAB'P'A'B$ lying upon the surface, and having the opposite angles A, A' in the section θ, and the opposite angles B, B' in the section ϕ. It is, however, clear from what precedes, that the points P, P' lie in a section passing through the points of intersection of θ, ϕ, and thus the proper transformation so obtained is not the general proper transformation.

2 *Stone Buildings, January* 11, 1854.