626.

ON THE GENERAL DIFFERENTIAL EQUATION $\dfrac{dx}{\sqrt{X}}+\dfrac{dy}{\sqrt{Y}}=0$, WHERE X, Y ARE THE SAME QUARTIC FUNCTIONS OF x, y RESPECTIVELY.

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WRITE $\Theta = a + b\theta + c\theta^2 + d\theta^3 + e\theta^4$, the general quartic function of θ; and let it be required to integrate by Abel's theorem the differential equation

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0.$$

We have

$$\begin{vmatrix} x^2, & x, & 1, & \sqrt{X} \\ y^2, & y, & 1, & \sqrt{Y} \\ z^2, & z, & 1, & \sqrt{Z} \\ w^2, & w, & 1, & \sqrt{W} \end{vmatrix} = 0,$$

a particular integral of

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}+\frac{dw}{\sqrt{W}}=0;$$

and consequently the above equation, taking therein z, w as constants, is the general integral of

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0,$$

viz. the two constants z, w must enter in such wise that the equation contains only a single constant; whence also, attributing to w any special value, we have the general integral with z as the arbitrary constant.

Take $w=\infty$; the equation becomes

$$\begin{vmatrix} x^2, & x, & 1, & \sqrt{X} \\ y^2, & y, & 1, & \sqrt{Y} \\ z^2, & z, & 1, & \sqrt{Z} \\ 1, & 0, & 0, & \sqrt{e} \end{vmatrix}=0,$$

a relation between x, y, z which may be otherwise expressed by means of the identity

$$e(\theta^2+\beta\theta+\gamma)^2-(e\theta^4+d\theta^3+c\theta^2+b\theta+a)=(2\beta e-d)(\theta-x)(\theta-y)(\theta-z),$$

or, what is the same thing,

$$\begin{aligned} e(2\gamma+\beta^2)-c &= -(2\beta e-d)(x+y+z),\\ e\ 2\beta\gamma \qquad -b &= \quad (2\beta e-d)(yz+zx+xy),\\ e\ \gamma^2 \qquad -a &= -(2\beta e-d)\,xyz, \end{aligned}$$

where β, γ are indeterminate coefficients which are to be eliminated.

Write

$$x^2-\frac{\sqrt{X}}{\sqrt{e}}=P,\quad y^2-\frac{\sqrt{Y}}{\sqrt{e}}=Q;$$

then we have

$$\beta x+\gamma+P=0,\quad \beta y+\gamma+Q=0;$$

giving

$$\beta:\gamma:1=Q-P:Py-Qx:x-y.$$

Substituting these values in the first of the preceding three equations, we have

$$e\,\frac{2(Py-Qx)(x-y)+(Q-P)^2}{(x-y)^2}-c=-\left\{\frac{2(Q-P)e}{x-y}-d\right\}(x+y+z),$$

that is,

$$e\left\{\frac{2(Qy-Px)}{x-y}+\frac{(Q-P)^2}{(x-y)^2}+\frac{2(Q-P)}{x-y}z\right\}=c+d(x+y+z);$$

or, reducing by

$$Qy-Px=y^3-x^3+\frac{x\sqrt{X}-y\sqrt{Y}}{\sqrt{e}},$$

$$Q-P=y^2-x^2+\frac{\sqrt{X}-\sqrt{Y}}{\sqrt{e}},\quad =y^2-x^2+(y-x)\frac{M}{\sqrt{e}},\ \text{if}\ M=\frac{\sqrt{X}-\sqrt{Y}}{x-y},$$

this is

$$e\left\{\frac{2(x\sqrt{X}-y\sqrt{Y})}{\sqrt{e}(x-y)}+2xy+\frac{M^2}{e}-2(x+y)\frac{M}{\sqrt{e}}-2(x+y)z+2z\frac{M}{\sqrt{e}}\right\}$$
$$=c+d(x+y+z)+e(x+y)^2.$$

We have Euler's solution in the far more simple form

$$M^2=C+d(x+y)+e(x+y)^2,$$

where C is the arbitrary constant. It is to be observed that, in the particular case where $e=0$, the first equation becomes

$$M^2=c+d(x+y+z);$$

and the two results for this case agree on putting $C=c+dz$.

But it is required to identify the two solutions in the general case where e is not $=0$. I remark that I have, in my *Treatise on Elliptic Functions*, Chap. XIV., further developed the theory of Euler's solution, and have shown that, regarding C as variable, and writing

$$\mathfrak{C}=ad^2+b^2e-2bcd+C[-4ae+bd+(C-c)^2],$$

then the given equation between the variables x, y, C corresponds to the differential equation

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dC}{\sqrt{\mathfrak{C}}}=0,$$

a result which will be useful for effecting the identification. The Abelian solution may be written

$$e\left\{\frac{2(x\sqrt{X}-y\sqrt{Y})}{\sqrt{e}(x-y)}-x^2-y^2+\frac{M^2}{e}-2(x+y)\frac{M}{\sqrt{e}}\right\}-c-d(x+y)=z\{d+2e(x+y)-2M\sqrt{e}\};$$

and substituting for M its value, and multiplying by $(x-y)^2$, the equation becomes

$$\begin{aligned}2\sqrt{e}(x-y)(x\sqrt{X}-y\sqrt{Y})-e(x^2+y^2)(x-y)^2+(\sqrt{X}-\sqrt{Y})^2\\ -2(x^2-y^2)(\sqrt{X}-\sqrt{Y})\sqrt{e}-c(x-y)^2-d(x+y)(x-y)^2\\ =z(x-y)\{d(x-y)+2e(x^2-y^2)-2(\sqrt{X}-\sqrt{Y})\sqrt{e}\}.\end{aligned}$$

On the left-hand side, the rational part is

$$X+Y+c(-x^2+2xy-y^2)+d(-x^3+x^2y+xy^2-y^3)+e(-x^4+2x^3y-2x^2y^2+2xy^3-y^4),$$

which, substituting therein for X, Y their values, becomes

$$=2a+b(x+y)+c\,.\,2xy+d\,xy(x+y)+e\,.\,2xy(x^2-xy+y^2);$$

and the irrational part is at once found to be

$$=2\sqrt{e}(x-y)(x\sqrt{Y}-y\sqrt{X})-2\sqrt{XY}.$$

The equation thus is

$$z=\frac{\left\{\begin{array}{c}2a+b(x+y)+c\,.\,2xy+d\,xy(x+y)+e\,.\,2xy(x^2-xy+y^2)\\ +2\sqrt{e}(x-y)(x\sqrt{Y}-y\sqrt{X})-2\sqrt{XY}\end{array}\right\}}{(x-y)\{d(x-y)+2e(x^2-y^2)-2(\sqrt{X}-\sqrt{Y})\sqrt{e}\}},$$

which equation is thus a form of the general integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$, and also a particular integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0$.

Multiplying the numerator and the denominator by

$$d(x-y)+2e(x^2-y^2)+2(\surd X-\surd Y)\surd e,$$

the denominator becomes

$$=(x-y)^3\left[\{d+2e(x+y)\}^2-4e\left(\frac{\surd X-\surd Y}{x-y}\right)^2\right],$$

which, introducing herein the C of Euler's equation, is

$$=(x-y)^3(d^2-4eC).$$

We have therefore

$$z(x-y)^3(d^2-4eC)=\{2a+b(x+y)+c\,.\,2xy+d\,xy(x+y)+e\,.\,2xy(x^2-xy+y^2)$$
$$+2\surd e(x-y)(x\surd Y-y\surd X)-2\surd\overline{XY}\}\times\{d(x-y)+2e(x^2-y^2)+2\surd e(\surd X-\surd Y)\}.$$

Using $\mathfrak{C}$ to denote the same value as before, the function on the right-hand is, in fact,

$$=(x-y)^3\{2be-cd+dC+2\surd e\surd\mathfrak{C}\};$$

and, this being so, the required relation between z, C is

$$z(d^2-4eC)=\{2be-cd+dC+2\surd e\surd\mathfrak{C}\}.$$

To prove this, we have first, from the equation

$$\left(\frac{\surd X-\surd Y}{x-y}\right)^2=C+d(x+y)+e(x+y)^2,$$

to express $\mathfrak{C}$ as a function of x, y. This equation, regarding therein C as a variable, gives

$$\frac{dx}{\surd X}+\frac{dy}{\surd Y}+\frac{dC}{\surd\mathfrak{C}}=0;$$

and we have therefore

$$-\surd\mathfrak{C}=\surd X\frac{dC}{dx}=\surd Y\frac{dC}{dy},$$

viz. $\surd X\dfrac{dC}{dx}$ will be a symmetrical function of x, y. Putting, as before

$$M=\frac{\surd X-\surd Y}{x-y},$$

we have

$$C=M^2-d(x+y)-e(x+y)^2,$$

and thence

$$\frac{dC}{dx}=2M\frac{dM}{dx}-d-2e(x+y).$$

We have

$$\frac{dM}{dx}=\frac{1}{x-y}\,\frac{X'}{2\surd X}-\frac{\surd X-\surd Y}{(x-y)^2},$$

and hence

$$\sqrt{\mathfrak{C}}\,(x-y)^3 = -\sqrt{X}\,(x-y)^3\left\{2M\frac{dM}{dx} - d - 2e\,(x+y)\right\}$$

$$= -(x-y)\,X'\,(\sqrt{X}-\sqrt{Y}) + 2\,(X+Y-2\sqrt{XY})\,\sqrt{X}$$
$$+ (d+2e\,\overline{x+y})\,(x-y)^3\,\sqrt{X}$$

$$= \quad [(x-y)\,X' + 2X + 2Y + (d+2e\,\overline{x+y})\,(x-y)^3]\,\sqrt{X}$$
$$+ [(x-y)\,X' - 4X]\,\sqrt{Y}.$$

We obtain at once the coefficient of $\sqrt{Y}$, and with little more difficulty that of $\sqrt{X}$; and the result is

$$\sqrt{\mathfrak{C}}\,(x-y)^3 = -[4a+3bx+2cx^2+dx^3+y\,(b+2cx+3dx^2+4ex^3)]\,\sqrt{Y}$$
$$+[4a+3by+2cy^2+dy^3+x\,(b+2cy+3dy^2+4ey^3)]\,\sqrt{X}.$$

We have also

$$C\,(x-y)^2 = (\sqrt{X}-\sqrt{Y})^2 - d\,(x+y)\,(x-y)^2 - e\,(x+y)^2\,(x-y)^2$$
$$= X+Y-d\,(x^3-x^2y-xy^2+y^3) - e\,(x^4-2x^2y^2+y^4) - 2\sqrt{XY}$$
$$= 2a+b\,(x+y)+c\,(x^2+y^2)+d\,xy\,(x+y)+2e\,x^2y^2-2\sqrt{XY},$$

or, say

$$C\,(x-y)^3 = 2a\,(x-y)+b\,(x^2-y^2)+c\,(x^3-x^2y+xy^2-y^3)+d\,xy\,(x^2-y^2)$$
$$+2e\,x^2y^2\,(x-y)-2\,(x-y)\sqrt{XY}.$$

We can hence form the expression of

$$(x-y)^3\,\{2be-cd+dC+2\sqrt{e}\,\sqrt{\mathfrak{C}}\},$$

viz. this is

$$=(2be-cd)\,(x-y)^3+2ad\,(x-y)+bd\,(x^2-y^2)+cd\,(x^3-x^2y+xy^2-y^3)+d^2\,xy\,(x^2-y^2)$$
$$+2de\,x^2y^2\,(x-y)-2d\,(x-y)\sqrt{XY}$$
$$+2\sqrt{e}\,\{[-(4a+3bx+2cx^2+dx^3)-y\,(b+2cx+3dx^2+4ex^3)]\,\sqrt{Y}$$
$$+[(4a+3by+2cy^2+dy^3)+x\,(b+2cy+3dy^2+4ey^3)]\,\sqrt{X}\},$$

and this should be

$$=\{2a+b\,(x+y)+c\,.\,2xy+d\,xy\,(x+y)+e\,.\,2xy\,(x^2-xy+y^2)$$
$$+2\sqrt{e}\,(x-y)\,(x\sqrt{Y}-y\sqrt{X})-2\sqrt{XY}\}\times\{d\,(x-y)+2e\,(x^2-y^2)+2\sqrt{e}\,(\sqrt{X}-\sqrt{Y})\}.$$

The function on the right-hand is, in fact,

$$=\{2a+b\,(x+y)+c\,.\,2xy+d\,xy\,(x+y)+e\,.\,2xy\,(x^2-xy+y^2)-2\sqrt{XY}\}$$
$$\times\{d\,(x-y)+2e\,(x^2-y^2)\}+4e\,(x-y)\,(\sqrt{X}-\sqrt{Y})\,(x\sqrt{Y}-y\sqrt{X})$$
$$+2\sqrt{e}\,(\sqrt{X}-\sqrt{Y})\,\{2a+b\,(x+y)+c\,.\,2xy+d\,xy\,(x+y)+e\,.\,2xy\,(x^2-xy+y^2)-2\sqrt{XY}\}$$
$$+2\sqrt{e}\,(x-y)\,(x\sqrt{Y}-y\sqrt{X})\,\{d\,(x-y)+2e\,(x^2-y^2)\},$$

viz. this is

$$= \{2a + b(x+y) + c \, . \, 2xy + d\,xy(x+y) + e \, . \, 2xy(x^2 - xy + y^2)\} \times \{d(x-y) + 2e(x^2 - y^2)\} + 4e(x-y)(-xY - yX)$$

$$- 2\sqrt{XY}\{d(x-y) + 2e(x^2-y^2)\} + 4e(x-y)(x+y)\sqrt{XY}$$

$$+ 2\sqrt{e}\left(\begin{array}{l} \sqrt{X}\{2a + b(x+y) + c \, . \, 2xy + d\,xy(x+y) + e \, . \, 2xy(x^2 - xy + y^2) \\ \qquad + 2Y - (x-y)\,y\,[d(x-y) + 2e(x^2-y^2)]\} \\ -\sqrt{Y}\{2a + b(x+y) + c \, . \, 2xy + d\,xy(x+y) + e \, . \, 2xy(x^2 - xy + y^2) \\ \qquad + 2X - (x-y)\,x\,[d(x-y) + 2e(x^2-y^2)]\} \end{array}\right),$$

which is, in fact, equal to the expression on the left-hand side.

To complete the theory, we require to express $\sqrt{Z}$ as a function of x, y. It would be impracticable to effect this by direct substitution of the foregoing value of z; but, observing that the value in question is a solution of $\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} + \frac{dz}{\sqrt{Z}} = 0$, or, what is the same thing, that $\frac{1}{\sqrt{X}} + \frac{1}{\sqrt{Z}}\frac{dz}{dx} = 0$, $\frac{1}{\sqrt{Y}} + \frac{1}{\sqrt{Z}}\frac{dz}{dy} = 0$, we can from either of these equations, considering therein z as a given function of x, y, calculate $\sqrt{Z}$.

Writing for shortness

$$z = \frac{J - 2\sqrt{e}\,y(x-y)\sqrt{X} + 2\sqrt{e}\,x(x-y)\sqrt{Y} - 2\sqrt{XY}}{R - 2\sqrt{e}(x-y)\sqrt{X} + 2\sqrt{e}(x-y)\sqrt{Y}},$$

where

$$R = (x-y)^2\{d + 2e(x+y)\},$$

$$J = 2a + b(x+y) + 2c\,xy + d\,xy(x+y) + 2e\,xy(x^2 - xy + y^2);$$

or, if for a moment $z = \frac{N}{D}$, then

$$\frac{dz}{dx} = \frac{1}{D^2}\left(D\frac{dN}{dx} - N\frac{dD}{dx}\right) = -\frac{\sqrt{Z}}{\sqrt{X}},$$

that is,

$$\sqrt{Z} = \frac{\sqrt{X}}{D^2}\left(N\frac{dD}{dx} - D\frac{dN}{dx}\right), \; = \frac{\Omega}{D^2} \text{ suppose};$$

or, writing for shortness X', R', J to denote the derived functions $\frac{dX}{dx}$, $\frac{dR}{dx}$, $\frac{dJ}{dx}$, (Y' is afterwards written to denote $\frac{dY}{dy}$, but as the final formulæ contain only X', $=\frac{dX}{dx}$, and Y', $=\frac{dY}{dy}$, this does not occasion any defect of symmetry), we find

$$\Omega = \quad N\{R'\sqrt{X} - 2\sqrt{e}\,X - \sqrt{e}\,(x-y)\,X' + 2\sqrt{e}\sqrt{XY}\}$$

$$- D\,\{J'\sqrt{X} - 2\sqrt{e}\,yX - \sqrt{e}(x-y)\,yX' + 2\sqrt{e}\,(2x-y)\sqrt{XY} - X'\sqrt{Y}\};$$

and substituting herein for N, D their values, and arranging the terms, we find

$$\Omega = \sqrt{e}\,\mathfrak{A} + \mathfrak{B}\sqrt{X} + \mathfrak{C}\sqrt{Y} + \sqrt{e}\,\mathfrak{D}\sqrt{XY},$$

where

$$\begin{aligned}\mathfrak{A} = &- J\{2X + (x-y)X'\}\\ &- 2(x-y)\,yR'X\\ &- 4XY\\ &+ Ry\{2X + (x-y)X'\}\\ &+ 2(x-y)XJ'\\ &+ 2(x-y)X'Y,\end{aligned}$$

$$\begin{aligned}\mathfrak{B} = &\ \ JR'\\ &+ 2e(x-y)\,y\{2X + (x-y)X'\}\\ &+ 4ex(x-y)\,Y\\ &- RJ'\\ &- 2e(x-y)\,y\{2X + (x-y)X'\}\\ &- 4e(x-y)(2x-y)\,Y,\end{aligned}$$

$$\begin{aligned}\mathfrak{C} = &- 4ey(x-y)X\\ &- 2e(x-y)\,x\{2X + (x-y)X'\}\\ &- 2R'X\\ &+ RX'\\ &+ 2e(x-y)\,y\{2X + (x-y)X'\}\\ &+ 4e(x-y)(2x-y)X,\end{aligned}$$

$$\begin{aligned}\mathfrak{D} = &\ \ 2J\\ &+ 2(x-y)\,xR'\\ &+ 2\{2X + (x-y)X'\}\\ &- 2(2x-y)R\\ &- 2(x-y)X'\\ &- 2(x-y)J',\end{aligned}$$

where the terms have been written down as they immediately present themselves; but, collecting and arranging, we have

$$\begin{aligned}\mathfrak{A} &= \ \ 2X(-J + Ry - 2Y) + (x-y)\{2XJ' + 2X'Y - X'J - 2yR'X + yRX'\},\\ \mathfrak{B} &= \ \ JR' - J'R - 4e(x-y)^2\,Y,\\ \mathfrak{C} &= -2XR' + X'R + 4e(x-y)^2X - 2e(x-y)^3X',\\ \mathfrak{D} &= \ \ 2J + 4X - 2Rx + 2(x-y)(xR' - R - J').\end{aligned}$$

To reduce these expressions, writing

$$\begin{aligned}M &= d + 2e(x+y),\\ \Lambda &= c + \ d(x+y) + e(x^2+y^2),\end{aligned}$$

we have $R = (x-y)^2M$, and therefore $R' = 2(x-y)M + 2e(x-y)^2$; also

$$J = X + Y - (x-y)^2\Lambda;$$

also, from the original form,

$$J' = b + 2cy + d(2xy + y^2) + e(6x^2y - 4xy^2 + 2y^3).$$

The final values are

$$\begin{aligned}\mathfrak{A} &= \qquad - X^2 - 6XY - Y^2 + (x-y)^4\{\Lambda^2 + (-b + dxy)M + xyM^2\},\\ \mathfrak{B} &= \ \ (x-y)M\{4Y + (x-y)Y'\} + 2e(x-y)^3Y',\\ \mathfrak{C} &= -(x-y)M\{4X - (x-y)X'\} - 2e(x-y)^3X',\\ \mathfrak{D} &= \qquad 4(X+Y) + 4e(x-y)^4,\end{aligned}$$

which, once obtained, may be verified without difficulty.

Verification of 𝔄.—The equation is

$$-X^2-6XY-Y^2+(x-y)^4\{\Lambda^2+(-b+dxy)M+xyM^2\}$$
$$=2X(-J+Ry-2Y)+(x-y)\{2XJ'+2X'Y-X'J-2yR'X+yRX'\};$$

or, putting for shortness

$$\Lambda^2+(-b+dxy)M+xyM^2=\nabla,$$

this is

$$\begin{aligned}(x-y)^4\nabla= {} & X^2+6XY+Y^2\\ & +2X\{-X-3Y+(x-y)^2\Lambda+(x-y)^2yM\}\\ & +(x-y)\{2XJ'+2X'Y-X'J-2yR'X+yRX'\},\\ = {} & -X^2+Y^2+2(x-y)^2X\Lambda+2(x-y)^2yXM\\ & +(x-y)\{2XJ'+2X'Y-X'J-2yR'X+yRX'\};\end{aligned}$$

we have $-X^2+Y^2=-(X-Y)(X+Y)$, where $X-Y$ divides by $x-y$, $=(x-y)\Omega$ suppose; hence, throwing out the factor $x-y$, the equation becomes

$$\begin{aligned}(x-y)^3\nabla= {} & -\Omega(X+Y)+2(x-y)X\Lambda+2(x-y)yXM\\ & +2XJ'+2X'Y-X'\{X+Y-(x-y)^2\Lambda\}\\ & -2yX\{2(x-y)M+2(x-y)^2e\}+(x-y)^2yMX',\\ = {} & -\Omega(X+Y)+2XJ'-X'(X-Y)\\ & +2(x-y)X\Lambda-2(x-y)yXM\\ & +(x-y)^2X'\Lambda-4(x-y)^2eyX+(x-y)^2yMX'.\end{aligned}$$

We have $2XJ'=J'(X+Y)+J'(X-Y)$, and hence the first line is

$$=(-\Omega+J')(X+Y)+J'(X-Y);$$

$-\Omega+J'$, as will be shown, divides by $x-y$, or say it is $=(x-y)\Phi$, and, as before, $X-Y$ is $=(x-y)\Omega$; hence, throwing out the factor $x-y$, the equation becomes

$$(x-y)^2\nabla=\Phi(X+Y)+\Omega(J'-X')+2X\Lambda-2yXM+(x-y)\{X'\Lambda-4eyX+yMX'\}.$$

We have

$$\Omega=b+c(x+y)+d(x^2+xy+y^2)+e(x^3+x^2y+xy^2+y^3),$$

and thence

$$-\Omega+J'=c(-x+y)+d(-x^2+xy)+e(-x^3+5x^2y-5xy^2+y^3);$$

or, dividing this by $(x-y)$, we find

$$\Phi=-c-dx-e(x^2-4xy+y^2),$$

or, as this may be written,

$$\Phi=-\Lambda+dy+4exy.$$

We find, moreover,

$$J'-X'=2c(-x+y)+d(-3x^2+2xy+y^2)+e(-4x^3+6x^2y-4xy^2+2y^3),$$

which divides by $(x-y)$, the quotient being

$$-2c-d(3x+y)-e(4x^2-2xy+2y^2),$$

viz. this is

$$=-2\Lambda-(x-y)(d+2ex).$$

Hence the equation now is

$$\begin{aligned}(x-y)^2\nabla=(X+Y)\{-\Lambda+dy+4exy\}&+2X\Lambda-2yXM\\ &+(x-y)\,\Omega\,\{-2\Lambda-(x-y)(d+2ex)\}\\ &+(x-y)\quad\{\;X'\Lambda-4eyX+yMX'\;\}.\end{aligned}$$

The first line is

$$(X+Y)\{-\Lambda+yM+2(x-y)ye\}+2X\Lambda-2yXM,$$

which is

$$=(\Lambda-yM)(X-Y)+2(x-y)ey(X+Y);$$

hence, throwing out the factor $x-y$, the equation becomes

$$\begin{aligned}(x-y)\nabla&=(\Lambda-yM)\,\Omega+2ey(X+Y)-2\Lambda\Omega+X'\Lambda-4eyX+yMX'-(x-y)\,\Omega\,(d+2ex)\\ &=(\Lambda+yM)(-\Omega+X')-2ey(X-Y)-(x-y)\,\Omega\,(d+2ex).\end{aligned}$$

We have

$$-\Omega+X'=c(x-y)+d(2x^2-xy-y^2)+e(3x^3-x^2y-xy^2-y^3),$$

which is $=(x-y)(\Lambda+xM)$: also $(X-Y)=(x-y)\,\Omega$, as before; whence, throwing out the factor $x-y$, the equation is

$$\nabla=(\Lambda+xM)(\Lambda+yM)-2ey\Omega-(d+2ex)\,\Omega,$$

that is,

$$\nabla=(\Lambda+xM)(\Lambda+yM)-M\Omega;$$

viz. substituting for ∇ its value, reducing, and throwing out the factor M, the equation becomes

$$-b+dxy=(x+y)\,\Lambda-\Omega,$$

which is right.

Verification of 𝔅.—The equation is

$$\begin{aligned}J\{2(x-y)M+2e(x-y)^2\}-J'(x-y)^2M-4e(x-y)^2\,Y&\\ =4(x-y)MY+(x-y)^2MY'&+2e(x-y)^3\,Y',\end{aligned}$$

which, throwing out the factor $x-y$, is

$$0=2M(-J+2Y)+(x-y)M(J'+Y')+2e(x-y)(-J+2Y)+2e(x-y)^2\,Y'.$$

Here $-J+2Y, =-(X-Y)+(x-y)^2\Lambda$, is divisible by $(x-y)$: hence, throwing out the factor $x-y$, the equation is

$$0=M\{-2b-2c(x+y)-2d(x^2+xy+y^2)-2e(x^3+x^2y+xy^2+y^3)\}$$
$$+M(J'+Y')+2M(x-y)\Lambda+2e(-J+2Y)+2e(x-y)Y'.$$

In the first and second terms, the factor which multiplies M is

$$c(-2x+2y)+d(-2x^2+2y^2)+e(-2x^3+4x^2y-6xy^2+4y^3),$$

which is divisible by $x-y$; also $-J+2Y, =-(X-Y)+(x-y)^2\Lambda$, is divisible by $(x-y)$: hence, throwing this factor out, the equation is

$$0=M\{-2c+d(-2x-2y)+e(-2x^2+2xy-4y^2)\}+2M\Lambda$$
$$+2e\{-b-c(x+y)-d(x^2+xy+y^2)-e(x^3+x^2y+xy^2+y^3)\}$$
$$+2e(x-y)\Lambda+2eY'.$$

Here in the first line the coefficient of M is $=e(2xy-2y^2)$: hence, throwing out the constant factor $2e$, the equation is

$$0=-b-c(x+y)-d(x^2+xy+y^2)-e(x^3+x^2y+xy^2+y^3)+Y'+(x-y)yM+(x-y)\Lambda.$$

The first five terms are

$$=c(-x+y)+d(-x^2-xy+2y^2)+e(-x^3-x^2y-xy^2+3y^3),$$

which is divisible by $x-y$; throwing out this factor, the equation is

$$0=-c-d(x+2y)-e(x^2+2xy+3y^2)+\Lambda+yM,$$

which is right.

Verification of $\mathfrak{C}$.—We have

$$-2X\{2(x-y)M+2e(x-y)^2\}+(x-y)^2X'M+4e(x-y)^2X-2e(x-y)^3X'$$
$$=-(x-y)M\{4X-(x-y)X'\}-2e(x-y)^3X',$$

which is, in fact, an identity.

Verification of $\mathfrak{D}$.—The equation may be written

$$4X+4Y+4e(x-y)^4$$
$$=2X+2Y-2(x-y)^2\Lambda$$
$$+4X-2x(x-y)^2M$$
$$+2(x-y)\{2(x-y)xM+2ex(x-y)^2-M(x-y)^2-J'\},$$

viz. this is

$$0=2X-2Y-4e(x-y)^4-2(x-y)^2\Lambda+2x(x-y)^2M$$
$$+4ex(x-y)^3-2M(x-y)^3-2(x-y)J'.$$

The first term $2(X-Y)$ is divisible by $2(x-y)$; throwing this factor out, the equation becomes

$$0=b+c(x+y)+d(x^2+xy+y^2)+e(x^3+x^2y+xy^2+y^3)-J'$$
$$-2e(x-y)^3-(x-y)\Lambda+x(x-y)M+2ex(x-y)^2-M(x-y)^2.$$

Substituting for J' its value, the first line becomes

$$c(x-y)+d(x^2-xy)+e(x^3-5x^2y+5xy^2-y^3),$$

which is divisible by $(x-y)$; hence, throwing out this factor, the equation is

$$0=c+dx+e(x^2-4xy+y^2)-\Lambda+xM-2e(x-y)^2+2ex(x-y)-M(x-y),$$

where the sum of all the terms but the last is $=d(x-y)+e(2x^2-2xy)$: hence, again throwing out the factor $x-y$, the equation becomes

$$0=d+2ex-2e(x-y)+2ex-M,$$

which is right.

Recapitulating, we have for the general integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$, or for a particular integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0$,

$$z=\frac{J-2\sqrt{e}(x-y)y\sqrt{X}+2\sqrt{e}(x-y)x\sqrt{Y}-2\sqrt{XY}}{(x-y)^2M-2\sqrt{e}(x-y)\sqrt{X}+2\sqrt{e}(x-y)\sqrt{Y}},$$

the corresponding value of $\sqrt{Z}$ being

$$\sqrt{Z}=\frac{\begin{aligned}&\sqrt{e}\,[-X^2-6XY-Y^2+(x-y)^4\{\Lambda^2+(-b+dxy)M+xyM^2\}]\\&+[\{4Y+(x-y)Y'\}M+2e(x-y)^2Y'](x-y)\sqrt{X}\\&-[\{4X-(x-y)X'\}M+2e(x-y)^2X'](x-y)\sqrt{Y}\\&+[\quad 4(X+Y)+4e(x-y)^4]\quad\sqrt{XY}\end{aligned}}{\{(x-y)^2M-2\sqrt{e}(x-y)\sqrt{X}+2\sqrt{e}(x-y)\sqrt{Y}\}^2},$$

where, as before,

$$M=d+2e(x+y),$$
$$\Lambda=c+d(x+y)+e(x^2+y^2),$$
$$J=2a+b(x+y)+2cxy+dxy(x+y)+exy(x^2-xy+y^2):$$

also X is the general quartic function $a+bx+cx^2+dx^3+ex^4$, and Y, Z are the same functions of y, z respectively.

In connexion with what precedes, I give some investigations relating to the more simple form $\Theta=a+c\theta^2+e\theta^4$, or, as it will be convenient to write it, $\Theta=1-l\theta^2+\theta^4$.

We have

$$\left.\begin{array}{l} \begin{vmatrix} x, & \sqrt{X} \\ y, & \sqrt{Y} \end{vmatrix} = 0 \text{ a particular integral} \\ \begin{vmatrix} x^3, & x, & \sqrt{X} \\ y^3, & y, & \sqrt{Y} \\ z^3, & z, & \sqrt{Z} \end{vmatrix} = 0 \text{ the general integral} \end{array}\right\} \text{of } \frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0,$$

$$\left.\begin{array}{l} \begin{vmatrix} x^3, & x, & \sqrt{X} \\ y^3, & y, & \sqrt{Y} \\ z^3, & z, & \sqrt{Z} \end{vmatrix} \text{ a particular integral} \\ \begin{vmatrix} x^3, & x, & x^2\sqrt{X}, & \sqrt{X} \\ y^3, & y, & y^2\sqrt{Y}, & \sqrt{Y} \\ z^3, & z, & z^2\sqrt{Z}, & \sqrt{Z} \\ w^3, & w, & w^2\sqrt{W}, & \sqrt{W} \end{vmatrix} = 0 \text{ the general integral} \end{array}\right\} \text{of } \frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} + \frac{dz}{\sqrt{Z}} = 0,$$

$$\left.\begin{array}{l} \begin{vmatrix} x^3, & x, & x^2\sqrt{X}, & \sqrt{X} \\ y^3, & y, & y^2\sqrt{Y}, & \sqrt{Y} \\ z^3, & z, & z^2\sqrt{Z}, & \sqrt{Z} \\ w^3, & w, & w^2\sqrt{W}, & \sqrt{W} \end{vmatrix} \text{ a particular integral} \\ \quad \cdots \quad \cdots \end{array}\right\} \text{of } \frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} + \frac{dz}{\sqrt{Z}} + \frac{dw}{\sqrt{W}} = 0,$$

and so on; viz. in taking

$$\begin{vmatrix} x^3, & x, & \sqrt{X} \\ y^3, & y, & \sqrt{Y} \\ z^3, & z, & \sqrt{Z} \end{vmatrix} = 0 \text{ as the general integral of } \frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0,$$

we consider z as the constant of integration: and so in other cases.

It is to be remarked that it is an essentially different problem to verify a particular integral and to verify a general integral, and that the former is the more difficult one. In fact, if $U = 0$ is a particular integral of the differential equation $Mdx + Ndy = 0$, then we must have $N\frac{dU}{dx} - M\frac{dU}{dy} = 0$, not identically but in virtue of the relation $U = 0$, or we have to consider whether two given relations between x and y are in fact one and the same relation. In the case of a general solution, this is theoretically reducible to the form $c = U$, c being the constant of integration, and we have then the equation $N\frac{dU}{dx} - M\frac{dU}{dy} = 0$, satisfied identically, or, what is the same thing, U a solution of this partial differential equation.

Hence it is theoretically easier to verify that

$$\begin{vmatrix} x^3, & x, & \sqrt{X} \\ y^3, & y, & \sqrt{Y} \\ z^3, & z, & \sqrt{Z} \end{vmatrix} = 0$$

is a general solution, than to verify that

$$\begin{vmatrix} x, & \sqrt{X} \\ y, & \sqrt{Y} \end{vmatrix} = 0$$

is a particular solution of the differential equation $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$. Moreover, taking the first equation in the before mentioned form

$$-z=\frac{x^2-y^2}{x\sqrt{Y}-y\sqrt{X}},$$

and writing therein $z=\infty$, we see that the second equation

$$\begin{vmatrix} x, & \sqrt{X} \\ y, & \sqrt{Y} \end{vmatrix}=0$$

is, in fact, a particular case of the first equation, so that we only require to verify the first equation; or, what is the same thing, to verify that

$$-z=\frac{x^2-y^2}{x\sqrt{Y}-y\sqrt{X}}$$

is the general integral of

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}.$$

To verify this, we have to show that $dz=\Omega\left(\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}\right)$, viz. that $\sqrt{X}\frac{dz}{dx}=\Omega$, a symmetrical function of (x, y); for then $\sqrt{Y}\frac{dz}{dy}=\Omega$, and we have the relation in question.

We have

$$\begin{aligned}(x\sqrt{Y}-y\sqrt{X})^2\sqrt{X}\frac{dz}{dx}&=\sqrt{X}\left\{(x^2-y^2)\left(\sqrt{Y}-\frac{yX'}{2\sqrt{X}}\right)-2x(x\sqrt{Y}-y\sqrt{X})\right\}\\ &=\sqrt{X}\left\{(x^2-y^2-2x^2)\sqrt{Y}-\frac{(x^2-y^2)yX'}{2\sqrt{X}}+2xy\sqrt{X}\right\}\\ &=-(x^2+y^2)\sqrt{XY}+2xyX-\tfrac{1}{2}(x^2-y^2)yX'.\end{aligned}$$

Writing here $X=1-lx^2+x^4$, then $X'=-2lx+4x^3$, and we have the last two terms

$$\begin{aligned}&=2xy(1-lx^2+x^4)+(x^2-y^2)xy(l-2x^2)\\ &=xy\{2-2lx^2+2x^4+(x^2-y^2)(l-2x^2)\}\\ &=xy\{2-l(x^2+y^2)+2x^2y^2\}.\end{aligned}$$

Hence the equation is

$$(x\sqrt{Y}-y\sqrt{X})^2\sqrt{X}\frac{dz}{dx}=-(x^2+y^2)\sqrt{XY}+xy\{2-l(x^2+y^2)+2x^2y^2\},$$

or we have

$$\Omega=\frac{1}{(x\sqrt{Y}-y\sqrt{X})^2}\{-(x^2+y^2)\sqrt{XY}+xy(2-l(x^2+y^2)+2x^2y^2)\},$$

which is symmetrical in (x, y), as it should be. And observe, further, that since the equation is a particular solution of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0$, we must have $\Omega=-\sqrt{Z}$; viz. we have

$$\sqrt{Z}\,(x\sqrt{Y}-y\sqrt{X})^2=-(x^2+y^2)\sqrt{XY}+xy\,\{2-l\,(x^2+y^2)+2x^2y^2\}.$$

Proceeding to the next case, where we have between x, y, z, w a relation which may be written

$$(x^3,\ x,\ x^2\sqrt{X},\ \sqrt{X})=0,$$

then here a, b, c, d can be determined so that

$$(c\theta^2+d)^2\,(1+\beta\theta^2+\gamma\theta^4)-(a\theta^3+b\theta)^2=c^2\gamma\,(\theta^2-x^2)\,(\theta^2-y^2)\,(\theta^2-z^2)\,(\theta^2-w^2),$$

viz. we have $d^2=c^2\gamma\,x^2y^2z^2w^2$, or say $d=c\sqrt{\gamma}\,xyzw$. And, supposing the ratios of a, b, c, d determined by the three equations which contain (x, y, z) respectively, we have

$$a : b : c : d=(x,\ x^2\sqrt{X},\ \sqrt{X}) : -(x^3,\ x^2\sqrt{X},\ \sqrt{X}) : (x^3,\ x,\ \sqrt{X}) : -(x^3,\ x,\ x^2\sqrt{X}),$$

or in particular

$$\frac{d}{c}=\frac{-(x^3,\ x,\ x^2\sqrt{X})}{(x^3,\ x,\ \sqrt{X})},\quad =\frac{-xyz\,(x^2,\ 1,\ x\sqrt{X})}{(x^3,\ x,\ \sqrt{X})};$$

whence we have

$$w=-\frac{(x^2,\ 1,\ x\sqrt{X})}{(x^3,\ x,\ \sqrt{X})}$$

as a new form of the integral equation; viz. written at full length, this is

$$-w=\begin{vmatrix} x^2, & 1, & x\sqrt{X} \\ y^2, & 1, & y\sqrt{Y} \\ z^2, & 1, & z\sqrt{Z} \end{vmatrix} \div \begin{vmatrix} x^3, & x, & \sqrt{X} \\ y^3, & y, & \sqrt{Y} \\ z^3, & z, & \sqrt{Z} \end{vmatrix};$$

and taking $w=0$ and $=\infty$ respectively, we thus see how

$$\begin{vmatrix} x^2, & 1, & x\sqrt{X} \\ y^2, & 1, & y\sqrt{Y} \\ z^2, & 1, & z\sqrt{Z} \end{vmatrix}=0,\qquad \begin{vmatrix} x^3, & x, & \sqrt{X} \\ y^3, & y, & \sqrt{Y} \\ z^3, & z, & \sqrt{Z} \end{vmatrix}=0,$$

are each of them a particular integral of

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0.$$

Reverting to the general form

$$w=-\frac{(x^2,\ 1,\ x\sqrt{X})}{(x^3,\ x,\ \sqrt{X})},$$

this will be a general integral if only

$$dw=\Omega\left(\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}\right),$$

viz. if we have

$$-\sqrt{X}\frac{d}{dx}\frac{(x^2,\ 1,\ x\sqrt{X})}{(x^3,\ x,\ \sqrt{X})}=\Omega,\ \text{a symmetrical function of } (x,\ y,\ z).$$

The expression is

$$\Omega=\frac{1}{(x^3,\ x,\ \sqrt{X})^2}\left\{(x^2,\ 1,\ x\sqrt{X})\sqrt{X}\frac{d}{dx}(x^3,\ x,\ \sqrt{X})-(x^3,\ x,\ \sqrt{X})\sqrt{X}\frac{d}{dx}(x^2,\ 1,\ x\sqrt{X})\right\},$$

or, writing for shortness

$$\begin{aligned}\alpha&=x(y^2-z^2), & a&=yz(y^2-z^2),\\ \beta&=y(z^2-x^2), & b&=zx(z^2-x^2),\\ \gamma&=z(x^2-y^2), & c&=xy(x^2-y^2),\end{aligned}$$

we have

$$\begin{aligned}(x^2,\ 1,\ x\sqrt{X})&=\alpha\sqrt{X}+\beta\sqrt{Y}+\gamma\sqrt{Z},\\ (x^3,\ x,\ \ \sqrt{X})&=a\sqrt{X}+b\ \sqrt{Y}+c\ \sqrt{Z};\end{aligned}$$

and the formula is

$$\begin{aligned}&(x^3,\ x,\ \sqrt{X})^2\,\Omega\\ &=\ (\alpha\sqrt{X}+\beta\sqrt{Y}+\gamma\sqrt{Z})\left\{(y^3z-yz^3)\tfrac{1}{2}X'+(-3x^2z+z^3)\sqrt{XY}+(3x^2y-y^3)\sqrt{XZ}\right\}\\ &\quad-(a\sqrt{X}+b\sqrt{Y}+c\sqrt{Z})\left\{(y^2-z^2)(X+\tfrac{1}{2}X'x)-2xy\sqrt{XY}-2xz\sqrt{XZ}\right\}\\ &=\ (\alpha\sqrt{X}+\beta\sqrt{Y}+\gamma\sqrt{Z})(L+M\sqrt{XY}+N\sqrt{XZ})\\ &\quad-(a\sqrt{X}+b\sqrt{Y}+c\sqrt{Z})(P+Q\sqrt{XY}+R\sqrt{XZ}),\ \text{suppose},\end{aligned}$$

$$=\ \begin{array}{llll}\sqrt{X} & +\sqrt{Y} & +\sqrt{Z} & +\sqrt{XYZ}\\ \hline \alpha L & +\alpha MX & +\alpha NX & \\ +\beta MY & +\beta L & & +\beta N\\ +\gamma NZ & & +\gamma L & +\gamma M\\ -aP & -aQX & -aRX & \\ -bQY & -bP & & -bR\\ -cRZ & & -cP & -cQ\end{array};$$

viz. this is

$$\begin{aligned}=\ &\{\alpha L-aP+Y(\beta M-bQ)+Z(\gamma N-cR)\}\sqrt{X}\\ &+\{X(\alpha M-aQ)+\beta L-bP\}\sqrt{Y}\\ &+\{X(\alpha N-aR)+\gamma L-cP\}\sqrt{Z}\\ &+(\beta N+\gamma M-bR-cQ)\sqrt{XYZ}.\end{aligned}$$

The coefficient of $\sqrt{XYZ}$ is here

$$\begin{aligned}
= \quad & y(z^2-x^2)(3x^2y-y^3) & = \quad & y^2(z^2-x^2)(3x^2-y^2)\\
& + z(x^2-y^2)(-3x^2z+z^3) & & + z^2(x^2-y^2)(-3x^2+z^2)\\
& - zx(z^2-x^2)(2xz) & & - 2x^2z^2(z^2-x^2)\\
& - xy(x^2-y^2)(-2xy) & & + 2x^2y^2(x^2-y^2),
\end{aligned}$$

which is

$$= 6x^2y^2z^2 - y^2z^4 - y^4z^2 - z^2x^4 - z^4x^2 - x^4y^2 - x^2y^4.$$

The coefficient of $\sqrt{Y}$ is

$$\begin{aligned}
= \quad & [x(y^2-z^2)(-3x^2z+z^3) + yz(y^2-z^2)\,2xy]\,X\\
& + y(z^2-x^2)\tfrac{1}{2}X'(y^3z-yz^3) - zx(z^2-x^2)(y^2-z^2)(X+\tfrac{1}{2}X'x)\\
= & -2xz(x^2-y^2)(y^2-z^2)X - z(x^2-y^2)(y^2-z^2)(z^2-x^2)\tfrac{1}{2}X'\\
= & -(x^2-y^2)(y^2-z^2)\,z\,\{2xX+\tfrac{1}{2}(z^2-x^2)X'\},
\end{aligned}$$

where the term in { } is

$$\begin{aligned}
&= 2x(1-lx^2+x^4)+(z^2-x^2)(-lx+2x^3),\\
&= x\{2-l(z^2+x^2)+2z^2x^2\},
\end{aligned}$$

or the whole coefficient is

$$= -(x^2-y^2)(y^2-z^2)\,zx\,\{2-l(z^2+x^2)+2z^2x^2\}.$$

We obtain in like manner the coefficient of $\sqrt{Z}$, and with a little more trouble that of $\sqrt{X}$; and the final result is

$$\begin{aligned}
\Omega(x^3, x, \sqrt{X})^2 = & -(z^2-x^2)(x^2-y^2)\,yz\,\{2-l(y^2+z^2)+2y^2z^2\}\sqrt{X}\\
& -(x^2-y^2)(y^2-z^2)\,zx\,\{2-l(z^2+x^2)+2z^2x^2\}\sqrt{Y}\\
& -(y^2-z^2)(z^2-x^2)\,xy\,\{2-l(x^2+y^2)+2x^2y^2\}\sqrt{Z}\\
& +(6x^2y^2z^2-y^2z^4-y^4z^2-z^2x^4-z^4x^2-x^2y^4-x^4y^2)\sqrt{XYZ}.
\end{aligned}$$

And inasmuch as the equation is a solution of

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}+\frac{dw}{\sqrt{W}}=0,$$

it follows that $\Omega = -\sqrt{W}$, viz. that $\sqrt{W}$ is by the foregoing equation expressed as a function of x, y, z.

The equation $(x^3, x, x^2\sqrt{X}, \sqrt{X})=0$, that is,

$$\begin{vmatrix}
x^3, & x, & x^2\sqrt{X}, & \sqrt{X}\\
y^3, & y, & y^2\sqrt{Y}, & \sqrt{Y}\\
z^3, & z, & z^2\sqrt{Z}, & \sqrt{Z}\\
w^3, & w, & w^2\sqrt{W}, & \sqrt{W}
\end{vmatrix}=0,$$

gives

$$w=\frac{(x^2,\ 1,\ x\sqrt{X})}{(x^3,\ x,\ \ \sqrt{X})},$$

where the numerator and the denominator are determinants formed with the variables x, y, z.

Writing $\frac{1}{w}$ for w, it follows that the equation

$$\begin{vmatrix} x^3, & x, & x^2\sqrt{X}, & \sqrt{X} \\ y^3, & y, & y^2\sqrt{Y}, & \sqrt{Y} \\ z^3, & z, & z^2\sqrt{Z}, & \sqrt{Z} \\ w, & w^3, & \sqrt{W}, & w^2\sqrt{W} \end{vmatrix}=0$$

gives

$$w=\frac{(x^3,\ x,\ \ \sqrt{X})}{(x^2,\ 1,\ x\sqrt{X})},$$

which last equation is a transformation of

$$\begin{vmatrix} x^4, & x^2, & 1, & x\sqrt{X} \\ y^4, & y^2, & 1, & y\sqrt{Y} \\ z^4, & z^2, & 1, & z\sqrt{Z} \\ w^4, & w^2, & 1, & w\sqrt{W} \end{vmatrix}=0.$$

The two equations, involving these determinants of the order 4, are consequently equivalent equations.