623.

ON THREE-BAR MOTION.

[From the *Proceedings of the London Mathematical Society*, vol. VII. (1875—1876), pp. 136—166. Read March 10, 1876.]

THE discovery by Mr Roberts of the triple generation of a Three-Bar Curve, throws a new light on the whole theory, and is a copious source of further developments *. The present paper gives in its most simple form the theorem of the triple generation; it also establishes the relation between the nodes and foci; and it contains other researches. I have made on the subject a further investigation, which I give in a separate paper, "On the Bicursal Sextic," [624]; but the two papers are intimately related and should be read in connection.

The Three-Bar Curve is derived from the motion of a system of three bars of given lengths pivoted to each other, and to two fixed points, so as to form the three sides of a quadrilateral, the fourth side of which is the line joining the two fixed points; the curve is described by a point rigidly connected with the middle bar; or, what is more convenient, we take the middle bar to be a triangle pivoted at the extremities of the base to the other two bars (say, the radial bars), and having its vertex for the describing point.

Including the constants of position and magnitude, the Three-Bar Curve thus depends on nine parameters; viz. these are the coordinates of the two fixed points, the lengths of the connecting bars, and the three sides of the triangle. It is known that the curve is a tricircular trinodal sextic, and the equation of such a curve contains $27-6-6-3$, $=12$ constants. Imposing on the curve the condition that the three nodes lie upon a given curve, the number of constants is reduced to $12-3$, $=9$: and it is in this way that the Three-Bar Curve is distinguished from the general tricircular

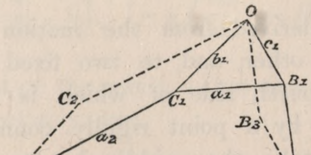
* See his paper "On Three-Bar Motion in Plane Space," *l.c.*, vol. VII., pp. 15—23, which contains more than I had supposed of the results here arrived at. There is no question as to Mr Roberts' priority in all his results.

trinodal sextic; viz. in the Three-Bar Curve the two fixed points are foci, and they determine a third focus*; and the condition is that the nodes are situate on the circle through the 3 foci.

The nodes are two of them arbitrary points on the circle; and the third of them is a point such that, measuring the distances along the circle from any fixed point of the circumference, the sum of the distances of the nodes is equal to the sum of the distances of the foci. Considering the two fixed points as given, the curve depends upon five parameters, viz. the lengths of the connecting bars and the sides of the triangle. Taking the *form* of the triangle as given, there are then only three parameters, say the lengths of the connecting bars and the base of the triangle; in this case the third focus is determined, and therefore the circle through the three foci; we may then take two of the nodes as given points on this circle, and thereby establish two relations between the three parameters, in fact, we thereby determine the differences of the squares of the lengths in question: but the third node is then an absolutely determined point on the circle, and we cannot make use of it for completing the determination of the parameters; viz. one parameter remains arbitrary. Or, what is the same thing, given the three foci and also the three nodes, consistently with the foregoing conditions, viz. the nodes lie in the centre through the three foci, the sum of the distances of the nodes being equal to the sum of the distances of the foci: we have a singly infinite series of three-bar curves.

In reference to the notation proper for the theorem of the triple generation, I shall, when only a single node of generation is attended to, take the curve to be generated as shown in the annexed Figure 1; viz. O is the generating point, OC_1B_1 the triangle, C, B the fixed points, CC_1 and BB_1 the radial bars. The sides of the

Fig. 1.



triangle are a_1, b_1, c_1; its angles are $O, =A$, $B_1, =B$, $C_1, =C$: the bars CC_1 and BB_1 are $=a_2$ and a_3 respectively, and the distance CB is $=a$. The sides a_1, b_1, c_1 may be put $=k_1(\sin A, \sin B, \sin C)$, and the lines a_1, a_2, $a_3=(k_1, k_2, k_3)\sin A$, viz. the original data a_1, b_1, c_1, a_1, a_2, a_3, may be replaced by the angles A, B, C $(A+B+C=\pi)$ and the lines k_1, k_2, k_3. And it is convenient to mention at once that the third focus A is then a point such that ABC is a triangle similar and congruent to OB_1C_1.

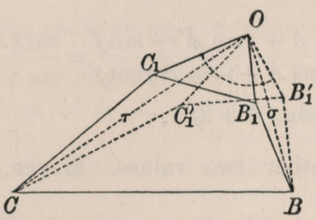
* A focus is a point, given as the intersection of a tangent to the curve from one circular point at infinity with a tangent from the other circular point at infinity; if the circular points are simple or multiple points on the curve, then the tangent or tangents *at* a circular point should be excluded from the tangents *from* the point; and the intersection of two such tangents at the two circular points respectively is not an ordinary focus; but, as the points in question are the only kind of foci occurring in the present paper, I have in the text called them foci.

It may be remarked that, producing CC_1 and BB_1 to meet in a point α, this is the centre of instantaneous rotation of the triangle, and therefore αO is the normal to the curve at O.

I proceed to show that the three nodes F, G, H are in the circle circumscribed about ABC, and that their positions are such that (the distances being measured along the circle as before) we have the property, Sum of the distances of F, G, H is equal to the Sum of the distances of A, B, C.

Supposing O to be at a node F, we have then the two equal triangles FB_1C_1, $FB_1'C_1'$, such that C_1, C_1' are equidistant from C, and B_1, B_1' equidistant from B. Hence the angles B_1FB_1', C_1FC_1' are equal; consequently the halves of these angles CFC_1' and

Fig. 2.



BFB_1' are equal; whence the angle CFB is equal to the angle $C_1'FB_1'$, that is, to the angle A; or F lies on a circle through B, C such that the segment upon BC contains the angle A, that is, upon the circle through A, B, C. To complete the investigation of the nodes, suppose $CF=\tau$, $BF=\sigma$: then the condition $\angle CFC_1' = \angle BFB_1'$ gives

$$\frac{b_1^2+\tau^2-a_2^2}{2b_1\tau}=\frac{c_1^2+\sigma^2-a_3^2}{2c_1\sigma},$$

that is,

$$c_1\sigma(b_1^2+\tau^2-a_2^2)-b_1\tau(c_1^2+\sigma^2-a_3^2)=0;$$

and the condition that F is on the circle gives

$$\sigma^2+\tau^2-2\sigma\tau\cos A=a^2.$$

These equations give six values of $(\sigma,\ \tau)$ corresponding in pairs to each other; viz. if $(\sigma_1,\ \tau_1)$ is a solution, then $(-\sigma_1,\ -\tau_1)$ is also a solution; and to each pair of solutions corresponds a single point on the circle, viz. we have thus the three nodes F, G, H.

Writing the foregoing equation in the form

$$\{c_1(b_1^2-a_2^2)\sigma-b_1(c_1^2-a_3^2)\tau\}(\sigma^2+\tau^2-2\sigma\tau\cos A)+a^2(c_1\sigma\tau^2-b_1\sigma^2\tau)=0,$$

and putting the left-hand side $=M(\sigma-p_1\tau)(\sigma-p_2\tau)(\sigma-p_3\tau)$; then, if α, β, γ denote $\cos A+i\sin A$, $\cos B+i\sin B$, $\cos C+i\sin C$ respectively, putting first $\sigma=\alpha\tau$ and next $\sigma=\dfrac{\tau}{\alpha}$, and dividing one of these results by the other, we find

$$\frac{c_1-b_1\alpha}{c_1\alpha-b_1}=\frac{\alpha-p_1\,.\,\alpha-p_2\,.\,\alpha-p_3}{1-p_1\alpha\,.\,1-p_2\alpha\,.\,1-p_3\alpha}.$$

The left-hand side is here

$$=\frac{\sin C-\alpha\sin B}{\alpha\sin C-\sin B}=\frac{\sin C-\sin B\,(\cos A+i\sin A)}{\sin C\,(\cos A+i\sin A)-\sin B}$$

$$=\frac{\sin A\,(\cos B-i\sin B)}{-\sin A\,(\cos C-i\sin C)}=-\frac{\gamma}{\beta},$$

or the equation is

$$\frac{\alpha-p_1\,.\,\alpha-p_2\,.\,\alpha-p_3}{1-p_1\alpha\,.\,1-p_2\alpha\,.\,1-p_3\alpha}=-\frac{\gamma}{\beta}.$$

Also, writing f for the angle FCB, we have $\sigma=\dfrac{\sin f}{\sin(A+f)}\tau$, viz. the values of p_1, p_2, p_3 are $\dfrac{\sin f}{\sin(A+f)}$, $\dfrac{\sin g}{\sin(A+g)}$, $\dfrac{\sin h}{\sin(A+h)}$. We thence find

$$\frac{\alpha-p_1}{1-\alpha p_1}=\frac{\sin(A+f)\,(\cos A+i\sin A)-\sin f}{\sin(A+f)-(\cos A+i\sin A)\sin f}=\frac{\cos(A+f)+i\sin(A+f)}{\cos f-i\sin f}$$

$$=\cos(A+2f)+i\sin(A+2f);$$

with the like values for the other two values. Hence, writing also

$$-\frac{\gamma}{\beta}=-\cos(C-B)-i\sin(C-B)=\cos(\pi+C-B)+i\sin(\pi+C-B),$$

the equation becomes

$$\cos(3A+2f+2g+2h)+i\sin(3A+2f+2g+2h)=\cos(\pi+C-B)+i\sin(\pi+C-B),$$

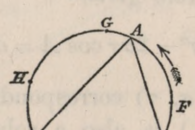
that is,

$$3A+2f+2g+2h=\pi+C-B,$$

or, what is the same thing,

$$2f+2g+2h=\pi+C-B-3A.$$

Fig. 3.



Reckoning the angles round the centre from a point Θ on the circumference, if A', B', C', F', G', H' are the angles belonging to the points A, B, C, F, G, H respectively, then

$$A'=\lambda+2C,\qquad F'=\lambda+2f,$$
$$B'=\lambda,\qquad G'=\lambda+2g,$$
$$C'=\lambda+2C+2B,\quad H'=\lambda+2h;$$

and therefore

$$A'+B'+C'=3\lambda+4C+2B,\quad F'+G'+H'=3\lambda+2f+2g+2h,\quad =3\lambda+\pi+C-B-3A;$$

that is, $A'+B'+C'-F'-G'-H'=-\pi+3(A+B+C)$; or, omitting an angle 2π, this is $A'+B'+C'=F'+G'+H'$, the equation which determines the relation between the three nodes on the circle ABC.

Reverting to the equation $c_1\sigma(b_1^2+\tau^2-a_2^2)-b_1\tau(c_1^2+\sigma^2-a_3^2)=0$, which belongs to a node: if we consider the form of the triangle as given, and write b_1, $c_1=k_1\sin B$, $k_1\sin C$, this becomes

$$\sigma\sin C(b_1^2-a_2^2)-\tau\sin B(c_1^2-a_3^2)+\sigma\tau(\tau\sin C-\sigma\sin B)=0;$$

viz. considering the node as given, then the values of σ, τ are given, and the equation establishes a relation between the values of $b_1^2-a_2^2$ and $c_1^2-a_3^2$*. If a second node be given, we have a second relation between these same quantities, and the two equations give the values of the two quantities, viz. the values of $k_1^2\sin^2 B-k_2^2\sin^2 A$, $k_1^2\sin^2 C-k_3^2\sin^2 A$, or, what is the same thing, the value of $\frac{k_1^2}{\sin^2 A}-\frac{k_2^2}{\sin^2 B}$, $\frac{k_1^2}{\sin^2 A}-\frac{k_3^2}{\sin^2 B}$. It thus appears that, if l_1, l_2, l_3 are any values of k_1, k_2, k_3 belonging to a given system of three nodes, the general values of k_1, k_2, k_3 belonging to the same system of three nodes are

$$k_1^2=l_1^2+u\sin^2 A,\quad k_2^2=l_2^2+u\sin^2 B,\quad k_3^2=l_3^2+u\sin^2 C,$$

where u is an arbitrary constant.

It may be added that there will be a node at B, if the equation is satisfied by $\tau=0$, $\sigma=a$, for the condition is $b_1^2-a_2^2=0$: that is, if $\frac{k_1}{\sin A}=\frac{k_2}{\sin B}$; similarly, there will be a node at C, if $c_1^2-a_3^2=0$, that is, if $\frac{k_1}{\sin A}=\frac{k_3}{\sin B}$; and a node at A, if $\frac{k_2}{\sin C}=\frac{k_3}{\sin B}$. If two of these equations are satisfied, the third equation is also satisfied, viz. we then have $\frac{k_1}{\sin A}=\frac{k_2}{\sin B}=\frac{k_3}{\sin C}$; and the three nodes coincide with the three foci respectively.

If, in Figure 2 (p. 553), the points C_1, C_1' coincide on the line CF, and therefore also the points B_1, B_1' coincide on the line BF, then, instead of a node at F, we have a cusp. We have in this case a triangle the sides of which are a_2+b_1, a_3+c_1, a, and the included angle between the first two sides is $=A$: we have, therefore, the relation

$$a^2=(a_2+b_1)^2+(a_3+c_1)^2-2(a_2+b_1)(a_3+c_1)\cos A.$$

Substituting herein for a, a_2, b_1, &c., the values $k\sin A$, $k_2\sin A$, $k_1\sin B$, &c., the equation is

$$k^2\sin^2 A=(k_1\sin B+k_2\sin A)^2+(k_1\sin C+k_3\sin A)^2$$
$$-2(k_1\sin B+k_2\sin A)(k_1\sin C+k_3\sin A)\cos A.$$

* Considering, in the equation, a_2 and a_3 as the distances of a variable point P from the points C and B respectively, the equation represents a circle having its centre on the line CB. Similarly, when a second node is given, the corresponding equation represents another circle, having its centre on the line CB, and the intersections of the two circles determine a_2 and a_3, the lengths of the radial bars, in order that the curve may have the given nodes.

Expanding the right-hand side and reducing by means of $A+B+C=\pi$, the whole becomes divisible by $\sin^2 A$, and we have

$$k^2=k_1^2+k_2^2+k_3^2-2k_2k_3\cos A+2k_3k_1\cos B+2k_1k_2\cos C;$$

viz. considering A, B, C, k_1, k_2, k_3 as given, this equation determines k so that the curve may have a cusp. The equation is one of the system of four equations

$$\begin{aligned}
k^2&=k_1^2+k_2^2+k_3^2-2k_2k_3\cos A+2k_3k_1\cos B+2k_1k_2\cos C,\\
k^2&=k_1^2+k_2^2+k_3^2+2k_2k_3\cos A-2k_3k_1\cos B+2k_1k_2\cos C,\\
k^2&=k_1^2+k_2^2+k_3^2+2k_2k_3\cos A+2k_3k_1\cos B-2k_1k_2\cos C,\\
k^2&=k_1^2+k_2^2+k_3^2-2k_2k_3\cos A-2k_3k_1\cos B-2k_1k_2\cos C,
\end{aligned}$$

which belong to the different arrangements CC_1F or CFC_1, BB_1F or BFB_1, of the three points on the lines BF and CF; if k has any of these four values, the curve will have a cusp. If two of the equations subsist together, we have a curve with two cusps. Taking k_1, k_2, k_3, and also $\cos A$, $\cos B$, $\cos C$, as positive, viz. assuming that the triangle is acute-angled, the fourth equation cannot subsist with any one of the others: but two of the others may subsist together, for instance, the first and second will do so, if $k_2k_3\cos A=k_3k_1\cos B$, that is, if $\dfrac{k_1}{\cos A}=\dfrac{k_2}{\cos B}$, and then $k^2=k_1^2+k_2^2+k_3^2+2k_1k_2\cos C$: the curve has then two cusps. Similarly, the three equations may subsist together, viz. we must then have

$$\frac{k_1}{\cos A}=\frac{k_2}{\cos B}=\frac{k_3}{\cos C},\quad k^2=k_1^2+k_2^2+k_3^2+2k_2k_3\cos A:$$

writing herein k_1, k_2, $k_3=\lambda\cos A$, $\lambda\cos B$, $\lambda\cos C$, we find

$$k^2=\lambda^2(\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C)=\lambda^2;$$

viz. if k_1, k_2, k_3 are respectively $=k\cos A$, $k\cos B$, $k\cos C$, the curve has then three cusps. It will be recollected that, if

$$k_1:k_2:k_3=\sin A:\sin B:\sin C,$$

the nodes coincide with the foci; the two sets of conditions subsist together, if $A=B=C=60°$; $k_1=k_2=k_3=\tfrac{1}{2}k$, viz. we have then a curve with three cusps coinciding with the three foci respectively.

Before going further, I will establish the theorem for the triple generation of the curve.

The theorem which gives the triple generation may be stated as follows. See Figures 4, 5, 6*.

Imagine a triangle ABC and a point O, through which point are drawn lines parallel to the sides dividing the triangle into three triangles OB_1C_1, OC_2A_2, OA_3B_3,

* Figure 6 (substantially the same as Fig. 5) belongs to the same curve as Figures 1 and 2, and it exhibits the triple generation of this curve: the generating point O being taken at a node (the same node as in Figure 2), and the two positions OB_1C_1 and $OB_1'C_1'$ of one of the triangles being shown in the figure.

similar *inter se* and to the original triangle, and into three parallelograms OA_2AA_3, OB_3BB_1, OC_1CC_2. Then, considering the three triangles as pivoted together at the point O, and replacing the exterior sides of the parallelograms by pairs of bars A_2AA_3, B_3BB_1, C_1CC_2 pivoted together at A, B, C, and to the triangles at A_2, A_3, B_3, B_1, C_1, C_2, the figure thus consisting of the three triangles and the six bars; let the

Fig. 4.

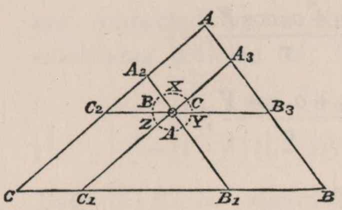


Fig. 5.

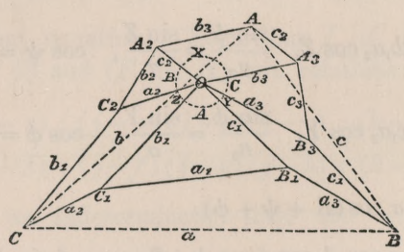
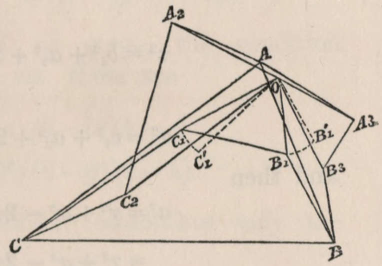


Fig. 6.



three triangles be turned at pleasure about the point O, so as to displace in any manner the points A, B, C: we have the theorem that the triangle ABC will remain always similar to the original triangle ABC, that is, to each of the three triangles OB_1C_1, OC_2A_2, OA_3B_3: and further, that, starting from any given positions of the three triangles, we may so move them as not to alter the triangle ABC in magnitude: whence, conversely, fixing the three points A, B, C, the point O will be moveable in a curve.

Assuming this, it is clear that the locus of the point O is simultaneously the locus given by

The triangle	OB_1C_1,	connected by bars	B_1B and C_1C	to fixed points	B, C,
„	OC_2A_2,	„	C_2C „ A_2A	„	C, A,
„	OA_3B_3,	„	A_3A „ B_3B	„	A, B;

or, that we have a triple generation of the same three-bar curve. It may be remarked that the intersection of the lines BB_1 and CC_1 is the axis of the instantaneous rotation of the triangle OB_1C_1, so that, joining this intersection with the point O, we have the normal at O to the locus; and similarly for the other two triangles. It of course follows that the intersections of BB_1 and CC_1, of CC_2 and AA_2, and of AA_3 and BB_3, lie on a line through O, viz. this line is the normal at O.

The result depends on the following theorem: viz. starting with the similar triangles OB_1C_1, A_2OC_2, A_3B_3O, say, the angles of these are A, B, C, so that the sides are

$$k_1(\sin A,\ \sin B,\ \sin C),\ k_2(\sin A,\ \sin B,\ \sin C),\ k_3(\sin A,\ \sin B,\ \sin C);$$

then it follows that the sides of the triangle ABC are

$$k(\sin A,\ \sin B,\ \sin C),$$

the value of k being given by the equation

$$k^2 = k_1^2 + k_2^2 + k_3^2 + 2k_2k_3\cos(X - A) + 2k_3k_1\cos(Y - B) + 2k_1k_2\cos(Z - C),$$

where X, Y, Z denote the angles A_2OA_3, B_3OB_1, C_1OC_2 respectively: whence, since $A+B+C=\pi$, we have also $X+Y+Z=\pi$. If therefore the angles X, Y, Z vary in any manner subject to this last relation and to the equation $k^2=\text{const.}$, the triangle ABC will be constant in magnitude.

There is no difficulty in proving the theorem. Writing $OC=\tau$, and $OB=\sigma$, also $\angle COC_1=\psi$, and $\angle BOB_1=\phi$, we have

$$\tau^2=b_1^2+a_2^2+2b_1a_2\cos Z,\quad \frac{\sin\psi}{a_2}=\frac{\sin Z}{\tau},\quad \cos\psi=\frac{b_1+a_2\cos Z}{\tau},$$

$$\sigma^2=c_1^2+a_3^2+2c_1a_3\cos Y,\quad \frac{\sin\phi}{a_3}=\frac{\sin Y}{\sigma},\quad \cos\phi=\frac{c_1+a_3\cos Y}{\sigma};$$

and then

$$\begin{aligned}
a^2 &= \tau^2+\sigma^2-2\tau\sigma\cos(A+\psi+\phi)\\
&= \tau^2+\sigma^2-2\tau\sigma\cos A\cos\phi\cos\psi+2\tau\sigma\cos A\sin\phi\sin\psi\\
&\qquad +2\tau\sigma\sin A\sin\psi\cos\phi+2\tau\sigma\sin A\cos\psi\sin\phi\\
&= b_1^2+c_1^2+a_2^2+a_3^2+2b_1a_2\cos Z+2c_1a_3\cos Y\\
&\quad -2\cos A\,(b_1+a_2\cos Z)(c_1+a_3\cos Y)\\
&\quad +2\cos A\,.\,a_2a_3\sin Y\sin Z\\
&\quad +2\sin A\sin Z\,.\,a_2(c_1+a_3\cos Y)\\
&\quad +2\sin A\sin Y\,.\,a_3(b_1+a_2\cos Z)\\
&= b_1^2+c_1^2-2b_1c_1\cos A+a_2^2+a_3^2\\
&\quad +2a_2a_3\,[-\cos A\,(-\sin Y\sin Z+\cos Y\cos Z)\\
&\qquad\qquad +\sin A\,(\sin Y\cos Z+\cos Y\sin Z)]\\
&\quad +2a_3\,[(c_1-b_1\cos A)\cos Y+b_1\sin A\sin Y]\\
&\quad +2a_2\,[(b_1-c_1\cos A)\cos Z+c_1\sin A\sin Z].
\end{aligned}$$

We have here $b_1^2+c_1^2-2b_1c_1\cos A=a_1^2$: the second line is $=-2a_2a_3\cos(A+Y+Z)$ which, by virtue of $Y+Z=\pi-X$, is $=2a_2a_3\cos(X-A)$: and in the third and fourth lines

$$c_1-b_1\cos A=a_1\cos B,\quad b_1\sin A=a_1\sin B,$$

$$b_1-c_1\cos A=a_1\cos C,\quad c_1\sin A=a_1\sin C;$$

whence these lines are $2a_3a_1\cos(Y-B)$, $2a_1a_2\cos(Z-C)$: the equation therefore is

$$a^2=a_1^2+a_2^2+a_3^2+2a_2a_3\cos(X-A)+2a_3a_1\cos(Y-B)+2a_1a_2\cos(Z-C),$$

which, putting therein for a_1, a_2, a_3 the values $k_1\sin A$, $k_2\sin A$, $k_3\sin A$, and assuming as above

$$k^2=k_1^2+k_2^2+k_3^2+2k_2k_3\cos(X-A)+2k_3k_1\cos(Y-B)+2k_1k_2\cos(Z-C),$$

becomes $a^2=k^2\sin^2 A$, or say $a=k\sin A$; and similarly $b=k\sin B$, $c=k\sin C$, that is, $(a, b, c)=k(\sin A, \sin B, \sin C)$, the required theorem.

Before proceeding to find the equation of the curve, I insert, by way of lemma, the following investigation:—

Three triads (A, B, C), (F, G, H), (I, J, K) of points in a line, or of lines through a point, may be in cubic involution; viz. representing A, B, &c. by the equations $x - ay = 0$, $x - by = 0$, &c., then this is the case when the cubic functions

$$(x-ay)(x-by)(x-cy),\ (x-fy)(x-gy)(x-hy),\ (x-iy)(x-jy)(x-ky),$$

are connected by a linear equation. Regarding I, J, K as given, the condition establishes between (A, B, C) and (F, G, H) two relations: viz. these are

$$\begin{aligned}&(i-a)(i-b)(i-c) : (j-a)(j-b)(j-c) : (k-a)(k-b)(k-c)\\ &=(i-f)(i-g)(i-h) : (j-f)(j-g)(j-h) : (k-f)(k-g)(k-h).\end{aligned}$$

But, if K be regarded as indeterminate, then the condition establishes only the single relation

$$\begin{aligned}&(i-a)(i-b)(i-c) : (j-a)(j-b)(j-c)\\ &=(i-f)(i-g)(i-h) : (j-f)(j-g)(j-h),\end{aligned}$$

which relation, if $i=0$, $j=\infty$, takes the form $abc=fgh$. When K is thus indeterminate, we may say that the triads (A, B, C), (F, G, H) are in cubic involution with the duad I, J.

If A, B, &c. are points on a conic, then, considering the pencils obtained by joining these points with a point Θ on the conic, if the cubic involution exists for any particular position of Θ, it will exist for every position whatever of Θ; hence, considering triads of points on a conic, we may have a cubic involution between three triads, or between two triads and a duad, as above.

Taking $x=0$, $y=0$ for the equations of the tangents at the points I, J respectively, and $z=0$ for the equation of the line joining these two points, the equation of the conic may be taken to be $xy - z^2 = 0$, and consequently the coordinates of any point A on the conic may be taken to be $x : y : z = \alpha : \frac{1}{\alpha} : 1$. It is then readily shown that α, β, γ, f, g, h referring to the points A, B, C, F, G, H respectively, the condition for the cubic involution of (A, B, C), (F, G, H) with the duad (I, J) is $\alpha\beta\gamma = fgh$.

And we thence at once prove the theorem, that there exists a cubic curve $J_A I_B I_C FGH$, viz. a cubic curve passing through J, and having there the tangent JA, having at I a node with the tangents IB, IC to the two branches respectively, and passing through the points F, G, H; viz. that the triads (A, B, C), (F, G, H) being in cubic involution with (I, J) as above, there exists a cubic curve satisfying these $2+5+3$, $=10$ conditions. In fact, the equation of the cubic curve is

$$\begin{aligned}J_A I_B I_C FGH;\ &\left(y-\frac{z}{\alpha}\right)(x-\beta z)(x-\gamma z)\\ &+\frac{z}{\alpha x}\{(x-\alpha z)(x-\beta z)(x-\gamma z)-(x-fz)(x-gz)(x-hz)\}=0,\end{aligned}$$

where observe that second term is an integral function $\frac{1}{\alpha}z^2(-Mx+Nz)$, if, for shortness,

$$M = \alpha+\beta+\gamma-f-g-h,$$
$$N = \beta\gamma+\gamma\alpha+\alpha\beta-gh-hf-fg.$$

In fact, the equations of the lines JA, IB, IC are $y-\frac{z}{\alpha}=0$, $x-\beta z=0$, $x-\gamma z=0$, respectively, and we at once see that these lines are tangents at the points I, J respectively; moreover, at the point F, we have $x, y, z = f, \frac{1}{f}, 1$. Substituting these values, the equation becomes

$$\left(\frac{1}{f}-\frac{1}{\alpha}\right)(f-\beta)(f-\gamma)+\frac{1}{\alpha f}(f-\alpha)(f-\beta)(f-\gamma)=0,$$

viz. the equation is satisfied identically, or the curve passes through F; and similarly the curve passes through G and H.

In precisely the same manner there exists a cubic curve $I_AJ_BJ_CFGH$; viz. this is

$$I_AJ_BJ_CFGH;\ (x-\alpha z)\left(y-\frac{z}{\beta}\right)\left(y-\frac{z}{\gamma}\right)$$
$$+\frac{\alpha z}{y}\left\{\left(y-\frac{z}{\alpha}\right)\left(y-\frac{z}{\beta}\right)\left(y-\frac{z}{\gamma}\right)-\left(y-\frac{z}{f}\right)\left(y-\frac{z}{g}\right)\left(y-\frac{z}{h}\right)\right\}=0,$$

where the second term is an integral function, $\alpha z^2(-M'y+N'z)$; if, for shortness,

$$M' = \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{f}-\frac{1}{g}-\frac{1}{h}=\frac{1}{\alpha\beta\gamma}N,$$
$$N' = \frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}+\frac{1}{\alpha\beta}-\frac{1}{gh}-\frac{1}{hf}-\frac{1}{fg}=\frac{1}{\alpha\beta\gamma}M,$$

in virtue of the relation $\alpha\beta\gamma=fgh$; so that the second term is in fact $=\frac{z^2}{\beta\gamma}(-Cx+Bz)$.

Writing for shortness J_A, I_A to denote these two cubics respectively, we have four other like cubics, $J_B(=J_BI_CI_AFGH)$, $I_B(=I_BJ_CJ_AFGH)$, $J_C(=J_CI_AI_BFGH)$, and $I_C(=I_CJ_AJ_BFGH)$; the equations being

$$J_A;\ \left(y-\frac{z}{\alpha}\right)(x-\beta z)(x-\gamma z)+\frac{z^2}{\alpha}\ (-Mx+Nz)=0,$$

$$J_B;\ \left(y-\frac{z}{\beta}\right)(x-\gamma z)(x-\alpha z)+\frac{z^2}{\beta}\ (-Mx+Nz)=0,$$

$$J_C;\ \left(y-\frac{z}{\gamma}\right)(x-\alpha z)(x-\beta z)+\frac{z^2}{\gamma}\ (-Mx+Nz)=0,$$

$$I_A;\ (x-\alpha z)\left(y-\frac{z}{\beta}\right)\left(y-\frac{z}{\gamma}\right)+\frac{z^2}{\beta\gamma}(-Ny+Mz)=0,$$

$$I_B;\ (x-\beta z)\left(y-\frac{z}{\gamma}\right)\left(y-\frac{z}{\alpha}\right)+\frac{z^2}{\gamma\alpha}(-Ny+Mz)=0,$$

$$I_C;\ (x-\gamma z)\left(y-\frac{z}{\alpha}\right)\left(y-\frac{z}{\beta}\right)+\frac{z^2}{\alpha\beta}(-Ny+Mz)=0.$$

We require the differences of the products $I_A J_A$, $I_B J_B$, $I_C J_C$. We find

$$I_B J_B = (x-\alpha z)(x-\beta z)(x-\gamma z)\left(y-\frac{z}{\alpha}\right)\left(y-\frac{z}{\beta}\right)\left(y-\frac{z}{\gamma}\right)+\frac{z^4}{\alpha\beta\gamma}(-Mx+Nz)(-Ny+Mz)$$

$$+\frac{z^2}{\gamma\alpha}(-Ny+Mz)\left(y-\frac{z}{\beta}\right)(x-\gamma z)(x-\alpha z)$$

$$+\frac{z^2}{\beta}(-Mx+Nz)(x-\beta z)\left(y-\frac{z}{\gamma}\right)\left(y-\frac{z}{\alpha}\right);$$

let Ω denote the sum of the two expressions in the first line. Similarly, we have

$$I_C J_C = \Omega + \frac{z^2}{\alpha\beta}(-My+Nz)\left(y-\frac{z}{\gamma}\right)(x-\alpha z)(x-\beta z)$$

$$+\frac{z^2}{\gamma}(-Mx+Nz)(x-\gamma z)\left(y-\frac{z}{\gamma}\right)\left(y-\frac{z}{\beta}\right).$$

We have thence

$$I_B J_B - I_C J_C = z^2\left\{\frac{1}{\alpha}(x-\alpha z)(-Ny+Mz)-\left(y-\frac{z}{\alpha}\right)(-Mx+Nz)\right\}$$

$$\times\left\{\frac{1}{\gamma}\left(y-\frac{z}{\beta}\right)(x-\gamma z)-\frac{1}{\beta}\left(y-\frac{z}{\gamma}\right)(x-\beta z)\right\}:$$

the factors in { } are respectively

$$=\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)(xy-z^2) \text{ and } \left(M-\frac{N}{\alpha}\right)(xy-z^2),$$

so that we have

$$I_B J_B - I_C J_C = \left(\frac{1}{\beta}-\frac{1}{\gamma}\right)\left(\frac{N}{\alpha}-M\right)z^2(xy-z^2)^2.$$

The constant factor

$$\left(\frac{1}{\beta}-\frac{1}{\gamma}\right)\left(\frac{N}{\alpha}-M\right)$$

is

$$=\left(\frac{N}{\alpha\beta}+\frac{M}{\gamma}\right)-\left(\frac{N}{\gamma\alpha}+\frac{M}{\beta}\right),\ = P_3 - P_2,$$

if P_1, P_2, P_3 denote respectively the functions

$$\frac{N}{\beta\gamma}+\frac{M}{\alpha},\ \frac{N}{\gamma\alpha}+\frac{M}{\beta},\ \frac{N}{\alpha\beta}+\frac{M}{\gamma}.$$

Attending to the equation $\alpha\beta\gamma = fgh$, it appears that we have

$$P_1 = \frac{1}{\alpha}(\alpha+\beta+\gamma-f-g-h)+\alpha\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{f}-\frac{1}{g}-\frac{1}{h}\right),$$

with like values for P_2 and P_3.

We have thus

$$I_B J_B - I_C J_C = -(P_2 - P_3)\, z^2 (xy - z^2)^2,$$

and similarly

$$I_C J_C - I_A J_A = -(P_3 - P_1)\, z^2 (xy - z^2)^2,$$

$$I_A J_A - I_B J_B = -(P_1 - P_2)\, z^2 (xy - z^2)^2.$$

Any function $I_A J_A + \lambda z^2 (xy - z^2)^2$, where λ is arbitrary, can of course be expressed in the form $I_A J_A + (\theta + P_1) z^2 (xy - z^2)^2$, where θ is arbitrary, and therefore in the three equivalent forms

$$I_A J_A + (\theta + P_1)\, z^2 (xy - z^2)^2,$$

$$I_B J_B + (\theta + P_2)\, z^2 (xy - z^2)^2,$$

$$I_C J_C + (\theta + P_3)\, z^2 (xy - z^2)^2.$$

We have $z = 0$, the line IJ: and $xy - z^2 = 0$, the conic $IJABCFGH$. The equation $I_A J_A + \lambda z^2 (xy - z^2)^2 = 0$ may thus be written in the more complete form

$$I_A J_B J_C FGH \,.\, J_A I_B I_C FGH + \lambda (IJ)^2 (IJABCFGH)^2 = 0,$$

and we hence see that it is the equation of a sextic curve, having a triple point at I, the tangents there being IA, IB, IC; having a triple point at J, the tangents there being JA, JB, JC; and having a node (double point) at each of the points F, G, H. There are thus in all $(6+3)+(6+3)+3+3+3$, $=27$ conditions, and these would in general be sufficient to determine the sextic. The data are, however, related in a special manner; viz. regarding the points I, J, F, G, H as arbitrary, the lines IA, IB, IC, JA, JB, JC are not arbitrary, but satisfy the conditions that A, B are arbitrary points, and C a determinate point, on the conic $IJABC$. And the foregoing result shows that, this being so, there exists a sextic satisfying the foregoing conditions, but containing in its equation an arbitrary constant λ or θ, and that the equation may be presented under the three forms

$$I_A J_B J_C FGH \,.\, J_A I_B I_C FGH + (\theta + P_1)(IJ)^2 (IJABCFGH)^2 = 0, \text{ \&c.},$$

corresponding to the partitions A, BC; B, CA; C, AB of the three points A, B, C.

In the case where I, J are the circular points at infinity, the conic $IJABCFGH$ is a circle passing through the six points A, B, C, F, G, H; and the condition of the cubic involution of the triads (A, B, C) and (F, G, H) with the points (I, J) is easily seen to be equivalent to the following relation, viz. the sum of the distances (measured along the circle from any fixed point of the circumference) of the three points A, B, C is equal to the sum of the distances of the three points F, G, H.

The sextic is a tricircular sextic having the three points A, B, C for foci, and having three nodes F, G, H, on the circle ABC, two of them being arbitrary points, and the third of them a determinate point on this circle. And it appears that there exists a sextic satisfying the foregoing conditions, and containing in its equation an arbitrary parameter.

I proceed to find the equation of the curve.

Consider the curve (see Fig. 1, p. 552) as generated by the point O, the vertex of the triangle OC_1B_1, connected by the bars C_1C and B_1B with the fixed points C and B respectively; and suppose, as before, $CB = a$, $C_1C = a_2$, $B_1B = a_3$, $B_1C_1 = a_1$, $OC_1 = b_1$, $OB_1 = c_1$; and draw as in the figure the parallelograms C_1CC_2O and B_1BB_3O; then O may be considered as the intersection of a circle, centre C_2 and radius C_2O, with a circle, centre B_3 and radius B_3O. Take $\angle C_2CB = \theta$, $\angle B_3BC = \phi$: the lines CC_2, BB_3 are parallel to OC_1, OB_1 respectively, and consequently $\theta + \phi = \pi - A$, a relation between the two variable angles θ, ϕ.

Taking the origin at C and the axis of x along the line CB, that of y being at right angles to it: the coordinates of C_2 are $(b_1 \cos\theta,\ b_1 \sin\theta)$, and those of B_3 are $(a - c_1\cos\phi,\ c_1 \sin\phi)$; the equations of the circles thus are

$$(x - b_1\cos\theta)^2 \qquad + (y - b_1\sin\theta)^2 = a_2^2,$$

$$(x - a + c_1\cos\phi)^2 + (y - c_1\sin\phi)^2 = a_3^2;$$

whence

$$+\,2b_1x\cos\theta + 2b_1y\sin\theta = \qquad x^2 + y^2 + b_1^2 - a_2^2,$$

$$-\,2c_1(x-a)\cos\phi + 2c_1y\sin\phi = (x-a)^2 + y^2 + c_1^2 - a_3^2,$$

which equations, writing therein for θ its value $= \pi - A - \phi$ and eliminating the single parameter ϕ, give the equation of the curve.

We in fact have

$$-\,2b_1x\cos(A+\phi) + 2b_1y\sin(A+\phi) = \qquad x^2 + y^2 + b_1^2 - a_2^2,$$

$$-\,2c_1(x-a)\cos\phi \qquad + 2c_1y\sin\phi \qquad = (x-a)^2 + y^2 + c_1^2 - a_3^2;$$

or say these are

$$P\cos\phi + Q\sin\phi = R,$$

$$P'\cos\phi + Q'\sin\phi = R',$$

where

$$P = -\,2b_1x\cos A + 2b_1y\sin A,\quad P' = -\,2c_1(x-a),$$

$$Q = \quad 2b_1x\sin A + 2b_1y\cos A,\quad Q' = \quad 2c_1y,$$

$$R = \quad x^2 + y^2 + b_1^2 - a_2^2, \qquad R' = (x-a)^2 + y^2 + c_1^2 - a_3^2.$$

The equations give therefore

$$\cos\phi : \sin\phi : -1 = QR' - Q'R : RP' - R'P : PQ' - P'Q,$$

whence

$$(QR' - Q'R)^2 + (RP' - R'P)^2 = (PQ' - P'Q)^2;$$

and it hence follows that the nodes are the common intersections of the three curves

$$QR' - Q'R = 0,\ RP' - R'P = 0,\ PQ' - P'Q = 0.$$

We have, retaining R and R' to denote their values,

$$QR' - Q'R = -2\,[(Rc_1 - R'b_1 \cos A)\, y - R'b_1 \sin A\,.\,x],$$

$$RP' - R'P = -2\,[(Rc_1 - R'b_1 \cos A)(x-a) + R'b_1 \sin A\,(y - a\cot A)],$$

$$PQ' - P'Q = -4\,b_1c_1\,[x(x-a) + y(y - a\cot A)].$$

Observing that $R=0$, $R'=0$ are circles; the equation $QR'-Q'R=0$ is a circular cubic through the point $x=0$, $y=0$; the equation $RP'-R'P=0$, a circular cubic through the point $x=a$, $y=a\cot A$; and the equation $PQ'-P'Q=0$, a circle through these two points (and also the points $x=0$, $y=a\cot A$; $x=a$, $y=0$). Hence the first and third curves intersect in the point $(x=0,\ y=0)$, in the circular points at infinity, and in three other points which are the nodes; viz. the curve has three nodes, say these are F, G, H. The second and third curves intersect in the point $(x=0,\ y=a\cot A)$, in the circular points at infinity, and in the three nodes. As regards the first and second curves, it is readily shown that these touch at the circular points at infinity; viz. they intersect in these points each twice, in the two finite intersections of the circles $R=0$, $R'=0$, and in the three nodes.

The three nodes F, G, H thus lie in the circle

$$x(x-a) + y(y - a\cot A) = 0,$$

which passes through the points $(x=0,\ y=0)$ and $(x=a,\ y=0)$, that is, the points C and B. Assuming $b = \dfrac{a\sin A}{\sin B}$, the circle also passes through the point $x = b\cos C$, $y = b\sin C$, that is, the point A of the figure. Thus the three nodes F, G, H lie in the circle circumscribed about the triangle ABC.

Writing, for greater convenience,

$$R = x^2 + y^2 - e^2,\quad R' = x^2 + y^2 - 2ax - f^2,$$

the nodes F, G, H lie on the two curves

$$c_1y(x^2+y^2-e^2) - b_1 \sin A\,(x + y\cot A)(x^2+y^2-2ax-f^2) = 0,$$

$$x^2 + y^2 = a(x + y\cot A).$$

The first of these is

$$\begin{aligned}&[c_1y - b_1\sin A\,(x + y\cot A)](x^2+y^2)\\ &+ [b_1 \sin A\,(x+y\cot A) f^2 - c_1e^2y]\\ &+ 2ab_1 \sin A\,(x + y\cot A)\,x = 0.\end{aligned}$$

We may combine these equations so as to obtain the equation of the triad of lines CF, CG, CH; viz. multiplying the second and the third terms of the first equation by $\dfrac{(x^2+y^2)^2}{a^2(x+y\cot A)^2}$ and $\dfrac{x^2+y^2}{a(x+y\cot A)}$ (each $=1$ in virtue of the second equation), the equation becomes divisible by x^2+y^2: and, throwing this out, the equation is

$$\begin{aligned}&c_1y - b_1\sin A\,(x+y\cot A)\\ &+ [b_1\sin A\,(x+y\cot A) f^2 - c_1e^2y]\,\frac{x^2+y^2}{a^2(x+y\cot A)^2}\\ &+ 2b_1x\sin A = 0,\end{aligned}$$

where the first and the third terms together are $=(c_1-b_1\cos A)\,y+b_1x\sin A$, viz. this is $=a_1\sin B\,(x+y\cot B)$. Hence, writing also in the second term $a_1\sin B$ for $b_1\sin A$, the equation is

$$(x+y\cot A)^2\,(x+y\cot B)+\frac{1}{a^2}\left\{(x+y\cot A)\,f^2-\frac{c_1e^2}{a_1\sin B}\,y\right\}(x^2+y^2)=0\,;$$

or say this is

$$(x\sin A+y\cos A)^2\,(x\sin B+y\cos B)$$
$$+\frac{\sin A\sin B}{a^2}\left\{(x\sin A+y\cos A)\,f^2-\frac{c_1e^2}{b_1}\,y\right\}(x^2+y^2)=0\,;$$

viz. there is a term in x^2+y^2, and another term

$$(x\sin A+y\cos A)^2\,(x\sin B+y\cos B).$$

Suppose for a moment that the angles FBC, GBC, HBC are called F, G, H; then the function on the left hand must be

$$=M\,(x\sin F-y\cos F)\,(x\sin G-y\cos G)\,(x\sin H-y\cos H).$$

Writing in the identity $x=iy$, we have

$$(\cos A+i\sin A)^2\,(\cos B+i\sin B)=-M\,(\cos F-i\sin F)\,(\cos G-i\sin G)\,(\cos H-i\sin H)\,;$$

and similarly, writing $x=-iy$, we have the like equation with $-i$ instead of $+i$; whence, dividing the two equations and taking the logarithms,

$$4A+2B=2m\pi-F-G-H,$$

which leads as before to the relation $A'+B'+C'=F'+G'+H'$.

In completion of the investigation, observe that M is determinately $+1$ or -1: and that

$$\frac{\sin A\sin B}{a^2}\left\{(x\sin A+y\cos A)\,f^2-\frac{c_1}{b_1}\,e^2y\right\}$$

is the linear factor of

$$M\,(x\sin F-y\cos F)\,(x\sin G-y\cos G)\,(x\sin H-y\cos H)$$
$$-(x\sin A+y\cos A)^2\,(x\sin B+y\cos B),$$

which remains after throwing out the factor x^2+y^2. Calling this linear factor $px+qy$, we have

$$\frac{a^2p}{\sin A\sin B}=f^2\sin A,\quad \frac{a^2q}{\sin A\sin B}=f^2\cos A-\frac{c_1}{b_1}\,e^2,$$

or, as this last equation may be written,

$$\frac{a^2q}{\sin A\sin B}=f^2\cos A-\frac{\sin C}{\sin B}\,e^2.$$

Hence, writing $a=k\sin A$, we have

$$f^2=\frac{k^2p}{\sin B},\quad e^2=\frac{k^2}{\sin C}\,(p\cos A-q\sin A)\,;$$

substituting for f^2 and e^2 their values, we have

$$-k_1^2\sin^2 C+k_3^2\sin^2 A=\frac{k^2p}{\sin B}+k^2\sin^2 A,$$

$$-k_1^2\sin^2 B+k_2^2\sin^2 A=\frac{k^2}{\sin C}(p\cos A-q\sin A),$$

or, what is the same thing,

$$-\frac{k_1^2}{\sin^2 A}+\frac{k_3^2}{\sin^2 C}=\frac{k^2}{\sin^2 A\sin B\sin^2 C}(p+\sin^2 A\sin B)$$

$$=\frac{k^2\sin F\sin G\sin H}{\sin A\sin C\,.\sin A\sin B\sin C},$$

$$-\frac{k_1^2}{\sin^2 A}+\frac{k_2^2}{\sin^2 B}=\frac{k^2}{\sin^2 A\sin^2 B\sin C}(p\cos A-q\sin A)$$

$$=\frac{k^2\sin(A-F)\sin(A-G)\sin(A-H)}{\sin A\sin B\,.\sin A\sin B\sin C},$$

which are the relations connecting k_1, k_2, k_3, when the foci and nodes are given.

It is to be remarked that if, for instance, $F=0$ and $G=A$, then $k_1 : k_2 : k_3 = \sin A : \sin B : \sin C$; the nodes in this case coincide with the foci. A simple example is when $A=B=C$; the three triangles are here equal equilateral triangles. The general equations show that, if l_1, l_2, l_3 are values of k_1, k_2, k_3 belonging to a given set of nodes and foci, then the values $k_1^2=l_1^2+u\sin^2 A$, $k_2^2=l_2^2+u\sin^2 B$, $k_3^2=l_3^2+u\sin^2 C$ (where u is arbitrary) will belong to the same set of nodes and foci.

I write the equation of the curve in the form

$$\{(QR'-Q'R)+i(RP'-R'P)\}\{QR'-Q'R-i(RP'-R'P)\}-(PQ'-P'Q)^2=0,$$

where

$$(QR'-Q'R)+i(RP'-R'P)=(Rc_1-R'b_1\cos A)\,i\,(x-a-iy)-R'b_1\sin A\,\{x-i(y-a\cot A)\}.$$

Calling I, J the circular points $(\infty,\ x+iy=0)$ and $(\infty,\ x-iy=0)$, this is a nodal circular cubic having I for an ordinary point, but J for a node. Moreover, one of the tangents at J is the line $x-iy=0$, that is, the line JB; in fact, writing as before

$$R=x^2+y^2-e^2,\quad R'=x^2+y^2-2ax-f^2,$$

then, when $x-iy=0$, we have $R=-e^2$, $R'=-2ax-f^2$, and the equation becomes

$$\{-ce^2+b_1\cos A\,(2ax+f^2)\}(-ia)+b_1\sin A\,(2ax+f^2)(ia\cot A)=0\,;$$

viz. the term in x here disappears, or the three intersections are at infinity. The other tangent at J is the line $x-a-iy=0$, that is, the line JC; in fact, when $x-a-iy=0$, that is, $y=-i(x-a)$, we have $R=2ax-a^2-e^2$, $R'=-a^2-f^2$, and the equation becomes

$$\{c_1(2ax-a^2-e^2)+b_1\cos A\,(a^2+f^2)\}\,.\,0+b_1\sin A\,(a^2+f^2)\,.\,a(1+i\cot A)=0,$$

viz. the three intersections are here at infinity. The tangent at I is the line $x - b\cos C + i(y - b\sin C) = 0$, that is, the line IA; in fact, writing this in the form

$$y = ix - ib(\cos C + i\sin C) = ix - ib\gamma,$$

(if for a moment $\cos C + i\sin C = \gamma$, and similarly $\cos A + i\sin A = \alpha$, $\cos B + i\sin B = \beta$); then, y having this value, we find

$$R = 2bxy - b^2\gamma^2 - e^2,\; R' = 2(b\gamma - a)x - b^2\gamma^2 - f^2,$$
$$= -\frac{2c}{\beta}x - b^2\gamma^2 - f^2;$$

and the equation becomes

$$\left.\begin{Bmatrix} C_1(2b\gamma x - b^2\gamma^2 - e^2) \\ -b_1\cos A\left(-\dfrac{2c}{\beta}x - b^2\gamma^2 - e^2\right)\end{Bmatrix}\right. i(2x - a - b\gamma)$$
$$-b_1\sin A\left(-\frac{2c}{\beta}x - b^2\gamma^2 - e^2\right)(2x + ia\cot A - b\gamma) = 0.$$

The coefficient of x^2 is here

$$2i\left(bc_1\gamma + \frac{b_1c}{\beta}\cdot\frac{1}{\alpha}\right),$$

or, since $b_1c = bc_1$, this is

$$= 2ibc_1\left(\gamma + \frac{1}{\alpha\beta}\right),\; = 0,$$

in virtue of the relation $A + B + C = \pi$, giving $\alpha\beta\gamma = -1$: hence there is only one finite intersection, or the line IA is a tangent.

The cubic in question

$$QR' - Q'R + i(RP' - R'P) = 0$$

is thus a nodal circular cubic which it is convenient to represent in the form

$$(I_AJ_BJ_CFGH) = 0;$$

viz. this is a cubic, through I with the tangent IA, having J as a node with the tangents JB, JC, and through the points F, G, H. Observe that, if F, G, H were arbitrary, this would be $2 + 5 + 3$, $= 10$ conditions. The before-mentioned relation is, in fact, the condition in order to the existence of the cubic.

Similarly the cubic

$$QR' - Q'R - i(RP' - R'P) = 0$$

is the cubic

$$(J_AI_BI_CFGH) = 0.$$

The circle $PQ' - P'Q = 0$ is the conic through I, J, A, B, C, F, G, H; or it may in like manner be written $(IJABCFGH) = 0$; and we may write $(IJ) = 0$, as the equation of the line infinity. The functions denoted as above contain implicitly constant multipliers which give, in the equation of the three-bar curve, one arbitrary parameter—and the equation thus is

$$(I_A J_B J_C FGH)(J_A I_B I_C FGH) - \theta (IJ)^2 (IJABCFGH)^2 = 0,$$

a form which puts in evidence that I, J are triple points having the tangents IA, IB, IC, and JA, JB, JC respectively (whence also A, B, C are foci), and that F, G, H are nodes; viz. the result is as follows:—

Taking A, B, C, F, G, H points in a circle, such that, Sum of the distances (being the angular distances from a fixed point in the circumference) of A, B, C is equal to the sum of the distances of F, G, H: then there exist the cubics $(I_A J_B J_C FGH) = 0$, $(J_A I_B I_C FGH) = 0$, and the sextic is as above.

Writing for shortness

$$(I_A J_B J_C FGH) = I_A, \quad (J_A I_B I_C FGH) = J_A,$$

then the above form is clearly one of three equivalent forms

$$\begin{aligned} U &= I_A J_A - \theta_1 \Omega^2, \\ &= I_B J_B - \theta_2 \Omega^2, \\ &= I_C J_C - \theta_3 \Omega^2. \end{aligned}$$

This implies an identical linear relation between the functions $I_A J_A$, $I_B J_B$, $I_C J_C$; whence also U and Ω^2 are each of them a linear function of any two of these quantities.

I originally obtained the equation of the curve in a form which, though far less valuable than the preceding one, is nevertheless worth preserving; viz. the equation

$$(QR' - Q'R)^2 + (RP' - R'P)^2 = (PQ' - P'Q)^2$$

may be written

$$(R^2 - P^2 - Q^2)(R'^2 - P'^2 - Q'^2) - (RR' - PP' - QQ')^2 = 0,$$

which equation, substituting therein for P, Q, R, P', Q', R' their values, gives the form in question.

Proceeding to the reduction, we have

$$\begin{aligned} R^2 - P^2 - Q^2 &= (x^2 + y^2 + b_1^2 - a_2^2)^2 - 4b_1^2 (x^2 + y^2) \\ &= (x^2 + y^2)^2 - 2(b_1^2 + a_2^2)(x^2 + y^2) + (b_1^2 - a_2^2)^2 \\ &= (x^2 + y^2 - \overline{b_1 + a_2}^2)(x^2 + y^2 - \overline{b_1 - a_2}^2); \end{aligned}$$

$$\begin{aligned} R'^2 - P'^2 - Q'^2 &= (\overline{x-a}^2 + y^2 + c_1^2 - a_3^2)^2 - 4c_1^2 (\overline{x-a}^2 + y^2) \\ &= (\overline{x-a}^2 + y^2)^2 - 2(c_1^2 + a_3^2)(\overline{x-a}^2 + y^2) + (c_1^2 - a_3^2)^2 \\ &= (\overline{x-a}^2 + y^2 - \overline{c_1 + a_3}^2)(\overline{x-a}^2 + y^2 - \overline{c_1 - a_3}^2). \end{aligned}$$

But the reduction of $RR' - PP' - QQ'$ is somewhat longer. We have

$$RR' - PP' - QQ' = (x^2 + y^2 + b_1^2 - a_2^2)(\overline{x-a}^2 + y^2 + c_1^2 - a_3^2)$$
$$- 4b_1c_1 (x\,\overline{x-a} + y^2) \cos A - 4b_1c_1\, ay \sin A :$$

and here

$$2b_1c_1 \cos A = b_1^2 + c_1^2 - a_1^2, \quad 2x(x-a) + 2y^2 = x^2 + y^2 + (x-a)^2 + y^2 - a^2,$$

also $b_1c_1 \sin A = a_1p_1$, if p_1 be the perpendicular distance of O from the base B_1C_1. Hence the second line is

$$-(b_1^2 + c_1^2 - a_1^2)(x^2 + y^2 + \overline{x-a}^2 + y^2 - a^2) - 4aa_1p_1y,$$

and the whole is

$$\begin{aligned} = \;& (x^2+y^2)(\overline{x-a}^2 + y^2) \\ &+ (x^2+y^2)(a_1^2 - b_1^2 - a_3^2) \\ &+ (\overline{x-a}^2 + y^2)(a_1^2 - c_1^2 - a_2^2) \\ &+ (b_1^2 - a_2^2)(c_1^2 - a_3^2) + a^2(b_1^2 + c_1^2 - a_1^2) - 4aa_1p_1y\,; \end{aligned}$$

whence, finally, we have

$$RR' - PP' - QQ' = (x^2 + y^2 + a_1^2 - a_2^2 - c_1^2)(\overline{x-a}^2 + y^2 + a_1^2 - a_3^2 - b_1^2)$$
$$+ (a^2 + a_1^2 - a_2^2 - a_3^2)(b_1^2 + c_1^2 - a_1^2) - 4aa_1p_1y.$$

Hence the equation of the curve is

$$(x^2 + y^2 - \overline{b_1 + a_2}^2)(x^2 + y^2 - \overline{b_1 - a_2}^2)(\overline{x-a}^2 + y^2 - \overline{c_1 + a_3}^2)(\overline{x-a}^2 + y^2 - \overline{c_1 - a_3}^2)$$
$$- \{(x^2 + y^2 + a_1^2 - a_2^2 - c_1^2)(\overline{x-a}^2 + y^2 + a_1^2 - a_3^2 - b_1^2)$$
$$+ (a^2 + a_1^2 - a_2^2 - a_3^2)(b_1^2 + c_1^2 - a_1^2) - 4aa_1p_1y\}^2 = 0,$$

where p_1 is given in terms of the constants a_1, b_1, c_1 by the equation

$$2a_1p_1 = \sqrt{2b_1^2c_1^2 + 2c_1^2a_1^2 + 2a_1^2b_1^2 - a_1^4 - b_1^4 - c_1^4}.$$

There are in the equation two terms, $(x^2+y^2)^2$, $(\overline{x-a}^2 + y^2)^2$, which destroy each other, and the remaining terms are of the order 6 at most. Hence the curve is a sextic; and it is, moreover, readily seen that the curve is tricircular. Assuming this, it appears at once that the lines $x + iy = 0$, $x - iy = 0$ are tangents to the curve at the two circular points at infinity. In fact, assuming either of these equations, we have $x^2 + y^2 = 0$, and the equation becomes

$$(b_1^2 - a_2^2)(-2ax + a^2 - \overline{c_1 + a_3}^2)(-2ax + a^2 - \overline{c_1 - a_3}^2)$$
$$- \{(a_1^2 - a_2^2 - c_1^2)(-2ax + a^2 + a_1^2 - a_3^2 - b_1^2)$$
$$+ (a^2 + a_1^2 - a_2^2 - a_3^2)(b_1^2 + c_1^2 - a_1^2) - 4aa_1p_1y\}^2 = 0,$$

a quadric equation. Hence there are on each of the two lines only two finite intersections, or the number of intersections at infinity is $= 4$; viz. the line is a tangent to one of the branches at the triple point. Similarly, the lines $x-a+iy=0$, $x-a-iy=0$ are tangents. Thus the points C and B are foci. It might with somewhat more difficulty be shown from the equation that the point $x=b\cos C$, $y=b\sin C$ $\left(\text{where, as before, } b=\dfrac{a\sin A}{\sin B}\right)$, viz. the point A of the figure, is a focus; but I have not verified this directly. It clearly follows, if we generate the curve by means of the triangle OA_2C_2 and the fixed points C, A. Hence A, B, C are a triad of foci, and the theorem as to the nodes is that these lie on the circle drawn through the three foci A, B, C.

I prove in a somewhat different manner, for the sake of the further theory which arises, the theorem of the triple generation; for this purpose, constructing the foregoing Figure 2 (p. 553) by means of the three triangles OB_1C_1, OC_2A_2, OA_3B_3, but without assuming anything as to the form or position of the triangle ABC, I draw through O a line Ox, the position of which is in the first instance arbitrary, say its inclination to OC_2 is $=v$; and drawing Oy at right angles to Ox, I proceed, in regard to these axes, to find the coordinates of the points C, B. We have, for C,

$$x=a_2\cos v+b_1\cos(v+Z),\quad y=a_2\sin v+b_1\sin(v+Z);$$

for B,

$$x=c_1\cos(v+A+Z)+a_3\cos(v+A+Z+Y),$$

$$y=c_1\sin(v+A+Z)+a_3\sin(v+A+Z+Y);$$

or, writing for $Y+Z$ the value $\pi-X$, so that

$$v+A+Z+Y=\pi+v+A-X,$$

the coordinates of B are

$$x=c_1\cos(v+A+Z)-a_3\cos(v+A-X),$$

$$y=c_1\sin(v+A+Z)-a_3\sin(v+A-X).$$

Taking the two values of y equal to each other, the equation to determine v is

$$a_2\sin v+b_1\sin(v+Z)-c_1\sin(v+A+Z)+a_3\sin(v+A-X)=0.$$

We make the line Ox parallel to BC, so that, writing

$$x=a_2\cos v\qquad\qquad +b_1\cos(v+Z),$$

$$x-a=c_1\cos(v+A+Z)-a_3\cos(v+A-X),$$

we have

$$a=a_2\cos v+b_1\cos(v+Z)-c_1\cos(v+A+Z)+a_3\cos(v+A-X),$$

which determines the distance BC, $=a$. And moreover, writing

$$y=a_2\sin v\qquad\qquad +b_1\sin(v+Z),$$

$$=c_1\sin(v+A+Z)-a_3\sin(v+A-X),$$

we have y as the perpendicular distance of O from BC, and x and $(a-x)$ as the two parts into which BC is divided by the foot of this perpendicular. In the reduction of the formulæ we assume that the three triangles are similar; viz. we write

$$(a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3)$$

$$= k_1(\sin A, \sin B, \sin C), k_2(\sin A, \sin B, \sin C), k_3(\sin A, \sin B, \sin C);$$

and we use when required the relation $A+B+C=\pi$.

The equation for v becomes

$$k_1 \sin(v-C+Z) + k_2 \sin v + k_3 \sin(v+A-X) = 0,$$

which may be written

$$L \sin v - M \cos v = 0,$$

where

$$L = k_2 + k_1 \cos(Z-C) + k_3 \cos(X-A),$$

$$M = \quad -k_1 \sin(Z-C) + k_3 \sin(X-A);$$

hence, putting

$$k^2 = k_1^2 + k_2^2 + k_3^2 + 2k_2k_3 \cos(X-A) + 2k_3k_1 \cos(Y-B) + 2k_1k_2 \cos(Z-C),$$

we have $L^2+M^2=k^2$, that is, $\sqrt{L^2+M^2}=k$, and therefore

$$k \sin v = M, \quad k \cos v = L,$$

which gives the value of v; and then, after all reductions,

$$\begin{aligned} kx = {} & k_1^2 \sin B \cos C + k_2^2 \sin A + k_3^2 . 0 + k_2k_3 \sin A \cos(X-A) \\ & + k_3k_1[-\sin B \cos(Y+A)] \\ & + k_1k_2[\sin(B-A)\cos(Z+A) + 2 \sin A \sin B \sin(Z+A)], \end{aligned}$$

$$\begin{aligned} k(a-x) = {} & k_1^2 \sin C \cos B + k_2^2 . 0 + k_3^2 \sin A + k_2k_3 \sin A \cos(X-A) \\ & + k_3k_1[\sin(C-A)\cos(Y+A) + 2 \sin A \sin C \sin(Y+A)] \\ & + k_1k_2[-\sin C \cos(Z+A)], \end{aligned}$$

and

$$ky = k_1^2 \sin B \sin C + k_2k_3 \sin A \sin(X-A) + k_3k_1 \sin B \sin(Y+A) + k_1k_2 \sin C \sin(Z+A).$$

The first and second equations give $ka = k^2 \sin A$, that is, $a = k \sin A$; and, similarly, $b = k \sin B$, $c = k \sin C$: viz. we have

$$(a, b, c) = k(\sin A, \sin B, \sin C),$$

or the triangle ABC is similar to the other three triangles, its magnitude being given by the foregoing equation for k^2. These are the properties which give the triple generation.

Changing the notation of the coordinates, and writing (x, y, z) for the perpendicular distances from O on the sides of the triangle ABC, we have, as above,

$$kx = k_1^2 \sin B \sin C + k_2k_3 \sin A \sin (X - A) + k_3k_1 \sin B \sin (Y + A) + k_1k_2 \sin C \sin (Z + A),$$

and therefore

$$ky = k_2^2 \sin B \sin C + k_2k_3 \sin A \sin (X + B) + k_3k_1 \sin B \sin (Y - B) + k_1k_2 \sin C \sin (Z + B),$$

$$kz = k_3^2 \sin C \sin A + k_2k_3 \sin A \sin (X + C) + k_3k_1 \sin B \sin (Y + C) + k_1k_2 \sin C \sin (Z - C),$$

values which give, as they should do,

$$x \sin A + y \sin B + z \sin C = k^2 \sin A \sin B \sin C.$$

Taking (x, y, z) as simply proportional (instead of equal) to the perpendicular distances, then (x, y, z) will be a system of trilinear coordinates in which the equation of the line infinity is

$$x \sin A + y \sin B + z \sin C = 0;$$

and considering (x, y, z) as proportional to the foregoing values, and in these X, Y, Z as connected by the equation $X + Y + Z = \pi$ and by the equation which determines k^2, the coordinates (x, y, z) are given as proportional to functions of a single parameter, so that the equations in effect determine the curve which is the locus of O.

But to determine the order, &c., the trigonometrical functions must be expressed algebraically; and this is done most readily by introducing instead of X, Y, Z the functions

$$\cos X + i \sin X, \quad \cos Y + i \sin Y, \quad \cos Z + i \sin Z, \quad = \xi, \ \eta, \ \zeta;$$

and we may at the same time, in place of A, B, C, introduce the functions

$$\cos A + i \sin A, \quad \cos B + i \sin B, \quad \cos C + i \sin C, \ = \alpha, \ \beta, \ \gamma.$$

The relation $X + Y + Z = \pi$ gives $\xi\eta\zeta = -1$; and similarly $A + B + C = \pi$ gives $\alpha\beta\gamma = -1$.

We have

$$\cos (X - A) = \tfrac{1}{2}\left(\frac{\xi}{\alpha} + \frac{\alpha}{\xi}\right), \quad i \sin (X - A) = \tfrac{1}{2}\left(\frac{\xi}{\alpha} - \frac{\alpha}{\xi}\right), \text{ \&c.};$$

the equation $k^2 = k_1^2 + \&c.$ becomes

$$k^2 = k_1^2 + k_2^2 + k_3^2 + k_2k_3\left(\frac{\xi}{\alpha} + \frac{\alpha}{\xi}\right) + k_3k_1\left(\frac{\eta}{\beta} + \frac{\beta}{\eta}\right) + k_1k_2\left(\frac{\zeta}{\gamma} + \frac{\gamma}{\zeta}\right),$$

or, as this may be written,

$$(-k^2 + k_1^2 + k_2^2 + k_3^2) + k_2k_3\left(\frac{\xi}{\alpha} - \alpha\eta\zeta\right) + k_3k_1\left(\frac{\eta}{\beta} - \beta\zeta\xi\right) + k_1k_2\left(\frac{\zeta}{\gamma} - \gamma\xi\eta\right) = 0.$$

Also the value of x is proportional to

$$k_1^2\left(\beta - \frac{1}{\beta}\right)\left(\gamma - \frac{1}{\gamma}\right) + k_2k_3\left(\alpha - \frac{1}{\alpha}\right)\left(\frac{\xi}{\alpha} - \frac{\alpha}{\xi}\right) + k_3k_1\left(\beta - \frac{1}{\beta}\right)\left(\alpha\eta - \frac{1}{\alpha\eta}\right) + k_1k_2\left(\gamma - \frac{1}{\gamma}\right)\left(\alpha\zeta - \frac{1}{\alpha\zeta}\right),$$

or, what is the same thing, to

$$k_1^2\left(\beta-\frac{1}{\beta}\right)\left(\gamma-\frac{1}{\gamma}\right)+k_2k_3\left(\alpha-\frac{1}{\alpha}\right)\left(\frac{\xi}{\alpha}+\alpha\eta\zeta\right)+k_3k_1\left(\beta-\frac{1}{\beta}\right)\left(\alpha\eta+\frac{\zeta\xi}{\alpha}\right)+k_1k_2\left(\gamma-\frac{1}{\gamma}\right)\left(\alpha\zeta+\frac{\xi\eta}{\alpha}\right);$$

with the like expressions as to the values of y and z. Introducing for homogeneity a quantity ω, viz. writing $\frac{\xi}{\omega}$, $\frac{\eta}{\omega}$, $\frac{\zeta}{\omega}$ in place of ξ, η, ζ, we have the parameters $(\xi, \eta, \zeta, \omega)$ connected by the homogeneous equations

$$\xi\eta\zeta+\omega^3=0,$$

$$(-k^2+k_1^2+k_2^2+k_3^2)\,\omega^2+k_2k_3\left(\frac{\xi\omega}{\alpha}-\alpha\eta\zeta\right)+k_3k_1\left(\frac{\eta\omega}{\beta}-\beta\zeta\xi\right)+k_1k_2\left(\frac{\zeta\omega}{\gamma}-\gamma\xi\eta\right)=0,$$

and the ratios of the coordinates are

$$\begin{aligned}
x : y : z = {} & k_1^2\left(\beta-\frac{1}{\beta}\right)\left(\gamma-\frac{1}{\gamma}\right)\omega^2+k_2k_3\left(\alpha-\frac{1}{\alpha}\right)\left(\frac{\xi\omega}{\alpha}+\alpha\eta\zeta\right)\\
& +k_3k_1\left(\beta-\frac{1}{\beta}\right)\left(\alpha\eta\omega+\frac{\xi\zeta}{\alpha}\right)+k_1k_2\left(\gamma-\frac{1}{\gamma}\right)\left(\alpha\zeta\omega+\frac{\xi\eta}{\alpha}\right)\\
& : k_2^2\left(\gamma-\frac{1}{\gamma}\right)\left(\alpha-\frac{1}{\alpha}\right)\omega^2+k_2k_3\left(\alpha-\frac{1}{\alpha}\right)\left(\beta\xi\omega+\frac{\eta\zeta}{\beta}\right)\\
& +k_3k_1\left(\beta-\frac{1}{\beta}\right)\left(\frac{\eta\omega}{\beta}+\beta\zeta\xi\right)+k_1k_2\left(\gamma-\frac{1}{\gamma}\right)\left(\beta\zeta\omega+\frac{\xi\eta}{\beta}\right)\\
& : k_3^2\left(\alpha-\frac{1}{\alpha}\right)\left(\beta-\frac{1}{\beta}\right)\omega^2+k_2k_3\left(\alpha-\frac{1}{\alpha}\right)\left(\gamma\xi\omega+\frac{\eta\zeta}{\gamma}\right)\\
& +k_3k_1\left(\beta-\frac{1}{\beta}\right)\left(\gamma\eta\omega+\frac{\zeta\xi}{\gamma}\right)+k_1k_2\left(\gamma-\frac{1}{\gamma}\right)\left(\frac{\zeta\omega}{\gamma}+\gamma\xi\eta\right).
\end{aligned}$$

Suppose, for shortness, these are $x : y : z = P : Q : R$. Observe that the form of the equations is $\xi\eta\zeta+\omega^3=0$, $\Omega=0$, and $x : y : z = P : Q : R$, where Ω and P, Q, R are each of them a quadric function of the form $(\omega^2, \omega\xi, \omega\eta, \omega\zeta, \eta\zeta, \zeta\xi, \xi\eta)$, the terms in ξ^2, η^2, ζ^2 being wanting.

Treating $(\xi, \eta, \zeta, \omega)$ as the coordinates of a point in space, the equation $\xi\eta\zeta+\omega^3=0$ is a cubic surface having a binode at each of the points $(\xi=0, \omega=0)$, $(\eta=0, \omega=0)$, $(\zeta=0, \omega=0)$, and the second equation is that of a quadric surface passing through these three points; hence the two equations together represent a sextic in space, or say a skew sextic, having a node at each of these three points. The equations $x : y : z = P : Q : R$ establish a (1, 1) correspondence between the locus of O and this skew sextic. To find the degree of the locus we intersect it by the arbitrary line $ax+by+cz=0$; viz. we intersect the skew sextic by the quadric surface $aP+bQ+cR=0$. This is a surface passing through the three nodes of the skew sextic, and it therefore besides intersects the skew sextic in $12-2.3, =6$ points. Hence the locus is (as it should be) a sextic.

I consider the point $\eta=0$, $\zeta=0$, $\omega=0$, or say the point $(1, 0, 0, 0)$, of the skew sextic. This is a node, and for the consecutive point on one branch we have $\eta : \zeta : \omega = m\epsilon : l\epsilon^2 : n\epsilon$, where ϵ is infinitesimal. The equation of the cubic surface gives $lm+n^2=0$, and the equation of the quadric surface gives $k_2k_3 \cdot \frac{\omega}{\alpha} - k_1k_2\gamma\eta = 0$, that is, $k_3\omega = \alpha\gamma k_1\eta$, which, in fact, determines the ratio $l : m$; but it will be convenient to retain the equation in this form. For the corresponding values of (x, y, z) we have

$$x : y : z = k_3\left(\alpha-\frac{1}{\alpha}\right)\frac{\omega}{\alpha} + k_1\left(\gamma-\frac{1}{\gamma}\right)\frac{\eta}{\alpha}$$

$$: k_3\left(\alpha-\frac{1}{\alpha}\right)\beta\omega + k_1\left(\gamma-\frac{1}{\gamma}\right)\frac{\eta}{\beta}$$

$$: k_3\left(\alpha-\frac{1}{\alpha}\right)\gamma\omega + k_1\left(\gamma-\frac{1}{\gamma}\right)\gamma\eta,$$

which, writing for $k_3\omega$ its value $=\alpha\gamma k_1\eta$, become

$$x : y : z = \left(\alpha-\frac{1}{\alpha}\right)\gamma + \left(\gamma-\frac{1}{\gamma}\right)\frac{1}{\alpha} = \alpha\gamma - \frac{\gamma}{\alpha} + \frac{\gamma}{\alpha} - \frac{1}{\gamma\alpha}$$

$$: \left(\alpha-\frac{1}{\alpha}\right)\alpha\beta\gamma + \left(\gamma-\frac{1}{\gamma}\right)\frac{1}{\beta} \quad : -\alpha + \frac{1}{\alpha} - \alpha\gamma^2 + \alpha$$

$$: \left(\alpha-\frac{1}{\alpha}\right)\alpha\gamma^2 + \left(\gamma-\frac{1}{\gamma}\right)\gamma \quad : \alpha^2\gamma^2 - \gamma^2 + \gamma^2 - 1,$$

the last set of values being obtained by aid of the relation $\alpha\beta\gamma=-1$; viz. we thus have

$$x : y : z = \frac{1}{\alpha\gamma}(-1+\alpha^2\gamma^2) : -\frac{1}{\alpha}(-1+\alpha^2\gamma^2) : (-1+\alpha^2\gamma^2),$$

that is,

$$x : y : z = \frac{1}{\gamma} : -1 : \alpha,$$

which are, in fact, the values belonging to one of the circular points at infinity. For the consecutive point on the other branch we should obtain in like manner $x : y : z = \gamma : -1 : \frac{1}{\alpha}$, which are the values belonging to the other circular point at infinity; viz. the node $(1, 0, 0, 0)$ of the skew sextic corresponds to the circular points at infinity. But, in like manner, the other two nodes $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$ each correspond to the circular points at infinity, or say we have in the skew sextic the three nodes each corresponding to one circular point at infinity, and the same three nodes each corresponding to the other circular point at infinity; viz. we thus prove that each of the circular points at infinity is a triple point on the locus of O.

In order not to interrupt the demonstration, I have assumed the formulæ which, in the system of coordinates defined by taking x, y, z proportional to the perpendiculars on the sides of a triangle ABC, or where the equation of the line infinity is

$$x\sin A + y\sin B + z\sin C = 0,$$

give the circular points at infinity; viz. writing

$$\cos A + i\sin A,\ \cos B + i\sin B,\ \cos C + i\sin C = \alpha,\ \beta,\ \gamma,$$

the coordinates for the two points respectively are

$$\begin{aligned} x : y : z &= -1 : \gamma : \frac{1}{\beta} \quad &\text{and} \quad x : y : z &= -1 : \frac{1}{\gamma} : \beta \\ &= \frac{1}{\gamma} : -1 : \alpha & &= \gamma : -1 : \frac{1}{\alpha} \\ &= \beta : \frac{1}{\alpha} : -1, & &= \frac{1}{\beta} : \alpha : -1, \end{aligned}$$

the three values for each point being equivalent in virtue of the relation $\alpha\beta\gamma = -1$. This is, in fact, under a different form, the theorem given in my Smith's Prize paper for 1875; viz. the theorem was: If λ, μ, ν are the inclinations to a fixed line of the perpendiculars let fall from an interior point on the sides of the fundamental triangle ABC, then, in the system of trilinear coordinates in which the coordinates of a point P are proportional to the triangles PBC, PCA, PAB (or where the equation of the line infinity is $x+y+z=0$), the coordinates of the circular points at infinity are proportional, those of the one point to $e^{i\lambda}\sin(\mu-\nu)$, $e^{i\mu}\sin(\nu-\lambda)$, $e^{i\nu}\sin(\lambda-\mu)$, and those of the other point to $e^{-i\lambda}\sin(\mu-\nu)$, $e^{-i\mu}\sin(\nu-\lambda)$, $e^{-i\nu}\sin(\lambda-\mu)$.

In the plane curve, the lines drawn from A, B, C to the circular points at infinity are:

	To the one point.	To the other point.
From A,	$\alpha y + z = 0$,	$y + \alpha z = 0$;
„ B,	$\beta z + x = 0$,	$z + \beta x = 0$;
„ C,	$\gamma x + y = 0$,	$x + \gamma y = 0$.

Each of these lines, *quà* tangent at a triple point, meets the curve in the circular point at infinity counted four times, and in two other points. The corresponding points on the skew sextic should be a node counted twice, the two other nodes counted each once, and two other points. The proof that this is so would show that the points A, B, C are a triad of foci. There is also the question of the determination of the values of $(\xi,\ \eta,\ \zeta,\ \omega)$ which correspond to the nodes of the plane curve. But I have not further pursued the theory.

ADDITION.—Since writing the foregoing paper, I have found that the relation between the nodes and foci (sum of angular distances of the foci = sum of angular distances of the nodes) may be expressed in a different form; viz. the triangle of the foci and the triangle of the nodes are circumscribed to a parabola (having its focus on the circle); and I have made in relation to the question the following further investigations:—

Considering a circle: and a parabola having its focus at K, a point of the circle; then if, as usual, I, J are the circular points at infinity, we have IJK a triangle inscribed in the circle and circumscribed to the parabola; hence there exists a singly-infinite series of in- and circumscribed triangles, so that, drawing from a point A of the circle tangents to the parabola again meeting the circle in the points B and C respectively, BC will be a tangent to the parabola; or, what is the same thing, starting with the triangle ABC inscribed in the circle, we can, with the arbitrary point K on the circle as focus, describe a parabola touching the three sides of the triangle ABC; viz. the parabola described to touch two of the sides of the triangle will touch the third side.

Taking, then, a circle radius $\tfrac{1}{2}k$, and upon it the three points A, B, C determined by the angles 2α, 2β, 2γ respectively (viz. the coordinates of A are $x, y = \tfrac{1}{2}k\cos 2\alpha$, $\tfrac{1}{2}k\sin 2\alpha$, &c.), and a point K determined by the angle 2κ (suppose for a moment the origin is at K), the equation of a parabola having K for its focus will be

$$x^2 + y^2 = (x\cos 2\theta + y\sin 2\theta - p)^2,$$

or, what is the same thing,

$$(x\sin 2\theta - y\cos 2\theta)^2 + 2p\,(x\cos 2\theta + y\sin 2\theta) - p^2 = 0,$$

where θ, p are in the first instance arbitrary; and the condition in order that $\xi x + \eta y + \zeta = 0$ may be a tangent is easily found to be

$$p\,(\xi^2 + \eta^2) + 2\xi\cos 2\theta + 2\eta\sin 2\theta = 0.$$

It is to be shown that p, θ can be determined so that the parabola shall touch each of the lines BC, CA, AB.

Taking the origin at the centre, the equation of BC is

$$x\cos(\beta+\gamma) + y\sin(\beta+\gamma) - \tfrac{1}{2}k\cos(\beta-\gamma) = 0,$$

as is at once verified by showing that this equation is satisfied by the values

$$x,\ y = \tfrac{1}{2}k\cos 2\beta,\ \tfrac{1}{2}k\sin 2\beta,\ \text{and } = \tfrac{1}{2}k\cos 2\gamma,\ \tfrac{1}{2}k\sin 2\gamma.$$

Hence, transforming to the point K as origin, the equation is

$$[x + \tfrac{1}{2}k\cos 2\kappa]\cos(\beta+\gamma) + [y + \tfrac{1}{2}k\sin 2\kappa]\sin(\beta+\gamma) - \tfrac{1}{2}k\cos(\beta-\gamma) = 0;$$

viz. this is

$$x\cos(\beta+\gamma) + y\sin(\beta+\gamma) - \tfrac{1}{2}k[\cos(\beta-\gamma) - \cos(\beta+\gamma-2\kappa)] = 0;$$

or, finally, it is

$$x\cos(\beta+\gamma) + y\sin(\beta+\gamma) - k\sin(\kappa-\beta)\sin(\kappa-\gamma) = 0.$$

Hence the condition of contact with the line BC is

$$p = 2k\sin(\kappa-\beta)\sin(\kappa-\gamma)\cos(2\theta-\beta-\gamma);$$

and, similarly, the condition of contact with the line CA is

$$p = 2k\sin(\kappa-\gamma)\sin(\kappa-\alpha)\cos(2\theta-\gamma-\alpha);$$

viz. these conditions determine the unknown quantities p, θ. It is at once seen that we have

$$2\theta-\beta-\gamma=\tfrac{1}{2}\pi-(\kappa-\alpha),\ \text{ that is, }\ 2\theta=\tfrac{1}{2}\pi-\kappa+\alpha+\beta+\gamma;$$

and then

$$p=2k\sin(\kappa-\alpha)\sin(\kappa-\beta)\sin(\kappa-\gamma);$$

from symmetry, we see that the parabola touches also the side AB.

Suppose, next, F, G are points on the circle determined by the angles $2f$, $2g$; retaining p and θ to denote their values,

$$p=2k\sin(\kappa-\alpha)\sin(\kappa-\beta)\sin(\kappa-\gamma),\ \text{ and }\ 2\theta=\tfrac{1}{2}\pi-\kappa+\alpha+\beta+\gamma,$$

the condition, in order that FG may be a tangent, is

$$p=2k\sin(\kappa-f)\sin(\kappa-g)\cos(2\theta-f-g);$$

viz. determining h by the equation

$$\alpha+\beta+\gamma=f+g+h,$$

this is

$$p=2k\sin(\kappa-f)\sin(\kappa-g)\sin(\kappa-h),$$

or, what is the same thing,

$$\sin(\kappa-\alpha)\sin(\kappa-\beta)\sin(\kappa-\gamma)=\sin(\kappa-f)\sin(\kappa-g)\sin(\kappa-h);$$

viz. this equation, considering therein h as standing for $\alpha+\beta+\gamma-f-g$, is the relation which must subsist between f and g, in order that the line FG may be a tangent to the parabola. And then, h being determined as above, and the point H on the circle being determined by the angle $2h$, it is clear that the lines GH, HF will also be tangents to the parabola; viz. FGH will be an in- and circumscribed triangle, provided only f, g, h satisfy the above-mentioned two equations. The latter of these, if f, g, h satisfy only the relation $\alpha+\beta+\gamma=f+g+h$, serves to determine κ; and then, θ and κ denoting as above, the equation of the parabola is

$$x^2+y^2=(x\cos 2\theta+y\sin 2\theta-p)^2;$$

and it thus appears that the condition in question, $\alpha+\beta+\gamma=f+g+h$, is equivalent to the condition that the triangles ABC, FGH shall be circumscribed to the same parabola.

It is to be remarked that the distances KA, KB, &c. are equal to $k\sin(\kappa-\alpha)$, $k\sin(\kappa-\beta)$, &c.; hence the condition

$$\sin(\kappa-\alpha)\sin(\kappa-\beta)\sin(\kappa-\gamma)=\sin(\kappa-f)\sin(\kappa-g)\sin(\kappa-h)$$

becomes

$$KA\,.\,KB\,.\,KC=KF\,.\,KG\,.\,KH;$$

viz. the focus K is a point on the circle such that the product of its (linear) distances from the foci A, B, C is equal to the product of its (linear) distances from the nodes F, G, H.

It is to be remarked that the foregoing equation in κ determines a single position of the point K; viz. it determines $\tan\kappa$, and therefore $\sin 2\kappa$ and $\cos 2\kappa$, linearly. The equation is, in fact, a cubic equation in $\tan\kappa$, satisfied identically by $\tan\kappa = i$ and $\tan\kappa = -i$, and therefore reducible to a linear equation.

Write for a moment $\tan\kappa = \omega$, and

$$(\tan\kappa - \tan\alpha)(\tan\kappa - \tan\beta)(\tan\kappa - \tan\gamma) = \omega^3 - p\omega^2 + q\omega - r,$$

$$(\tan\kappa - \tan f)(\tan\kappa - \tan g)(\tan\kappa - \tan h) = \omega^3 - p'\omega^2 + q'\omega - r';$$

also

$$M = \cos f \cos g \cos h \div \cos\alpha \cos\beta \cos\gamma.$$

Then we have

$$\omega^3 - p\omega^2 + q\omega - r = M(\omega^3 - p'\omega^2 + q'\omega - r'),$$

where

$$p = r + M(r' - p),\quad q = 1 + M(q' - 1).$$

Substituting these values, the equation becomes

$$\omega^3 - r\omega^2 + \omega - r = M(\omega^3 - r'\omega^2 + \omega - r'),$$

viz. dividing by $\omega^2 + 1$, this is $\omega - r = M(\omega - r')$; or substituting for r, r', M their values,

$$(\cos\alpha\cos\beta\cos\gamma - \cos f\cos g\cos h)\tan\kappa = (\sin\alpha\sin\beta\sin\gamma - \sin f\sin g\sin h),$$

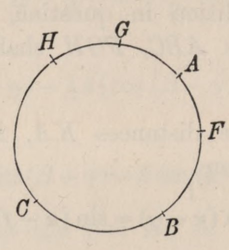
which is the value of $\tan\kappa$, and then

$$\sin 2\kappa = \frac{2\tan\kappa}{1+\tan^2\kappa},\quad \cos 2\kappa = \frac{1-\tan^2\kappa}{1+\tan^2\kappa}.$$

It may be further noticed that, if the parabola intersect the circle in a point L, and the tangent at L to the parabola again meet the circle in M, then, if $2l$, $2m$ are the angles for the points L, M, we have l, m, m for values of f, g, h, whence l, m are determined by the equations

$$l + 2m = \alpha + \beta + \gamma,\ \sin(\kappa - l)\sin^2(\kappa - m) = \sin(\kappa - \alpha)\sin(\kappa - \beta)\sin(\kappa - \gamma);$$

but as the circle intersects the parabola not only in two real points, but in two other imaginary points, there is no simple formula for the determination of l and m.



To determine the linkage when the nodes are given, suppose that, in the generation by O, the vertex of the triangle OB_1C_1, we have O at the node F: then, if τ, σ are the distances of C, B from the node in question, we have, as in the memoir,

$$(b_1^2 + \tau^2 - a_2^2)\,c_1\sigma = (c_1^2 + \sigma^2 - a_3^2)\,b_1\tau,$$

that is,

$$(b_1^2 - a_2^2)\, c_1\sigma - (c_1^2 - a_3^2)\, b_1\tau + \sigma\tau\,(c_1\tau - b_1\sigma) = 0,$$

or, what is the same thing,

$$(b_1^2 - a_2^2)\, c\sigma - (c_1^2 - a_3^2)\, b\tau + \sigma\tau\,(c\tau - b\sigma) = 0.$$

Suppose, as in the figure, that F is between B and A; then, if $AF = \rho$, we have $c\tau = b\sigma + a\rho$, and the equation becomes

$$(b_1^2 - a_2^2)\, c\sigma - (c_1^2 - a_3^2)\, b\tau + a\rho\sigma\tau = 0.$$

Similarly, if, as in the figure, G is on the other side of A, that is, between A and C, and if ρ', σ', τ' be the distances AG, BG, CG, then $b\sigma' = c\tau' + a\rho'$, that is, $c\tau' - b\sigma' = -a\rho'$, and the corresponding equation is

$$(b_1^2 - a_2^2)\, c\sigma' - (c_1^2 - a_3^2)\, b\tau' - a\rho'\sigma'\tau' = 0.$$

We hence find

$$(b_1^2 - a_2^2)\, c\,(\sigma\tau' - \sigma'\tau) + a\tau\tau'\,(\rho\sigma + \rho'\sigma') = 0,$$

$$(c_1^2 - a_3^2)\, b\,(\sigma\tau' - \sigma'\tau) + a\sigma\sigma'\,(\rho\tau + \rho'\tau') = 0.$$

But we have

$$BG\,.\,CF = BF\,.\,CG + BC\,.\,FG,$$

that is,

$$\sigma'\tau = \sigma\tau' + a\,.\,FG, \text{ or } \sigma\tau' - \sigma'\tau = -a\,.\,FG;$$

also,

$$\rho\sigma + \rho'\sigma' = AF\,.\,BF + AG\,.\,BG,\ = FG\,.\,CH,\ = FG\,.\,\tau'',$$

and

$$\rho\tau + \rho'\tau' = AF\,.\,CF + AG\,.\,CG,\ = FG\,.\,BH,\ = FG\,.\,\sigma'',$$

as may be shown without difficulty, ρ'', σ'', τ'' being the distances AH, BH, CH. Hence the equations become

$$c\,(b_1^2 - a_2^2) - \tau\tau'\tau'' = 0,$$

$$b\,(c_1^2 - a_3^2) - \sigma\sigma'\sigma'' = 0,$$

showing that, the foci being as in the figure, $b_1^2 - a_2^2$ and $c_1^2 - a_3^2$ are each of them positive; viz. that, in the generation by the triangle OC_1B_1, the radial bars a_2, a_3 are shorter than the sides b_1, c_1 respectively. Substituting for b_1, &c. the values $k_1 \sin B$, &c.; also, instead of k_1, k_2, k_3, introducing the quantities λ_1, λ_2, λ_3, where

$$k_1,\ k_2,\ k_3 = \lambda_1 \sin A,\ \lambda_2 \sin B,\ \lambda_3 \sin C,$$

these equations become

$$c\,(\lambda_1^2 - \lambda_2^2) \sin^2 A \sin^2 B = \tau\tau'\tau'',$$

$$b\,(\lambda_1^2 - \lambda_3^2) \sin^2 A \sin^2 C = \sigma\sigma'\sigma'';$$

or, as these may be written, putting for shortness $M = \sin A \sin B \sin C$,

$$M^2k^2\,(\lambda_1^2 - \lambda_2^2) = c\,\tau\tau'\tau'',$$

$$M^2k^2\,(\lambda_1^2 - \lambda_3^2) = b\,\sigma\sigma'\sigma''.$$

All the quantities have so far been regarded as positive, and the formulæ are applicable to the particular figure; but, to present them in a form applicable to any order of the nodes and foci, we have only to write the equations in the forms

$$M^2(\lambda_1^2-\lambda_2^2)=k^2\sin(\alpha-\beta)\sin(f-\gamma)\sin(g-\gamma)\sin(h-\gamma),$$

$$M^2(\lambda_1^2-\lambda_3^2)=k^2\sin(\alpha-\gamma)\sin(f-\beta)\sin(g-\beta)\sin(h-\beta);$$

and these may be replaced by the system

$$M^2(\lambda_2^2-\lambda_3^2)=k^2\sin(\beta-\gamma)\sin(f-\alpha)\sin(g-\alpha)\sin(h-\alpha),$$

$$M^2(\lambda_3^2-\lambda_1^2)=k^2\sin(\gamma-\alpha)\sin(f-\beta)\sin(g-\beta)\sin(h-\beta),$$

$$M^2(\lambda_1^2-\lambda_2^2)=k^2\sin(\alpha-\beta)\sin(f-\gamma)\sin(g-\gamma)\sin(h-\gamma),$$

since the first of these equations is implied in the other two; and then, reverting to the original form, we may write

$$M^2k^2(\lambda_2^2-\lambda_3^2)=BC\,.\,FA\,.\,GA\,.\,HA,$$

$$M^2k^2(\lambda_3^2-\lambda_1^2)=CA\,.\,FB\,.\,GB\,.\,HB,$$

$$M^2k^2(\lambda_1^2-\lambda_2^2)=AB\,.\,FC\,.\,GC\,.\,HC,$$

it being understood that the distances BC, FA, &c., which enter into these equations, are not all positive, but that they stand for $k\sin(\beta-\gamma)$, $k\sin(f-\alpha)$, &c., and that their signs are to be taken accordingly. Or, again, these may be written

$$BC\,(c_2^2-b_3^2)=FA\,.\,GA\,.\,HA,$$

$$CA\,(a_3^2-c_1^2)=FB\,.\,GB\,.\,HB,$$

$$AB\,(b_1^2-a_2^2)=FC\,.\,GC\,.\,HC,$$

where the signs are as just mentioned. We may say that $\pm(c_2^2-b_3^2)$ is the modulus for the focus A; and the formula then shows that this modulus, taken positively, is equal to the product of the distances FA, GA, HA of A from the three nodes respectively, divided by BC, the distance of the other two foci from each other.