619.

ON AN ALGEBRAICAL OPERATION.

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I CONSIDER

$$\Omega F(a, x),$$

an operation Ω performed upon $F(a, x)$ a rational function of (a, x); viz. F being first expanded or regarded as expanded in ascending powers of a, the coefficients of the several powers are then to be expanded or regarded as expanded in ascending powers of x, and the operation consists in the rejection of all negative powers of x.

In the cases intended to be considered, F contains only positive powers of a: but this restriction is not necessary to the theory.

The investigation has reference to the functions $A(x)$ of my "Ninth Memoir on Quantics," *Phil. Trans.*, t. CLXI. (1871), pp. 17—50, [462]; for instance, we there have as regards the covariants of a quadric

$$A(x) - \frac{1}{x^2} A\left(\frac{1}{x}\right) = \frac{1 - x^{-2}}{1 - ax^2 \,.\, 1 - a \,.\, 1 - ax^{-2}},$$

and consequently, in the present notation,

$$A(x) \qquad = \Omega \frac{1 - x^{-2}}{1 - ax^2 \,.\, 1 - a \,.\, 1 - ax^{-2}};$$

by a process of development and summation, the value of this expression was found to be

$$= \frac{1}{1 - ax^2 \,.\, 1 - a^2},$$

and in the other more complicated cases the value of $A(x)$ was found only by trial and verification. What I purpose now to show is that the operation Ω can be performed without any development in an infinite series; or say that it depends on finite algebraical operations only.

It is clear that if $F(a, x)$, considered to be developed as above contains only positive powers of x, then

$$\Omega F(a, x) = F(a, x);$$

and if it contains only negative powers of x, then $\Omega F(a, x) = 0$.

Consider now $\Omega \dfrac{\phi(x)}{x-a}$, where $\phi(x)$ is a function containing only positive powers of x; we have

$$\frac{\phi(x)}{x-a} = \frac{\phi(x)-\phi(a)}{x-a} + \frac{\phi(a)}{x-a},$$

and thence

$$\Omega \frac{\phi(x)}{x-a} = \Omega \frac{\phi(x)-\phi(a)}{x-a} + \Omega \frac{\phi(a)}{x-a}$$

$$= \frac{\phi(x)-\phi(a)}{x-a},$$

since $\dfrac{\phi(x)-\phi(a)}{x-a}$ is a rational and integral function of a, which when developed contains only positive powers of x, and $\dfrac{\phi(a)}{x-a}$ when developed contains only negative powers of x.

Consider next $\Omega \dfrac{\phi(x)}{x^2-a}$, where $\phi(x)$ is a rational and integral function of x; writing this $= f(x^2) + xg(x^2)$, we have

$$\Omega \frac{\phi(x)}{x^2-a} = \Omega \frac{f(x^2)}{x^2-a} + \Omega \frac{xg(x^2)}{x^2-a}$$

$$= \frac{f(x^2)-f(a)}{x^2-a} + \frac{x\{g(x^2)-g(a)\}}{x^2-a}.$$

As regards the last term, notice that

$$\Omega \frac{xg(x^2)}{x^2-a} = \Omega \frac{x\{g(x^2)-g(a)\}}{x^2-a} + \Omega \frac{xg(a)}{x^2-a},$$

in which $\dfrac{x\{g(x^2)-g(a)\}}{x^2-a}$ is a rational and integral function of (a, x), and therefore when developed contains only positive powers of x, while $\dfrac{xg(a)}{x^2-a}$ when developed contains only negative powers of x.

We thus have

$$\Omega \frac{\phi(x)}{x^2-a} = \frac{f(x^2)+xg(x^2)-f(a)-xg(a)}{x^2-a}$$

$$= \frac{\phi(x)-f(a)-xg(a)}{x^2-a}.$$

Similarly, if $\phi(x) = f(x^3) + xg(x^3) + x^2h(x^3)$, then

$$\Omega\,\frac{\phi(x)}{x^3-a} = \frac{\phi(x) - f(a) - xg(a) - x^2h(a)}{x^3-a};$$

and so on.

Consider now the above-mentioned function

$$A(x), = \Omega\,\frac{1-x^{-2}}{1-ax^2 \,.\, 1-a \,.\, 1-ax^{-2}}.$$

Writing

$$\frac{1-x^{-2}}{1-ax^2 \,.\, 1-a \,.\, 1-ax^{-2}} = \frac{P}{1-ax^2} + \frac{Q}{1-a} + \frac{R}{1-ax^{-2}},$$

we have

$$P = \left(\frac{1-x^{-2}}{1-a \,.\, 1-ax^{-2}}\right)_{a=x^{-2}}, \ = \frac{1}{1-x^{-4}}, \qquad = \frac{-x^4}{1-x^4},$$

$$Q = \left(\frac{1-x^{-2}}{1-ax^2 \,.\, 1-ax^{-2}}\right)_{a=1}, \ = \frac{1}{1-x^2},$$

$$R = \left(\frac{1-x^{-2}}{1-ax^2 \,.\, 1-a}\right)_{a=x^2}, \ = \frac{1-x^{-2}}{1-x^4 \,.\, 1-x^2}, \ = \frac{-1}{x^2(1-x^4)};$$

that is,

$$\frac{1-x^{-2}}{1-ax^2 \,.\, 1-a \,.\, 1-ax^{-2}} = \frac{-x^4}{1-x^4}\,\frac{1}{1-ax^2} + \frac{1}{1-x^2}\,\frac{1}{1-a} - \frac{1}{1-x^4}\,\frac{1}{x^2-a},$$

and thence

$$\Omega\,\frac{1-x^{-2}}{1-ax^2 \,.\, 1-a \,.\, 1-ax^{-2}} = \frac{-x^4}{1-x^4}\,\frac{1}{1-ax^2} + \frac{1}{1-x^2 \,.\, 1-a} + \Omega\,\frac{-1}{1-x^4}\,\frac{1}{x^2-a}.$$

Here, as regards the last term,

$$\phi(x^2) = \frac{-1}{1-x^4}, = f(x^2),\ g(x^2) = 0,$$

$$f(a) = -\frac{1}{1-a^2},$$

$$\frac{\phi(x^2) - f(a)}{x^2-a} = \frac{1}{x^2-a}\left(\frac{-1}{1-x^4} + \frac{1}{1-a^2}\right) = -\frac{x^4-a^2}{x^2-a \,.\, 1-x^4 \,.\, 1-a^2} = -\frac{x^2+a}{1-x^4 \,.\, 1-a^2},$$

and we have

$$\Omega\,\frac{1-x^{-2}}{1-ax^2 \,.\, 1-a \,.\, 1-ax^{-2}} = \frac{-x^4}{1-x^4 \,.\, 1-ax^2} + \frac{1}{1-x^2 \,.\, 1-a} - \frac{x^2+a}{1-x^4 \,.\, 1-a^2}.$$

The second term is $= \dfrac{1+x^2 \,.\, 1+a}{1-x^4 \,.\, 1-a^2}$: combining this with the third term, the two together are $= \dfrac{1+ax^2}{1-x^4 \,.\, 1-a^2}$.

Hence the value is

$$=\frac{1}{1-x^4}\left(\frac{-x^4}{1-ax^2}+\frac{1+ax^2}{1-a^2}\right),$$

which is

$$=\frac{1}{1-ax^2\,.\,1-a^2},$$

being, in fact, the expression for this function when decomposed into partial fraction of the denominators $1-ax^2$ and $1-a^2$ respectively. Hence finally

$$A(x)=\Omega\,\frac{1-x^{-2}}{1-ax^2\,.\,1-a\,.\,1-ax^{-2}}=\frac{1}{1-ax^2\,.\,1-a^2},$$

as it should be.

For the cubic function, we have

$$A(x)=\Omega\,\frac{1-x^{-2}}{1-ax^3\,.\,1-ax\,.\,1-ax^{-1}\,.\,1-ax^{-3}};$$

the function operated upon, when decomposed into partial fractions, is

$$=\frac{x^{10}}{1-x^4\,.\,1-x^6}\,\frac{1}{1-ax^3}-\frac{x^4}{1-x^2\,.\,1-x^4}\,\frac{1}{1-ax}$$

$$+\frac{x}{1-x^2\,.\,1-x^4}\,\frac{1}{x-a}+\frac{-x}{1-x^4\,.\,1-x^6}\,\frac{1}{x^3-a}.$$

Hence we require

$$\Omega\,\frac{x}{1-x^2\,.\,1-x^4}\,\frac{1}{x-a}+\Omega\,\frac{-x}{1-x^4\,.\,1-x^6}\,\frac{1}{x^3-a}.$$

The first of these is

$$=\frac{1}{x-a}\left\{\frac{x}{1-x^2\,.\,1-x^4}-\frac{a}{1-a^2\,.\,1-a^4}\right\},$$

which is

$$=\frac{1}{1-x^2\,.\,1-x^4\,.\,1-a^2\,.\,1-a^4}\left\{\begin{array}{l}1\\+x\,(a+a^3-a^5)\\+x^2\,(a^2-a^4)\\+x^3\,(a-a^3)\\-x^4a^2\\-x^5a\end{array}\right\}.$$

As regards the second, the function operated on may be expressed in the form

$$\frac{-x^9-x-x^5}{1-x^{12}\,.\,1-x^6}\,\frac{1}{x^3-a},$$

whence $f(a)$, $g(a)$, $h(a)$, and therefore $f(a)+xg(a)+x^2h(a)$, respectively, are $=-a^2$, -1, $-a$, $-a^3-x-ax^2$, each divided by $1-a^2 . 1-a^4$; or the term is

$$=\frac{1}{x^3-a}\left\{\frac{-x}{1-x^4 . 1-x^6}+\frac{a^3+x+ax^2}{1-a^2 . 1-a^4}\right\},$$

which is

$$=\frac{1}{1-x^4 . 1-x^6 . 1-a^2 . 1-a^4}\left\{\begin{array}{l}-a^2\\ +x(-a-a^3+a^5)\\ +x^2 . -1\\ +x^3 . -a\\ +x^4 . (a^4-1)\\ +x^5 . 0\\ +x^6 . a^2\\ +x^7 . a^3\\ +x^8 . 1\\ +x^9 . a\end{array}\right\}.$$

To combine the two terms, we multiply the numerator and denominator of the first by $1+x^2+x^4$, thereby reducing its denominator to $1-x^4 . 1-x^6 . 1-a^2 . 1-a^4$, the denominator of the second term; then the sum of the numerators is found to be

$$\begin{array}{ll}=1-a^2 & \text{viz. this is } =(1-a^2)\left(\begin{array}{l}1\\ +a^2x^2\\ +(a+a^3)x^3\\ +(a+a^3)x^5\\ +a^2x^6\\ +x^8.\end{array}\right.\\ +x\ (a^2-a^4) & \\ +x^3(a\ -a^5) & \\ +x^5(a\ -a^5) & \\ +x^6(a^2-a^4) & \\ +x^8(1\ -a^2), & \end{array}$$

Hence we have

$$A(x)=\Omega\frac{1-x^{-2}}{1-ax^3 . 1-ax . 1-ax^{-1} . 1-ax^{-3}}$$

$$=\frac{x^{10}}{1-x^4 . 1-x^6}\frac{1}{1-ax^3}$$

$$+\frac{-x^4}{1-x^2 . 1-x^4}\frac{1}{1-ax}$$

$$+\frac{1+x^8+(x^3+x^5)a+(x^2+x^6)a^2+(x^3+x^5)a^3}{1-x^4 . 1-x^6}\frac{1}{1-a^4},$$

which is, in fact, the expression for $\dfrac{1-ax+a^2x^2}{1-ax^3 . 1-ax . 1-a^4}$ decomposed into partial fractions with the denominators $1-ax^3$, $1-ax$, $1-a^4$ respectively. This is most easily seen by completing the decomposition, viz. we have

$$4\{1+x^8+(x^3+x^5)a+(x^2+x^6)a^2+(x^3+x^5)a^3\}$$
$$=(1+x^3)^2(1+x^2)(1+a)(1+a^2)+(1-x^3)^2(1+x^2)(1-a)(1+a^2)+2(1-x^2)(1-x^6)(1-a^2),$$

and thence the expression is

$$= \frac{x^{10}}{1-x^4\,.\,1-x^6}\frac{1}{1-ax^3}$$
$$+\frac{-x^4}{1-x^2\,.\,1-x^4}\frac{1}{1-ax}$$
$$+\tfrac{1}{4}\frac{1+x^3}{1-x^2\,.\,1-x^3}\frac{1}{1-a}+\tfrac{1}{4}\frac{1-x^3}{1-x^2\,.\,1+x^3}\frac{1}{1+a}+\tfrac{1}{2}\frac{1}{1+x^2}\frac{1}{1+a^2}$$
$$= \frac{1-ax+a^2x^2}{1-ax^3\,.\,1-ax\,.\,1-a^4},$$

as above. Hence finally

$$A(x)=\Omega\,\frac{1-x^{-2}}{1-ax^3\,.\,1-ax\,.\,1-ax^{-1}\,.\,1-ax^{-3}}$$
$$=\frac{1-ax+a^2x^2}{1-ax^3\,.\,1-ax\,.\,1-a^4}$$
$$=\frac{1+a^3x^3}{1-ax^3\,.\,1-a^2x^2\,.\,1-a^4}$$
$$=\frac{1-a^6x^6}{1-ax^3\,.\,1-a^2x^2\,.\,1-a^3x^3\,.\,1-a^4}.$$