107.

ON THE THEORY OF SKEW SURFACES.

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A SURFACE of the n^{th} order is a surface which is met by an indeterminate line in n points. It follows immediately that a surface of the n^{th} order is met by an indeterminate plane in a curve of the n^{th} order.

Consider a skew surface or the surface generated by a singly infinite series of lines, and let the surface be of the n^{th} order. Any plane through a generating line meets the surface in the line itself and in a curve of the $(n-1)^{th}$ order. The generating line meets this curve in $(n-1)$ points. Of these points one, viz. that adjacent to the intersection of the plane with the consecutive generating line, is a unique point; the other $(n-2)$ points form a system. Each of the $(n-1)$ points are *sub modo* points of contact of the plane with the surface, but the proper point of contact is the unique point adjacent to the intersection of the plane with the consecutive generating line. Thus every plane through a generating line is an ordinary tangent plane, the point of contact being a point on the generating line. It is not necessary for the present purpose, but I may stop for a moment to refer to the known theorems that the anharmonic ratio of any four tangent planes through the same generating line is equal to the anharmonic ratio of their points of contact, and that the locus of the normals to the surface along a generating line is a hyperbolic paraboloid. Returning to the $(n-2)$ points in which, together with the point of contact, a generating line meets the curve of intersection of the surface and a plane through the generating line, these are fixed points independent of the particular plane, and are the points in which the generating line is intersected by other generating lines. There is therefore on the surface a double curve intersected in $(n-2)$ points by each generating line of the surface—a property which, though insufficient to determine the order of this double curve, shows that the order cannot be less than $(n-2)$. (Thus for $n=4$, the above reasoning shows that the double-curve must be

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at least of the second order: assuming for a moment that it is in any case precisely of this order, it obviously cannot be a plane curve, and must therefore be two non-intersecting lines. This suggests at any rate the existence of a class of skew surfaces of the fourth order generated by a line which always passes through two fixed lines and by some other condition not yet ascertained; and it would appear that surfaces of the second order constitute a degenerate species belonging to the class in question.)

In particular cases a generating line will be intersected by the consecutive generating line. Such a generating line touches the double curve.

Consider now a point not on the surface; the planes determined by this point and the generating lines of the surface are the tangent planes through the point; the intersections of consecutive tangent planes are the tangent lines through the point; and the cone generated by these tangent lines or enveloped by the tangent planes is the tangent cone corresponding to the point. This cone is of the n^{th} class. For considering a line through the point, this line meets the surface in n points, i.e. it meets n generating lines of the surface; and the planes through the line and these n generating lines, are of course tangent planes to the cone: that is, n tangent planes can be drawn to the cone through a given line passing through the vertex. The cone has not in general any lines of inflexion, or, what is the same thing, stationary tangent planes. For a stationary tangent plane would imply the intersection of two consecutive generating lines of the surface. And since the number of generating lines intersected by a consecutive generating line, and therefore the number of planes through two consecutive generating lines, is finite, no such plane passes through an indeterminate point. The tangent cone will have in general a certain number of double tangent planes; let this number be x. We have therefore a cone of the class n, number of double tangent planes x, number of stationary tangent planes 0. Hence, if m be the order of the cone, α the number of its double lines, and β the number of its cuspidal or stationary lines,

$$m = n(n-1) - 2x,$$

$$\beta = 3n(n-2) - 6x,$$

$$\alpha = \tfrac{1}{2}n(n-2)(n^2-9) - 2x(n^2-n-6) + 2x(x-1).$$

This is the proper tangent cone, but the cone through the double curve is *sub modo* a tangent cone, and enters as a square factor into the equation of the general tangent cone of the order $n(n-1)$. Hence, if X be the order of the double curve, and therefore of the cone through this curve,

$$m + 2X = n(n-1), \quad \text{and therefore } X = x;$$

that is, the number of double tangent planes to the tangent cone is equal to the order of the double curve. It does not appear that there is anything to determine x; and if this is so, skew surfaces of the n^{th} order may be considered as forming different families according to the order of the double curve upon them.

To complete the theory, it should be added that a plane intersects the surface in a curve of the n^{th} order having x double points but no cusps.

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