102.

ON A DOUBLE INFINITE SERIES.

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THE following completely paradoxical investigation of the properties of the function Γ (which I have been in possession of for some years) may perhaps be found interesting from its connexion with the theories of expansion and divergent series.

Let $\Sigma_r \phi r$ denote the sum of the values of ϕr for all integer values of r from $-\infty$ to ∞. Then writing

$$u = \Sigma_r [n-1]^r x^{n-1-r}, \quad \ldots\ldots \quad (1)$$

(where n is any number whatever), we have immediately

$$\frac{du}{dx} = \Sigma_r [n-1]^{r+1} x^{n-2-r} = \Sigma_r [n-1]^r x^{n-1-r} = u;$$

that is,
$$\frac{du}{dx} = u, \text{ or } u = C_n e^x,$$

(the constant of integration being of course in general a function of n). Hence

$$C_n e^x = \Sigma_r [n-1]^r x^{n-1-r}; \quad \ldots\ldots \quad (2)$$

or e^x is expanded in general in a *doubly infinite necessarily divergent series of fractional powers of x*, (which resolves itself however in the case of n a positive or negative integer, into the ordinary singly infinite series, the value of C_n in this case being immediately seen to be Γn).

The equation (2) in its general form is to be considered as a definition of the function C_n. We deduce from it

$$\Sigma_r [n-1]^r (ax)^{n-1-r} = C_n e^{ax},$$
$$\Sigma_{r'} [n'-1]^{r'} (ax')^{n-1-r'} = C_{n'} e^{ax'};$$
$$\vdots$$

and also

$$\Sigma_k [n+n' \ldots -1]^k \{a(x+x' \ldots)\}^{n+n' \ldots -1-k} = C_{n+n' \ldots} e^{a(x+x' \ldots)}.$$

Multiplying the first set of series, and comparing with this last,

$$C_{n+n' \ldots} \Sigma_{r, r' \ldots} [n-1]^r [n'-1]^{r'} \ldots x^{n-1-r} x'^{n'-1-r'} \ldots$$
$$= C_n C_{n'} \ldots [n+n' \ldots -1]^k (x+x' \ldots)^{n+n' \ldots -1-k}, \ldots\ldots\ldots\ldots (3)$$

(where r, r' denote any positive or negative integer numbers satisfying $r+r'+\ldots = k+1-p$, p being the number of terms in the series $n, n', \ldots$). This equation constitutes a multinomial theorem of a class analogous to that of the exponential theorem contained in the equation (2).

In particular

$$C_{n+n' \ldots} \Sigma_{r, r' \ldots} [n-1]^r [n'-1]^{r'} \ldots = C_n C_{n'} \ldots [n+n' \ldots -1]^k p^{n+n' \ldots -1-k}, \ldots\ldots\ldots (4)$$

and if $p=2$, writing also m, n for n, n', and $k-1-r$ for r',

$$C_{m+n} \Sigma_r [m-1]^r [n-1]^{k-1-r} = C_m C_n [m+n-1]^k \, 2^{m+n-1-k}, \ldots\ldots\ldots\ldots (5)$$

or putting $k=0$ and dividing,

$$C_m C_n \div C_{m+n} = \frac{1}{2^{m+n-1}} \Sigma_r [m-1]^r [n-1]^{-1-r}. \ldots\ldots\ldots\ldots (6)$$

Now the series on the second side of this equation is easily seen to be convergent (at least for positive values of m, n). To determine its value write

$$\mathsf{F}(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx;$$

then

$$\mathsf{F}(m, n) = \int_0^{\frac{1}{2}} x^{m-1} (1-x)^{n-1} \, dx + \int_0^{\frac{1}{2}} x^{n-1} (1-x)^{m-1} \, dx;$$

and by successive integrations by parts, the first of these integrals is reducible to $\frac{1}{2^{m+n-1}} \Sigma_r [m-1]^r [n-1]^{-1-r}$, r extending from -1 to $-\infty$ inclusively, and the second to $\frac{1}{2^{m+n-1}} \Sigma_r [m-1]^r [n-1]^{-1-r}$, r extending from 0 to ∞; hence

$$\mathsf{F}(m, n) = \frac{1}{2^{m+n-1}} \Sigma_r [m-1]^r [n-1]^{-1-r},$$

or

$$C_m C_n \div C_{m+n} = \mathsf{F}(m, n), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

which proves the identity of C_m with the function $\Gamma(m)$. {Substituting in two of the preceding equations, we have

$$\Gamma n \Gamma n' \ldots \div \Gamma(n+n'\ldots) = \frac{1}{[n+n'\ldots-1]^k p^{n+n'\ldots-1-k}} \Sigma_{r,r'\ldots} [n-1]^r [n'-1]^{r'} \ldots, \quad \ldots(8)$$

(where, as before, p denotes the number of terms in the series $n, n', \ldots$ and $r+r'+\ldots=k+1-p$), the first side of which equation is, it is well known, reducible to a multiple definite integral by means of a theorem of M. Dirichlet's. And

$$\mathsf{F}(m, n) = \frac{1}{[m+n-1]^k 2^{m+n-1-k}} \Sigma_r [m-1]^r [n-1]^{k-1-r}, \quad \ldots\ldots\ldots\ldots(9)$$

where r extends from $-\infty$ to $+\infty$, and k is arbitrary. By giving large negative values to this quantity, very convergent series may be obtained for the calculation of $\mathsf{F}(m, n)$}.