34.

NOTE ON CAPTAIN MAC MAHON'S TRANSFORMATION OF THE THEORY OF INVARIANTS.

[*Messenger of Mathematics*, XIII. (1884), pp. 163—165.]

THE whole question as is well known consists in finding the free forms of $\Omega^{-1}0$, where

$$\Omega = a_0\delta a_1 + 2a_1\delta a_2 + \ldots + ia_{i-1}\delta a_i;$$

but, as long ago noticed by me* in the *Am. Math. Journal*, $\Omega^{-1}0$ is only a deformation of $V^{-1}0$, where

$$V = a_0\delta a_1 - a_1\delta a_2 + \ldots \pm a_{i-1}\delta a_i,$$

$\Omega^{-1}0$ being deducible from $V^{-1}0$ by altering the dimensions of the a elements which it contains in known numerical proportions, so that $\Omega^{-1}0$ may be said to be $V^{-1}0$ subjected to a known *strain*†.

To fix the ideas let $i = 3$ and call the a's by the names a, b, c, d or, for greater simplicity, 1, b, c, d.

Let
$$b = r + s + t,$$
$$c = rs + rt + st,$$
$$d = rst.$$

Then the matrix

$$\frac{D(b, c, d)}{D(r, s, t)} = \begin{matrix} 1 & 1 & 1\ , \\ s+t & t+r & r+s, \\ st & tr & rs\ , \end{matrix}$$

so that

$$\frac{D(r, s, t)}{D(b, c, d)} = \begin{matrix} \dfrac{r^2}{(r-s)(r-t)} & \dfrac{s^2}{(s-r)(s-t)} & \dfrac{t^2}{(t-r)(t-s)}, \\ \dfrac{r}{(r-s)(r-t)} & \dfrac{s}{(s-r)(s-t)} & \dfrac{t}{(t-r)(t-s)}, \\ \dfrac{1}{(r-s)(r-t)} & \dfrac{1}{(s-r)(s-t)} & \dfrac{1}{(t-r)(t-s)}. \end{matrix}$$

[* Vol. III. of this Reprint, p. 570.]

† In fact the numerical multipliers of the terms in Ω may be taken perfectly arbitrary without producing any effect upon the form $\Omega^{-1}0$ than what may be represented by a *strain*.

Consequently

$$V = \Sigma \frac{r^2 - (r+s+t)r + (rs+rt+st)}{(r-s)(r-t)} \delta_r = \Sigma \frac{st}{(r-s)(r-t)} \delta_r.$$

In like manner in general for 1, a_1, a_2, ... a_i we shall find, on writing

$$a_1 = r_1 + r_2 + \ldots + r_i,$$

$$a_2 = r_1 r_2 + r_2 r_3 + \ldots + r_{i-1} r_i,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$a_i = r_1 r_2 \ldots r_i,$$

$$V = \delta a_1 - a_1 \delta a_2 + \ldots \pm a_{i-1} \delta a_i = \Sigma \frac{r_2 r_3 \ldots r_i}{(r_1 - r_2)(r_1 - r_3) \ldots (r_1 - r_i)} \delta r_1.$$

Hence $\quad V^{-1} 0 = F(s_1, s_2, \ldots s_i),$

where, in general, $\quad s_\omega = r_1^\omega + r_2^\omega \ldots + r_i^\omega;$

and consequently the theory of invariants, which endoscopically treated in the ordinary way hinges upon symmetrical functions of the differences of a set of letters, is made to depend upon functions of the simple sums of powers commencing with the second power and ending with a power whose index is the order of any given finite quantic, but in the case of *perpetuants* taking in all the powers except the first.

It goes without saying that the same method applied to the *constrained* V will show that it is equal to $\Sigma \delta r_1$, so that V_0^{-1} is an arbitrary function of the differences of the r's corresponding to that hypothesis, as we know ought to be the case.

What has been established in the foregoing investigation is a principle of correspondence whose importance as a simplifying agent recalls Ivory's use of such principle in Attractions, namely, the remarkable algebraical law that any symmetrical function of the differences of a set of i quantities is a symmetrical function of the sums of the 2nd, 3rd, ..., ith powers of another equi-numerous set.

By virtue of this principle the numerical part of the Calculus of Invariants is capable of being entirely divorced from all question of algebraical content and a Zahl-Invariant theory comes into being, in its fundamental conception analogous to the Zahl-Geometrie of Schubert.

Further remarks on this subject will be found in the *Comptes Rendus de l'Institut* presumably for March 31 and April 7 of this year [p. 163 above].