39.

NOTE ON CERTAIN ELEMENTARY GEOMETRICAL NOTIONS AND DETERMINATIONS.

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A CURVE, as every one knows, may be regarded as a *locus* of points or as an *assembly* of directions, every point being common to two consecutive directions of the assembly, and every direction to two consecutive points of the locus; the locus is called the *envelop* of the assembly (that is part of the accepted language of geometry), and, conversely, the assembly may be called the *environment* of the locus. So we may regard a surface as an assembly of tangent planes or as a locus of points standing to each other in the relation of envelop and environment, and extend these definitions to space of any number of dimensions.

By a *plasm*, waiting a better word, we may understand a figure analogous to a point-pair in a line, a triangle in a plane, a pyramid in space, etc.; and an n-gonal plasm or n-gon will signify a plasm having n vertices and n faces themselves $(n-1)$-gons.

It is easy and desirable to find the general value of the content of a regular n-gon, say *abcde*, all whose edges we may call unity.

If $$b\beta = \tfrac{1}{2}ab, \quad c\gamma = \tfrac{2}{3}c\beta, \quad d\delta = \tfrac{3}{4}d\gamma \ldots,$$

it is easily seen by an elementary process of integration that $\beta, \gamma, \delta \ldots$ are the centres of figure to the successive plasms ab, abc, $abcd$, ..., and, making

$$ba = p_1, \quad c\beta = p_2, \quad d\gamma = p_3 \ldots,$$

each term in $p_1, p_2, p_3 \ldots$ will be perpendicular to the one which precedes it, so that, if V_n is the content of the plasm,

$$(1, 2, 3 \ldots n)^2\, V_n = p_1 p_2 \ldots p_n.$$

Moreover, we shall have

$$p_n^2 = 1 - \left(\frac{n-1}{n}\right)^2 p^2_{n-1},$$

of which the general integral is

$$p_n^2 = \frac{n+1}{2 \,.\, n} + C(-)^n \frac{1}{n^2};$$

in the present case, since $p_1 = 1$, $C = 0$, so that

$$V_n^2 = \frac{n+1}{(1 \,.\, 2 \ldots n)^2\, 2^n}.$$

If a, b, c be the angles of a fixed triangle, and A, B, C are proportional to the distances of a variable line from a, b, c, respectively, we may denote the line by $A : B : C$; as regards a variable point, it will presently be seen to be advantageous to denote its proportional coordinates, not, as is rather more usually done, by equimultiples of its distances from the three sides, but as equimultiples of these distances multiplied by the sides of the triangle from which they are measured*; so that, calling these coordinates a, b, c, the image† of the line at infinity becomes $a + b + c$.

Consider now the universal mixed concomitant (which it will be convenient to call a *mutuant*) $Aa + Bb + Cc$ (where a, b, c, A, B, C are used in lieu of the more usual letters x, y, z, ξ, η, ζ); it will readily be seen that, when a, b, c vary, and A, B, C are fixed, the mutuant images the line $A : B : C$, and that, when A, B, C vary and a, b, c are fixed, the mutuant images the *radiant* point $a : b : c$; that is to say, $Aa + Bb + Cc = 0$ is true for every point in the point-containing line $A : B : C$ in the one case, and to every line through the *radiant point* $a : b : c$ in the other.

Supposing, then, that the two kinds of coordinates are chosen in this manner, we see (what would not be the case if the simple distances were taken) that a form F and its " polar-reciprocal " ϕ image the self-same curve referred to the self-same fundamental triangle.

These consequences would moreover continue to subsist if, calling the distances of a line from the vertices P, Q, R, and of a point from the sides p, q, r, we took $\Lambda P : \mathrm{M}Q : \mathrm{N}R$, $\lambda p : \mu q : \nu r$ for the two sets of coordinates, provided only that $\lambda\Lambda F = \mu \mathrm{M} G = \nu \mathrm{N} H$; F, G, H being the distances of the sides from the vertices of the fundamental triangle, in which case the line at infinity would no longer be imaged by $a + b + c$. I shall, however, adhere in what follows to the convention above laid down. I need hardly add that in like manner, in space taking $A : B : C : D$ (the distances of a plane from the

* Or rather divided by the distances of these sides from the opposite angles of the fundamental triangle, whose coordinates thus become 1, 0, 0, 0, 1, 0, 0, 0, 1.

† If $F = 0$ is the equation to any *locus* or *assembly*, I call F the *image*, and such locus or assembly the *object*; to a given image responds in general an absolutely definite object, but, when the object is given, the image is only determined to a *constant* factor *près*.

vertices of a fundamental pyramid) as the coordinate-representation of such plane, and $a:b:c:d$ (the contents of the volumes which any variable point makes with the respective faces) as the coordinate-representation of such point, the mutuant $aA+bB+cC+dD$ will be the image of the radiant point $a:b:c:d$ when the capital letters are the variables, and of the plane $A:B:C:D$ when the small letters are the variables, meaning of course that $Aa+Bb+Cc+Dd=0$ will be true of every point in the plane $A:B:C:D$ and of every plane through the point $a:b:c:d$, and, as before, F and ϕ polar-reciprocals to each other will image the self-same surface (referred to the self-same fundamental pyramid) viewed as a locus or envelop on the one hand, as an assembly or environment on the other.

If a, b, c, d be used to signify the actual as distinguished from the proportional coordinates of a point, a linear function of these is constant, whereas it is a quadratic function of A, B, C, D..., when used to signify the actual distances of a variable line, plane, &c., from the vertices of the fundamental plasm which is constant; and it is the principal object of this note to determine the form of this quadratic function, which, as Prof. Cayley was the first to show, may be expressed by the determinant to a matrix standing in close relation to the well-known "invertebrate symmetrical matrix," the determinant to which represents a numerical multiple of any plasm in terms of its edges, as, for example:

$$\begin{vmatrix} . & ab & ac & ad & 1 \\ ba & . & bc & bd & 1 \\ ca & cb & . & cd & 1 \\ da & db & dc & . & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix}$$

where ab, ac, bc... are used for brevity to signify the measure of absolute distance between a, b, a, c, b, c..., that is, stand for what in ordinary notation would be denoted by $(ab)^2$, $(ac)^2$, $(bc)^2$, ... This may be quoted as the mutual-distance matrix; its determinant, besides representing a numerical multiplier of the squared content of the pyramid when equated to zero, expresses the conditions of the four points a, b, c, d lying in a plane, the former property being a consequence immediately deducible by *strict* algebraical reasoning from the latter.

That this determinant does image the condition of the plasm to which the points a, b, c, d ... are the vertices, losing one dimension of space, may be shown in a somewhat striking manner as follows. If for a moment we use x, y, z, the distances of any point in the plane of abc from bc, ca, ab as coordinates, the equation to a circle circumscribed about abc will be of the form $fyz+gzx+hxy$, and, calling the sides of the triangle a, b, c respectively,

$ax+by+cz$ is constant. Hence, substituting for z its value in terms of x and y, the image of the circle may be put under a form in which fb and ga will be the coefficients of y^2 and x^2 respectively; but, since x and y are *proportional* to the Cartesian coordinates y and x respectively, the coefficients of x^2 and y^2 must be equal. Hence $f:g:h::a:b:c$, and if now ax, by, cz, instead of x, y, z, be used as the coordinates of the variable point, the image to the circumscribing circle becomes $\Sigma\dfrac{ayz}{bc}$, or if we please Σa^2yz, that is, $\Sigma bcyz$, where bc stands as convened for $(bc)^2$.

Hence, if a, b, c, d be the vertices of a pyramid, $\Sigma abyz$ will be the image of the circumscribing sphere, for when any coordinate t is made zero the image becomes that of a circle; and so universally for a plasm of any number of dimensions.

Consider the case of a circle, and suppose that

$$\begin{vmatrix} . & ab & ac & 1 \\ ba & . & bc & 1 \\ ca & cb & . & 1 \\ 1 & 1 & 1 & . \end{vmatrix}$$

vanishes; this means that the line $x+y+z$ touches the circle

$$abxy+bcyz+cazx.$$

But, if $x+y+z$ images the line at infinity, it must *cut* this (as it cuts any other circle) in two distinct points, namely, the so-called circular points at infinity. Hence $x+y+z$ must, when the above determinant vanishes, cease to be the line at infinity, which can only come to pass by the triangle abc losing a dimension of space, and a, b, c coming into a straight line, in which case $x+y+z=0$, instead of being true of a particular line, is true of every point in the plane.

Just in like manner, if

$$\begin{vmatrix} . & ab & ac & ad & 1 \\ ba & . & bc & bd & 1 \\ ca & cb & . & cd & 1 \\ da & db & dc & . & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix}$$

vanishes, unless $x+y+z+t$ ceases to image the plane at infinity, this plane would touch the sphere $\Sigma abxy$, that is, would cut it in a pair of straight lines, whereas it intersects it in a circle. Consequently the plasm $abcd$ must, as before, lose one dimension, and so in general. The content of a plasm vanishes when the mutual-distance determinant does so, and the latter as well as the former may be expressed rationally in terms of ordinary Cartesian coordinates; but the expression for the content (being linear in each set of coordinates) is obviously indecomposable, and must therefore be a numerical multiple of some power of the mutual-distance determinant; a comparison of dimensions shows at once that this power is the square root.

As regards the numerical multiplier, when the plasm has all its edges equal to unity (say a triangle, for example), the mutual-distance determinant becomes

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix},$$

which is easily transformable into

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & \bar{1} & 0 & 0 \\ 1 & 0 & \bar{1} & 0 \\ 1 & 0 & 0 & \bar{1} \end{vmatrix},$$

of which the value is -3; and so in general for a regular plasm with $(n+1)$ vertices; that is, in space of n dimensions the mutual-distance determinant, say D_n, becomes $(-)^{n+1}(n+1)$, whereas the (volume)², say V_n^2, has been shown to be $\dfrac{n+1}{2^n(1\,.\,2 \ldots n)^2}$.

Hence, universally,

$$D_n = (-)^{n+1}\, 2^n\, (1\,.\,2 \ldots n)^2\, V_n^2.$$

It may be here noticed that, if p be the perpendicular from any vertex on an opposite face of the plasm whose content is V_{n-1}, we shall have

$$V_{n-1}p = nV_n.$$

Consequently, $$D_{n-1}p^2 = (-)^n\, 2^{n-1}\, \{1\,.\,2 \ldots (n-1)\}^2\, V^2_{n-1}\, p^2$$
$$= (-)^n\, 2^{n-1}\, (1\,.\,2 \ldots n)^2\, V_n^2 = -\tfrac{1}{2}D_n.$$

I now pass on to the leading motive of this note, namely, the determination of the connection between the coordinates $A, B, C \ldots$ drawn from $a, b, c \ldots$.

It is clear *à priori* that the form of the condition will be in all cases that a homogeneous quadratic function of the distances must be constant. Thus, for example, when there are four points, if A, B, C be assumed, we may describe three spheres with these quantities as radii, and the fourth point will be determined by means of one of the pairs of tangent planes drawn to them, the particular pair depending on the relative signs attributed to

A, B, C. Hence, if $F(A, B, C, D) = \infty$ be the general equation, each of the quantities must enter in the second and no higher degree; moreover, since by transporting the plane from which the distances are measured parallel to itself, A, B, C, D will be all increased by the same quantity, F must express a function of their differences, and consequently, since any two distances may be interchanged, F can contain no terms of the first order in the variables, so that $F = 0$ must amount to the predication of a homogeneous quadratic function of the distances being constant.

Thus, for example, in the case of three points, we have the well-known equation

$$\Sigma (ab)(A-C)(B-C) = \tfrac{1}{4}(abc)^2.$$

Suppose now that A, B, C are taken in proportions consistent with making

$$\Sigma (ab)^2 (A-C)(B-C) = 0.$$

Let $\Sigma (ab)^2 (A-C)(B-C) = P \,.\, Q$, where P, Q are two linear functions of A, B, C; then P, Q image two radiant points, each of which will have the property that any of its rays is at an infinite distance from a, b, c, or at all events, if it should pass through one of them, from the other two, and it is easy to anticipate that these two points must be the circular points at infinity. That such is the fact is obvious, because (using Cartesian co-ordinates) the perpendicular distance from any point upon $x \pm \sqrt{-1} \,.\, y$ contains zero in its denominator; so that the two points of the absolute may be regarded as the centres of two points of rays, all of them infinitely distant from the finite region.

But these two points are the intersections of the circumscribing circle with the line at infinity, and consequently their collective equation will be found by taking the resultant of $\Sigma ab\,xy$, Σx, ΣAx, which is well known to be the determinant of the quadratic function bordered by the coefficients of the two linear ones. Hence the constant quadratic function in A, B, C, namely, $\Sigma ab(A-B)(A-C)$, ought to be a numerical multiple of the determinant

$$\begin{vmatrix} . & A & B & C & . \\ A & . & ab & ac & 1 \\ B & ba & . & bc & 1 \\ C & ca & cb & . & 1 \\ . & 1 & 1 & 1 & . \end{vmatrix},$$

as is the case, the value of this determinant being

$$-2\Sigma ab(A-C)(B-C).$$

The same thing may be shown in a more elementary manner as follows. Combining

$$x + y + z = 0, \quad abxy + bcyz + cazx = 0,$$

we have $$acx^2 + (bc + ca - ab)\,xy + bcy^2 = 0,$$

at each point of the absolute. And, taking $x_1y_1z_1$, $x_2y_2z_2$ as the coordinates at these two points, it follows that

$$x_1x_2 : y_1y_2 : z_1z_2 : x_1y_2 + x_2y_1 : y_1z_2 + y_2z_1 : z_1x_2 + z_2x_1$$
$$:: bc : ca : ab : - bc - ca + ab : - ca - ab + bc : - ab - bc + ca.$$

And, as the two points will be imaged by

$$x_1A + y_1B + z_1C, \quad x_2A + y_2B + z_2C,$$

respectively, it follows that their collective image will be

$$\Sigma\,\{bcA^2 + (bc - ab - ac)\,BC\},$$

which is easily seen to be identical with

$$\Sigma bc\,(A - B)\,(A - C).$$

The universal algebraical theorem upon which the first method of proof depends is the well-known one that, if Q is a quadratic function and $L_1, L_2, \ldots L_i$ i linear functions of j variables, and if Q' (where j is not less than $i + 1$) is what Q becomes when i of its variables are expressed in terms of the rest, then the necessary and sufficient condition of the discriminant of *every* such Q' vanishing is that the determinant to Q bordered by the coefficients of the i linear functions shall vanish. When j is equal to $i + 1$, the theorem shows that the resultant of the quadratic and its i attendant linear functions will be the bordered determinant in question. In the above example we had $j = 3$, $i = 2$.

Let us now proceed to apply a similar principle to the case of four points a, b, c, d in space.

If we take the case $x^2 + y^2 + z^2 + t^2 = 0$, any tangent plane to it at x', y', z', t' will be

$$x'x + y'y + z'z + t't,$$

and, as $$x'^2 + y'^2 + z'^2 + t'^2 = 0,$$

it follows that every tangent plane will be at infinite distance from any point external to it; and, as this is true wherever the centre of the cone be placed, and all the cones so obtained have the "circle at infinity" in common, —it follows that every tangent plane to the circle at infinity is infinitely distant from any external point in the finite region,—the infinitely-infinite system of planes thus obtained one may regard, if one pleases, as consisting of sheaves of planes whose axes form the environment to the circle at infinity, and will be the correlative to the infinitely-infinite system of points in the plane at infinity, which are infinitely distant from all external planes in the finite region. We see, then, that the coordinates to each such plane must satisfy the condition that, on making $\Sigma x = 0$ and $\Sigma Ax = 0$, and expressing any two of the variables x, y, z, t in terms of the two others, the discriminant of the form then assumed by $\Sigma abxy$ must vanish, and consequently, as before, the mutual-distance determinant to the points a, b, c, d, bordered with a row and column of units and a row and column consisting of the letters A, B, C, D, will represent to a numerical factor *près* the constant quadratic function of distances, that is, this function will be

$$\begin{vmatrix} . & A & B & C & D & . \\ A & . & ab & ac & ad & 1 \\ B & ba & . & bc & bd & 1 \\ C & ca & cb & . & cd & 1 \\ D & da & db & dc & . & 1 \\ . & 1 & 1 & 1 & 1 & . \end{vmatrix},$$

and obviously a similar algebraical conclusion will continue to apply, whatever may be the number of points n in a space of $n-1$ dimensions.

As regards the value of the constant, in any case, that may be obtained by taking a face of the plasm as the *term* (line, plane, etc.) from which the distances $A, B, C \ldots$, are measured; that is, we may make $B=0$, $C=0$, $D=0 \ldots$, provided we make A equal to the perpendicular from a on the opposite face. The value of the bordered determinant then becomes the *negative* of the squared perpendicular from a on $bcd \ldots$ multiplied by the mutual-distance determinant to $bcd \ldots$; that is, by virtue of what has previously been shown, will be half of the mutual-distance determinant of $abcd \ldots$.

Hence the complete relation between A, B, C, D may be exhibited by making

$$\begin{vmatrix} -\frac{1}{2} & A & B & C & D & . \\ A & . & ab & ac & ad & 1 \\ B & ba & . & bc & bd & 1 \\ C & ca & cb & . & cd & 1 \\ D & da & db & dc & . & 1 \\ . & 1 & 1 & 1 & 1 & . \end{vmatrix} = 0,$$

and similarly for any number of points.

Professor Cayley has obtained the same result by a more direct but not more instructive process, as follows. Taking, by way of example, three points, $A+k$, $B+k$, $C+k$, (where k is infinite,) may be regarded as the distances of a, b, c from a fourth point at an infinite distance, and accordingly we may write

$$\begin{vmatrix} . & ab & ac & (A+k)^2 & 1 \\ ba & . & bc & (B+k)^2 & 1 \\ ca & cb & . & (C+k)^2 & 1 \\ (A+k)^2 & (B+k)^2 & (C+k)^2 & . & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix} = 0.$$

For the *gnomon* bordering the square formed by the small letters and dots, we may substitute

$$\begin{vmatrix} . & . & . & 2kA+A^2 & 1 \\ . & . & . & 2kB+B^2 & 1 \\ . & . & . & 2kC+C^2 & 1 \\ 2kA+A^2 & 2kB+B^2 & 2kC+C^2 & -2k^2 & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix},$$

without altering the value of the determinant, which therefore, remembering that k is infinite, is in a ratio of equality to $(2k)^2$ multiplied into the determinant

$$\begin{vmatrix} . & ab & ac & A & 1 \\ ba & . & bc & B & 1 \\ ca & cb & . & C & 1 \\ A & B & C & -\frac{1}{2} & . \\ 1 & 1 & 1 & . & . \end{vmatrix}.$$

This last determinant therefore must vanish, agreeing with what has been shown above by a more purely geometrical method*. I will now proceed to develop this determinant deprived of its constant term, expressing it as a function of the differences of the capital letters.

It is obvious that it may be expressed as a sum of terms of which each variable part will be of one or the other of these three forms

$$(A-B)^2,\quad (A-B)(A-C),\quad (A-B)(C-D);$$

and accordingly we may distribute the totality of the terms of the constant function of difference into three families depending on the form of the variable argument.

In general, if we consider any *invertebrate* symmetrical determinant expressed by the *umbral* notation

$$\begin{vmatrix} aa & ab & ac \dots al \\ ba & bb & bc \dots bl \\ \dots & \dots & \dots \quad \dots \\ la & lb & lc \dots ll \end{vmatrix},$$

* As a corollary, we may infer, from the vanishing of this determinant, that, using the notation previously employed,

$$\frac{D_n}{V_n{}^2} = -\tfrac{1}{2}n^2\frac{D_{n-1}}{V^2_{n-1}},$$

and consequently that $D_n = -(2)^n(1.2\dots n)^2 V_n{}^2$,

and that thus the content of a regular plasm with unit edges and $(n+1)$ vertices is

$$\frac{n+1}{2^n(1.2\dots n)^2},\ \text{namely,}\ \frac{3}{16},\ \frac{1}{72},\ \frac{5}{9.2^{10}}\dots$$

for triangle, pyramid, plu-pyramid, etc.

where $aa = bb = cc = ll \ldots = 0$ and $pq = qp$, we have this simple rule of proceeding:

Divide the letters $a \ldots l$ in every possible manner into cyclical sets, each set containing at least two letters.

Any cycle $a_1 a_2 \ldots a_i$ is to be interpreted as meaning

$$a_1 a_2 \,.\, a_2 a_3 \ldots a_{i-1} a_i \,.\, a_i a_1,$$

which, by virtue of the supposed condition $ab = ba$, will be the same in whichever direction the cycle is read, the effect of the inversion of the cycle being merely to give the same product over again, written under the form $a_1 a_i \,.\, a_2 a_1 \ldots a_1 a_{i-1}$.

The cycle of two letters $a_1 a_2$ must be interpreted to mean $(a_1 a_2)^2$. If now $C_1 C_2 \ldots C_i$ are cycles of two letters each, and $\chi_1 \chi_2 \cdots \chi_j$ cycles of three or more letters, the total value of the determinant will be

$$\Sigma\, (-)^{n+i+j}\, 2^j\, C_1 C_2 \ldots C_i \chi_1 \chi_2 \cdots \chi_j.$$

If, the principal diagonal terms remaining zero, the other terms were general, then the expression of the value of the determinant, calling the cycles $C_1 C_2 \ldots C_\nu$, and making no distinction between the case of their being binary or super-binary, would be $\Sigma\,(-)^{n+\nu} C_1 C_2 \ldots C_\nu$; only it would have to be understood that each cycle of two letters, as (ab), would mean $(ab)^2$, but a cycle of three or more letters, as (abc), would mean $ab \,.\, bc \,.\, ca + ac \,.\, cb \,.\, ba$.

This being premised, it is easy to deduce the following rule for the determination of the *three* different families of terms belonging to the constant determinant of distances, which, to avoid prolixity, must be left to the reader to verify.

FAMILY I.—Omitting any two letters, and forming all possible cyclical products with the remaining $(n-2)$ letters, if $C_1 C_2 \ldots C_\nu$ be any set thereof, and ν' the number of them containing more than two letters, the general term will be $\Sigma\Sigma\,(-)^{n+\nu} 2^{\nu'} C_1 \,.\, C_2 \ldots C_\nu (A-B)^2$, a, b being the two letters which do not occur in the cycles $C_1 C_2 \ldots C_\nu$.

FAMILY II.—Omitting any one letter, and forming with the remaining $n-1$ letters, in every possible way, *a chain* χ containing two or more letters, and cycles $C_1 C_2 \ldots C_\nu$, then, supposing the chain to be $bcd \ldots kl$, and understanding by (χ) the product $bc \,.\, cd \ldots kl$, the general term will be

$$\Sigma\Sigma\,(-)^{n+\nu}\, 2^{\nu'+1} C_1 C_2 \ldots C_\nu\, (\chi)\, (A-B)\, (A-L),$$

a being the letter which does not appear in the chain or any of the cycles, and ν' meaning as before the number of the cycles which contain at least three elements.

FAMILY III.—Form all the letters in every possible way into two *chains* (each containing two or more letters) χ, χ', and into cycles $C_1, C_2, \ldots C_\nu$;

then, supposing the initial and final letters of χ to be a, h, and of χ' to be k, l, the general term of this family will be

$$2\Sigma(-)^{n+\nu}2^{\nu'+1}C_1C_2\ldots C_\nu(\chi)(\chi')\{(A-K)(H-L)+(A-L)(H-K)\}.$$

I subjoin in the following table the *types* of the coefficients of the several families for all the values of n from 2 up to 7; the vacant cycle ( ) of course means unity, and a cycle (ab) means $(ab)^2$; that is, the *fourth* power of the length ab.

Every cycle enclosed in a parenthesis of three or more letters, will be understood to be affected with a coefficient 2, and for greater brevity the variable part of each term is *left to be supplied.* A round parenthesis indicates a cycle, a square parenthesis a chain.

Number of Letters	Types	Name of Family
2	$(\ )$	1st
3	(bc)	2nd
4	$-(cd)$	1st
„	$2[bcd]$	2nd
„	$2[ab].[cd]$	3rd
5	(cde)	1st
„	$-2[bcde]:2(bc)[de]$	2nd
„	$-2[ab][cde]$	3rd
6	$-(cdef):(cd)(ef)$	1st
„	$-2(bcd)[ef]:-2(bc)[def]:-2[bcdef]$	2nd
„	$-2(ab)[cd][ef]:[abc][def]:[ab][cdef]$	3rd
7	$(cdefg)-(cd)(efg)$	1st
„	$2(bcde)[fg]:2(bcd)[efg]:-2(bc)(de)[fg]\ 2(bc)[defg]:-2[bcdefg]$	2nd
„	$2(abc)[de][fg]:2(ab)[cd][efg]:-2[abc][defg]:-2[abcdefg]$	3rd

Thus, for example, the constant function of distances for three points in a plane is $2\Sigma bc(A-B)(A-C)$; for four points in space is

$$-\Sigma cd(A-B)^2+2\Sigma bc.cd(A-B)(A-D)$$
$$+2\Sigma ab.cd\{(A-C)(B-D)+(A-D)(B-C)\};$$

for five points in hyper-space is

$$2\Sigma(cd.de.ec)(A-B)^2-2\Sigma(bc.cd.de)(A-B)(A-E)$$
$$+2(bc)^2(de)(A-D)(B-E)$$
$$-2\Sigma ab.cd.de.ec\{(A-C)(B-E)+(A-E)(B-C)\}.$$

The part of the constant function of distances for seven points belonging to the 2nd family of terms will be

$$4\Sigma bc.cd.de.eb.fg(A-B)(A-E)+4\Sigma bc.cd.db.ef.fg(A-E)(A-G)$$
$$-2(bc)^2(de)^2fg(A-F)(A-G)+2(bc)^2(de.ef.fg)(A-D)(A-G)$$
$$-2\,bc.cd.de.ef.fg(A-B)(A-G).$$

The number of types in each family for n points is easily expressible by a generating function.

Obviously in the 1st family this number is the number of ways of resolving n into parts none less than 2; that is, it is the coefficient of x^{n-2} in

$$\frac{1}{1-x^2 \,.\, 1-x^3 \,.\, 1-x^4 \dots}.$$

In the 2nd family, it is the sum of the number of ways of decomposing $n-3$, $n-4$, ... into parts none less than 2; that is, it is the coefficient of x^{n-3} in

$$\frac{1+x+x^2+\dots}{(1-x^2)(1-x^3)\dots}, \text{ that is, in } \frac{1}{(1-x)(1-x^2)(1-x^3)\dots}.$$

In the 3rd family, if the number of ways of dividing r into two parts, neither of them less than 2, is called (r), and of dividing $(n-r)$ into any number of parts, none less than 2, is called $[n-r]$, the number of types is $\Sigma\,(r)\,[n-r]$; that is, it is the coefficient of x^{n-4} in

$$\frac{1+x+2x^2+2x^3+3x^4+3x^5+\dots}{(1-x^2)(1-x^3)(1-x^4)\dots}, \text{ that is, in } \frac{1}{(1-x)(1-x^2)^2(1-x^3)(1-x^4)\dots}.$$

Hence the total number of types in all three families combined will be the coefficient of x^{n-2} in

$$\frac{(1-x)(1-x^2)+x(1-x^2)+x^2}{1-x \,.\, 1-x^2 \,.\, 1-x^3 \dots}, \text{ that is, in } \frac{1}{1-x \,.\, (1-x^2)^2 \,.\, 1-x^3 \,.\, 1-x^4 \dots}.$$

Consequently, the indefinite partitions of 0, 1, 2, 3, 4, 5, 6, 7, ... being 1, 1, 2, 3, 5, 7, 11, 15, ..., the series for the type-number will be found by summing all the terms in the odd and even places successively. We thus obtain the series 1, 1, 3, 4, 8, 11, 19, 26, ... for the number of types in the constant-distance function for 2, 3, 4, 5, 6, 7, 8, 9, ... points respectively.

It may be worth while to exhibit the rule for the formation of the constant function of distances under a slightly different aspect.

As before, by the reading of any cycle, understand the product of its successive duads affected with the multiplier -1 or -2, according as the number of letters in the cycle is two or more than two.

By a *modified* reading of a cycle, understand what the reading becomes on substituting for any two duads pq, rs the product $(P-Q)(R-S)$, as for instance $(A-B \between C-D)$ in lieu of $ab \,.\, cd$, $(A-B \between B-C)$ in lieu of $ab \,.\, bc$, and (which can only happen in the case of a cycle of two letters), $(A-B)(B-A)$, that is, $-(A-B)^2$ in lieu of $ab \,.\, ba$.

Then, to find the constant function of distances to any given set of letters, we must begin with distributing the letters in every possible way into cycles containing between them two or more letters. Each such combination of cycles we may call a distribution.

In each distribution the cycle is to be taken (each in its turn), and the *sum* of its modified readings is to be multiplied by the product of the readings of the remaining cycles, if there are any. The sum of these sums (or the single sum, if there is but one cycle) is the portion of the quadratic function sought, due to the particular distribution dealt with; and the sum of these double sums, taken for each distribution in succession, is the total value of the function, and will be equal exactly to its representative determinant when the number of letters is odd, and to the same with its sign changed when that number is even.

As an example for five letters a, b, c, d, e, there will be ten distributions of the form $(ab)(cde)$, and twelve distributions of the form $(abcde)$.

From any one of the first ten distributions, as $(ab)(cde)$, by modifying first (ab) and then (cde), we obtain

(1) $2(cd\,.\,de\,.\,ec)(A-B)(B-A)$,

(2) $2(ab)^2\{ce(C-D)(D-E)+dc(D-E)(E-C)+ed(E-C)(C-D)\}$.

And from a distribution of the form $(abcde)$ we obtain, by operating on consecutive duads,

5 terms of the form $-2\{cd\,.\,de\,.\,ea(A-B)(B-C)\}$,

and, by operating on non-consecutive duads,

5 terms of the form $-2\{bc\,.\,de\,.\,ea(A-B)(C-D)\}$*.

The sum of all the sums of terms due to the twenty-two distributions is the constant function of distances for the five given letters.

In the case of six letters the distributions into cycles will be of four kinds, corresponding to the partitions 6; 4, 2; 3, 3; 2, 2, 2.

The first kind will contain two types of the 3rd family and one of the 2nd family; the second kind will contain one type of each of the three families, and the third and fourth kinds single types of the 2nd and 1st families respectively, thus giving eight distinct types of terms in all, as should be the case according to the rule.

* It will be observed that the distribution $(acbde)$ will give a term

$$-2\{cb\,.\,de\,.\,ea(A-C)(B-D)\},$$

in which the literal part $cb\,.\,de\,.\,ea$ is equal to the literal part $bc\,.\,de\,.\,ea$ in the term above expressed. This is how it comes to pass that the terms of the 3rd family may be grouped in pairs, as stated in the prior mode of arranging the result according to families instead of according to cycles.