112.

DEMONSTRATION OF A THEOREM RELATING TO THE PRODUCTS OF SUMS OF SQUARES.

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Mr Kirkman, in his paper "On Pluquaternions and Homoid Products of Sums of n Squares" (*Phil. Mag.* vol. XXXIII. [1848] pp. 447—459 and 494—509), quotes from a note of mine the following passage:—"The complete test of the possibility of the product of 2^n squares by 2^n squares reducing itself to a sum of 2^n squares is the following: forming the complete systems of triplets for (2^n-1) things, if *eab, ecd, fac, fdb* be any four of them, we must have, paying attention to the signs alone,

$$(\pm\, eab)\,(\pm\, ecd) = (\pm\, fac)\,(\pm\, fdb)\,;$$

i.e. if the first two are of the same sign, the last two must be so also, and *vice versâ;* I believe that, for a system of seven, two conditions of this kind being satisfied would imply the satisfaction of all the others: it remains to be shown that the complete system of conditions cannot be satisfied for fifteen things." I propose to explain the meaning of the theorem, and to establish the truth of it, without in any way assuming the existence of imaginary units.

The identity to be established is

$$(w^2 + a^2 + b^2 + \ldots)\,(w_,^2 + a_,^2 + b_,^2 \ldots) = w_{,,}^2 + a_{,,}^2 + b_{,,}^2 + \ldots$$

where the 2^n quantities $w, a, b, c, \ldots$ and the 2^n quantities $w_,, a_,, b_,, c_,, \ldots$ are given quantities in terms of which the 2^n quantities $w_{,,}, a_{,,}, b_{,,}, c_{,,}, \ldots$ have to be determined.

Without attaching any meaning whatever to the symbols $a_\circ, b_\circ, c_\circ \ldots$ I write down the expressions

$$w + aa_\circ + bb_\circ + cc_\circ \ldots,\quad w_, + a_,a_\circ + b_,b_\circ + c_,c_\circ \ldots,$$

and I multiply as if $a_\circ$, $b_\circ$, $c_\circ \ldots$ really existed, taking care to multiply without making any transposition in the order *inter se* of two symbols $a_\circ$, $b_\circ$ combined in the way of multiplication. This gives a quasi-product

$$\begin{aligned} ww_, &+ (aw_, + a_,w)\, a_\circ + (bw_, + b_,w)\, b_\circ + \ldots \\ &+ aa_,a_\circ^2 \;\; + bb_,b_\circ^2 \;\; + \ldots \\ &+ ab_,a_\circ b_\circ + a_,bb_\circ a_\circ + \ldots. \end{aligned}$$

Suppose, now, that a quasi-equation, such as

$$a_\circ b_\circ c_\circ = +,$$

means that in the expression of the quasi-product

$$b_\circ c_\circ, \quad c_\circ a_\circ, \quad a_\circ b_\circ, \quad c_\circ b_\circ, \quad a_\circ c_\circ, \quad b_\circ a_\circ$$

are to be replaced by

$$a_\circ, \quad b_\circ, \quad c_\circ, \quad -a_\circ, \quad -b_\circ, \quad -c_\circ;$$

and that a quasi-equation, such as $a_\circ b_\circ c_\circ = -$, means that in the expression of the quasi-product

$$b_\circ c_\circ, \quad c_\circ a_\circ, \quad a_\circ b_\circ, \quad c_\circ b_\circ, \quad a_\circ c_\circ, \quad b_\circ a_\circ$$

are to be replaced by

$$-a_\circ, \quad -b_\circ, \quad -c_\circ, \quad a_\circ, \quad b_\circ, \quad c_\circ.$$

It is in the first place clear that the quasi-equation, $a_\circ b_\circ c_\circ = +$, may be written in any one of the six forms

$$\begin{aligned} a_\circ b_\circ c_\circ = +, \quad b_\circ c_\circ a_\circ = +, \quad c_\circ a_\circ b_\circ = +, \\ a_\circ c_\circ b_\circ = -, \quad c_\circ b_\circ a_\circ = -, \quad b_\circ a_\circ c_\circ = -; \end{aligned}$$

and so for the quasi-equation $a_\circ b_\circ c_\circ = -$. This being premised, if we form a system of quasi-equations, such as

$$a_\circ b_\circ c_\circ = \pm, \quad a_\circ d_\circ e_\circ = \pm, \text{ \&c.}$$

where the system of triplets contains each duad once, and once only, and the arbitrary signs are chosen at pleasure; if, moreover, in the expression of the quasi-product we replace $a_\circ^2$, $b_\circ^2, \ldots$ each by -1, it is clear that the quasi-product will assume the form

$$w_{,,} + a_{,,}a_\circ + b_{,,}b_\circ + c_{,,}c_\circ + \ldots,$$

$w_{,,}$, $a_{,,}$, $b_{,,}$, $c_{,,} \ldots$ being determinate functions of $w, a, b, c, \ldots$; $w_,$, $a_,$, $b_,$, $c_, \ldots$, homogeneous of the first order in the quantities of each set; the value of $w_{,,}$ being obviously in every case

$$w_{,,} = ww_, - aa_, - bb_, - cc_, \ldots,$$

and $a_{,,}$, $b_{,,}$, $c_{,,}, \ldots$ containing in every case the terms $aw_, + a_,w$, $bw_, + b_,w$, $cw_, + c_,w, \ldots$ but the form of the remaining terms depending as well on the triplets entering into the system of quasi-equations as on the values given to the signs $\pm$; *the quasi-equations serving, in fact, to prescribe a rule for the formation of certain functions* $w_{,,}, a_{,,}, b_{,,}, c_{,,}, \ldots$, the properties of which functions may afterwards be investigated.

Suppose, now, that the system of quasi-equations is such that

$$e_\circ a_\circ b_\circ, \quad e_\circ c_\circ d_\circ$$

being any two of its triplets, with a common symbol $e_\circ$, there occur also in the system the triplets

$$f_\circ a_\circ c_\circ, \quad f_\circ d_\circ b_\circ, \quad g_\circ a_\circ d_\circ, \quad g_\circ b_\circ c_\circ;$$

and suppose that the corresponding portion of the system is

$$e_\circ a_\circ b_\circ = \epsilon, \quad e_\circ c_\circ d_\circ = \epsilon',$$
$$f_\circ a_\circ c_\circ = \zeta, \quad f_\circ d_\circ b_\circ = \zeta',$$
$$g_\circ a_\circ d_\circ = \iota, \quad g_\circ b_\circ c_\circ = \iota',$$

where $\epsilon, \zeta, \iota, \epsilon', \zeta', \iota'$ each of them denote one of the signs + or −; then $e_{,,}, f_{,,}, g_{,,}$ will contain respectively the terms

$$\epsilon(ab_, - a_, b) + \epsilon'(cd_, - c_, d),$$
$$\zeta(ac_, - a_, c) + \zeta'(db_, - d_, b),$$
$$\iota(ad_, - a_, d) + \iota'(bc_, - b_, c);$$

and $e_{,,}{}^2 + f_{,,}{}^2 + g_{,,}{}^2$ contains the terms

$$\begin{aligned}(a^2+b^2+c^2+d^2)&(a_,{}^2+b_,{}^2+c_,{}^2+d_,{}^2) - a^2a_,{}^2 - b^2b_,{}^2 - c^2c_,{}^2 - d^2d_,{}^2\\ &+ 2\,[\epsilon\epsilon'(ab_, - a_, b)(cd_, - c_, d)\\ &+ \zeta\zeta'(ac_, - a_, c)(db_, - d_, b)\\ &+ \iota\iota'(ad_, - a_, d)(bc_, - b_, c)];\end{aligned}$$

and by taking account of the terms $ew_, + e_, w$, $fw_, + f_, w$, $gw_, + g_, w$ in $e_{,,}, f_{,,}, g_{,,}$ respectively, we should have had besides in $e_{,,}{}^2 + f_{,,}{}^2 + g_{,,}{}^2$ the terms

$$\begin{aligned}(e^2+f^2+g^2)\,w_,{}^2 &+ (e_,{}^2+f_,{}^2+g_,{}^2)\,w^2\\ &+ 2\,(ee_, + ff_, + gg_,)\,ww_,.\end{aligned}$$

Also $w_{,,}{}^2$ contains the terms

$$\begin{aligned}w^2w_,{}^2 &+ a^2a_,{}^2 + b^2b_,{}^2 + c^2c_,{}^2 + d^2d_,{}^2\\ &- 2\,(ee_, + ff_, + gg_,)\,ww_,;\end{aligned}$$

whence it is easy to see that

$$\begin{aligned}&w_{,,}{}^2 + a_{,,}{}^2 + b_{,,}{}^2 + c_{,,}{}^2 + \ldots =\\ &(w^2+a^2+b^2+c^2+\ldots)(w_,{}^2+a_,{}^2+b_,{}^2+c_,{}^2+\ldots)\\ &\quad + 2\Sigma\,[\epsilon\epsilon'(ab_, - a_, b)(cd_, - c_, d)\\ &\quad + \zeta\zeta'(ac_, - a_, c)(db_, - d_, b)\\ &\quad + \iota\iota'(ad_, - a_, d)(bc_, - b_, c)].\end{aligned}$$

7—2

where the summation extends to all the quadruplets formed each by the combination of two duads such as ab and cd, or ac and db, or ad and bc, i. e. two duads, which, combined with the same common letter (in the instances just mentioned e, or f, or g), enter as triplets into the system of quasi-equations—so that if $\nu = 2^n - 1$, the number of quadruplets is

$$\tfrac{1}{2}\{\tfrac{1}{2}(\nu-1)\,.\,\tfrac{1}{2}(\nu-3)\}\,\nu\,.\,\tfrac{1}{3}, = \tfrac{1}{24}\nu(\nu-1)(\nu-3),$$

and the terms under the sign Σ will vanish identically if only

$$\epsilon\epsilon' = \zeta\zeta' = \iota\iota'\,;$$

but the relation $\epsilon\epsilon' = \iota\iota'$ is of the same form as the equation $\epsilon\epsilon' = \zeta\zeta'$; hence if all the relations

$$\epsilon\epsilon' = \zeta\zeta'$$

are satisfied, the terms under the sign Σ vanish, and we have

$$(w_{\prime\prime}{}^2 + a_{\prime\prime}{}^2 + b_{\prime\prime}{}^2 + c_{\prime\prime}{}^2 + \ldots) = (w^2 + a^2 + b^2 + c^2 + \ldots)(w_{\prime}{}^2 + a_{\prime}{}^2 + b_{\prime}{}^2 + c_{\prime}{}^2 + \ldots)$$

which is thus shown to be true, upon the suppositions—

1. That the system of quasi-equations is such that

$$e_\circ a_\circ b_\circ,\quad e_\circ c_\circ d_\circ$$

being any two of its triplets with a common symbol $e_\circ$, there occur also in the system the triplets

$$f_\circ a_\circ c_\circ,\quad f_\circ d_\circ b_\circ,$$
$$g_\circ a_\circ d_\circ,\quad g_\circ b_\circ c_\circ.$$

2. That for any two pairs of triplets, such as

$$e_\circ a_\circ b_\circ,\quad e_\circ c_\circ d_\circ \quad\text{and}\quad f_\circ a_\circ c_\circ,\quad f_\circ d_\circ b_\circ,$$

the product of the signs of the triplets of the first pair is equal to the product of the signs of the triplets of the second pair.

In the case of fifteen things $a, b, c, \ldots$ the triplets may, as appears from Mr Kirkman's paper, be chosen so as to satisfy the first condition; but the second condition involves, as Mr Kirkman has shown, a contradiction; and therefore the product of two sums, each of them of sixteen squares, is *not* a sum of sixteen squares. It is proper to remark, that this demonstration, although I think rendered clearer by the introduction of the idea of the system of triplets furnishing the rule for the formation of the expressions $w_{\prime\prime}$, $a_{\prime\prime}$, $b_{\prime\prime}$, $c_{\prime\prime}$, &c., is not in principle different from that contained in Prof. Young's paper "On an Extension of a Theorem of Euler, &c.", *Irish Transactions*, vol. XXI. [1848 pp. 311—341].