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ON A PROBLEM OF PROJECTION.

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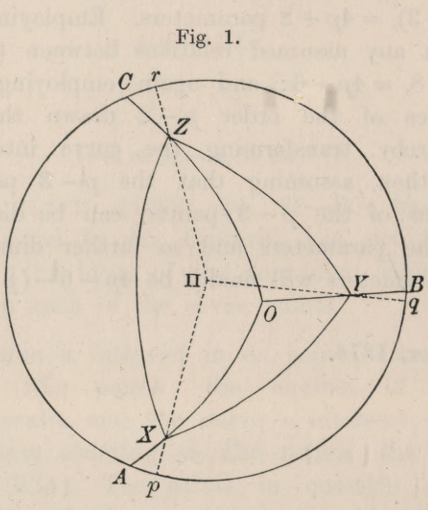
I MEASURE off on three rectangular axes the distances $\Omega X = \Omega Y = \Omega Z, = \theta$; and then, in a plane through Ω drawing in arbitrary directions the three lines ΩA, ΩB, ΩC, $= a, b, c$ respectively, I assume that A, B, C (fig. 1) are the parallel projections of X, Y, Z respectively; viz. taking ΩO as the direction of the projecting lines, then ΩA, ΩB, ΩC being given in position and magnitude, we have to find θ, and the position of the line ΩO.

Fig. 1.

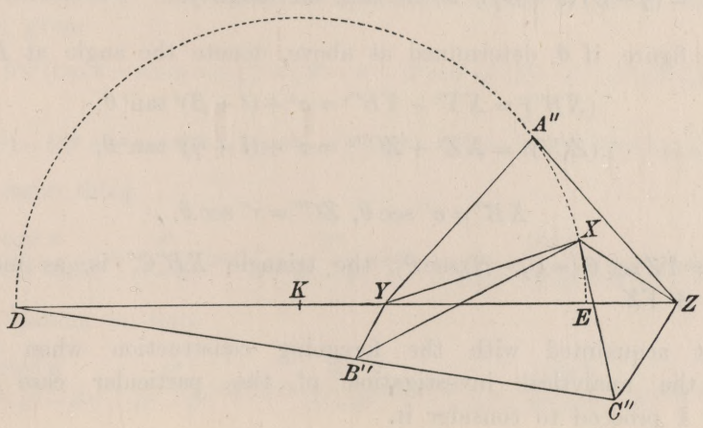


This is in fact a case of a more general problem solved by Prof. Pohlke in 1853, (see the paper by Schwarz, "Elementarer Beweis des Pohlke'schen Fundamentalsatzes der Axonometrie," *Crelle*, t. LXIII. (1864), pp. 309—314), viz. the three lines ΩX, ΩY, ΩZ may be any three axes given in magnitude and direction, and their parallel projection is to be similar to the three lines ΩA, ΩB, ΩC. Schwarz obtains a very elegant construction, which I will first reproduce. We may imagine through Ω a plane cutting at right angles the projecting lines, say in the points X', Y', Z'; we have then in plano a triad of lines $\Omega X'$, $\Omega Y'$, $\Omega Z'$ which are an orthogonal projection of ΩX, ΩY, ΩZ; and are also an orthogonal projection of a plane triad similar to ΩA, ΩB, ΩC; quà such last-mentioned projection, the triangles $\Omega Y'Z'$, $\Omega Z'X'$, $\Omega X'Y'$, must be proportional to the triangles ΩBC, ΩCA, ΩAB; that is, we have to find an orthogonal projection of ΩX, ΩY, ΩZ, such that the triangles $\Omega Y'Z'$, $\Omega Z'X'$, $\Omega X'Y'$, which are the projections of ΩYZ, ΩZX, ΩXY respectively, shall be in given ratios. There is no difficulty in the solution of this problem; referring everything to a sphere centre Ω, let the normals to the planes ΩYZ, ΩZX, ΩXY, meet the sphere in the points X'', Y'', Z'' respectively, and the projecting line through Ω meet the sphere in the point O, then the projection of ΩYZ is to ΩYZ as $\cos OX'' : 1$; and the like as to the projections of ΩZX and ΩXY; that is, in the given spherical triangle $X''Y''Z''$, we have to find a point O, such that the cosines of the distances OX'', OY'', OZ'' are in given ratios; we have at once, through X'', Y'', Z'' respectively, three arcs meeting in the required point O.

The projecting lines being thus obtained, say these are the three parallel lines X', Y', Z', we have next to draw through Ω a plane meeting these in the points A', B', C' such that the triangle $A'B'C'$ is similar to the given triangle ABC; for this being so, the triangles $\Omega B'C'$, $\Omega C'A'$, $\Omega A'B'$ being the projections of, and therefore proportional to $\Omega Y'Z'$, $\Omega Z'X'$, $\Omega X'Y'$, that is, proportional to ΩBC, ΩCA, ΩAB, will, it is clear, be similar to these triangles respectively; that is, we have the triad $\Omega A'$, $\Omega B'$, $\Omega C'$, a projection of ΩX, ΩY, ΩZ, and similar to the triad ΩA, ΩB, ΩC, which is what was required.

It remains only to show how the given three parallel lines X', Y', Z', not in the same plane, can be cut by a plane in a triangle similar to a given triangle ABC.

Fig. 2.



Imagine the three lines at right angles to the plane of the paper, meeting the plane of the paper in the given points X, Y, Z (fig. 2) respectively. On the base

YZ describe a triangle $A''YZ$ similar to the given triangle ABC; and through A'', X with centre on the line YZ, describe a circle meeting this line in the points D and E. Then in the plane, through YZ at right angles to the plane of the paper, we may draw a line meeting the lines Y, Z in the points B'', C'' respectively, such that joining XB'', XC'' we obtain a triangle $XB''C''$ similar to $A''YZ$, that is, to the given triangle ABC.

Taking K the centre of the circle, suppose that its radius is $=l$, and that we have $KY=\beta$, $KZ=\gamma$; also $YX=\sigma$, $ZX=\tau$; $YA''=\sigma''$, $ZA''=\tau''$. If for a moment x, y denote the coordinates of X, then

$$\sigma^2=(x-\beta)^2+y^2,\ =l^2+\beta^2-2\beta x,$$

$$\tau^2=(x-\gamma)^2+y^2,\ =l^2+\gamma^2-2\gamma x,$$

and thence

$$\gamma\sigma^2-\beta\tau^2=\gamma(l^2+\beta^2)-\beta(l^2+\gamma^2),$$

that is,

$$\gamma\sigma^2-\beta\tau^2=(\gamma-\beta)(l^2-\beta\gamma);$$

viz. this is the equation of the circle in terms of the vectors σ, τ; we have therefore in like manner

$$\gamma\sigma''^2-\beta\tau''^2=(\gamma-\beta)(l^2-\beta\gamma).$$

We may determine θ so as to satisfy the two equations

$$\sigma''^2=\sigma^2\cos^2\theta+(l+\beta)^2\sin^2\theta,$$

$$\tau''^2=\tau^2\cos^2\theta+(l+\gamma)^2\sin^2\theta;$$

in fact, these equations give

$$\gamma\sigma''^2-\beta\tau''^2=(\gamma\sigma^2-\beta\tau^2)\cos^2\theta+\{\gamma(l^2+\beta^2)-\beta(l^2+\gamma^2)\}\sin^2\theta,$$

which, the left-hand side and the coefficients of $\cos^2\theta$, and $\sin^2\theta$ on the right-hand side being each $=(\gamma-\beta)(l^2-\beta\gamma)$, is, in fact, an identity.

But in the figure, if θ, determined as above, denote the angle at D, then

$$(XB'')^2=XY^2+YB''^2=\sigma^2+(l+\beta)^2\tan^2\theta,$$

$$(ZC'')^2=XZ^2+ZC''^2=\tau^2+(l+\gamma)^2\tan^2\theta,$$

that is,

$$XB''=\sigma''\sec\theta,\ ZC''=\tau''\sec\theta,$$

or, since $B''C''=YZ\sec\theta\ \{=(\gamma-\beta)\sec\theta\}$, the triangle $XB''C''$ is, as mentioned, similar to the triangle $A''YZ$.

I was not acquainted with the foregoing construction when my paper was written; but the analytical investigation of the particular case is nevertheless interesting, and I proceed to consider it.

Taking (fig. 1) Ω as the centre of a sphere and projecting on this sphere, we have A, B, C given points on a great circle; and we have to find the point O, such that there may be a trirectangular triangle XYZ, the vertices of which lie in OA, OB, OC respectively, and for which

$$\frac{\sin OX}{\sin OA}=\frac{a}{\theta},\ \frac{\sin OY}{\sin OB}=\frac{b}{\theta},\ \frac{\sin OZ}{\sin OC}=\frac{c}{\theta}.$$

I take the arcs BC, CA, $AB=\alpha$, β, γ respectively, $\alpha+\beta+\gamma=2\pi$; and the required arcs OA, OB, OC are taken to be ξ, η, ζ respectively; these are connected by the relation

$$\sin\alpha\cos\xi+\sin\beta\cos\eta+\sin\gamma\cos\zeta=0,$$

to obtain which, observe that from the triangles OAB, OAC, we have

$$\cos A=\frac{\cos\eta-\cos\xi\cos\gamma}{\sin\xi\sin\gamma}=-\frac{\cos\zeta-\cos\xi\cos\beta}{\sin\xi\sin\beta},$$

that is,

$$\sin\beta(\cos\eta-\cos\xi\cos\gamma)+\sin\gamma(\cos\zeta-\cos\xi\cos\beta)=0,$$

which, with $\sin\alpha=-\sin(\beta+\gamma)$, gives the required relation. We have

$$\sin OX=\frac{a}{\theta}\sin\xi,\ \sin OY=\frac{b}{\theta}\sin\eta,\ \sin OZ=\frac{c}{\theta}\sin\zeta;$$

and then from the triangles OBC, OCA, OAB, and the quadrantal triangles OYZ, OZX, OXY, we have

$$\cos BOC=\frac{\cos\alpha-\cos\eta\cos\zeta}{\sin\eta\sin\zeta}=-\frac{\sqrt{\left(1-\frac{b^2}{\theta^2}\sin^2\eta\right)}\sqrt{\left(1-\frac{c^2}{\theta^2}\sin^2\zeta\right)}}{\frac{bc}{\theta^2}\sin\eta\sin\zeta},\text{ \&c.};$$

that is,

$$bc(\cos\alpha-\cos\eta\cos\zeta)=-\sqrt{(\theta^2-b^2\sin^2\eta)}\sqrt{(\theta^2-c^2\sin^2\zeta)},$$
$$ca(\cos\beta-\cos\zeta\cos\xi)=-\sqrt{(\theta^2-c^2\sin^2\zeta)}\sqrt{(\theta^2-a^2\sin^2\xi)},$$
$$ab(\cos\gamma-\cos\xi\cos\eta)=-\sqrt{(\theta^2-a^2\sin^2\xi)}\sqrt{(\theta^2-b^2\sin^2\eta)},$$

which, when rationalized, are quadric equations in $\cos\xi$, $\cos\eta$, $\cos\zeta$. The first equation, in fact, gives

$$b^2c^2(\cos\alpha-\cos\eta\cos\zeta)^2=(\theta^2-b^2+b^2\cos^2\eta)(\theta^2-c^2+c^2\cos^2\zeta),$$

that is,

$$(\theta^2-b^2)(\theta^2-c^2)-b^2c^2\cos^2\alpha+(\theta^2-b^2)c^2\cos^2\zeta+(\theta^2-c^2)b^2\cos^2\eta+2b^2c^2\cos\alpha\cos\eta\cos\zeta=0,$$

or, what is the same thing,

$$-\left(1-\frac{b^2c^2\cos^2\alpha}{b^2-\theta^2\,.\,c^2-\theta^2}\right)+\frac{c^2}{c^2-\theta^2}\cos^2\zeta+\frac{b^2}{b^2-\theta^2}\cos^2\eta-\frac{2b^2c^2}{(b^2-\theta^2)(c^2-\theta^2)}\cos\alpha\cos\eta\cos\zeta=0.$$

Completing the system, we have

$$-\left(1-\frac{c^2a^2\cos^2\beta}{c^2-\theta^2\,.\,a^2-\theta^2}\right)+\frac{a^2}{a^2-\theta^2}\cos^2\xi+\frac{c^2}{c^2-\theta^2}\cos^2\zeta-\frac{2c^2a^2}{(c^2-\theta^2)(a^2-\theta^2)}\cos\beta\cos\zeta\cos\xi=0,$$

$$-\left(1-\frac{a^2b^2\cos^2\gamma}{a^2-\theta^2\,.\,b^2-\theta^2}\right)+\frac{b^2}{b^2-\theta^2}\cos^2\eta+\frac{a^2}{a^2-\theta^2}\cos^2\xi-\frac{2a^2b^2}{(a^2-\theta^2)(b^2-\theta^2)}\cos\gamma\cos\xi\cos\eta=0,$$

and, as above,

$$\sin\alpha\cos\xi+\sin\beta\cos\eta+\sin\gamma\cos\zeta=0.$$

It seems difficult from these equations to eliminate ξ, η, ζ, so as to obtain an equation in θ; but I employ some geometrical considerations.

Taking Π as the pole of the circle ABC, and drawing ΠX, ΠY, ΠZ to meet the circle in p, q, r respectively, then, if α'', β'', γ'' are the cosine-inclinations of O to X, Y, Z respectively, we have

$$\sin Xp,\quad \sin Yq,\quad \sin Zr=\alpha'',\ \beta'',\ \gamma''.$$

From the right parallel triangles BYq and CZr, we have

$$\sin Yq=\sin BY\sin B,$$

$$\sin Zr=\sin CZ\ \sin C,$$

and, thence,

$$\frac{\sin Yq}{\sin Zr}=\frac{\sin BY}{\sin CZ}\cdot\frac{\sin OC}{\sin OB};$$

or, since

$$BY=OB-OY,\quad CZ=OC-OZ,$$

and thence

$$\sin BY=\frac{\sin\eta}{\theta}\{\sqrt{(\theta^2-b^2\sin^2\eta)}-b\cos\eta\},$$

$$\sin CZ=\frac{\sin\zeta}{\theta}\{\sqrt{(\theta^2-c^2\sin^2\zeta)}-c\cos\zeta\},$$

we obtain

$$\frac{\beta''}{\gamma''}=\frac{\sqrt{(\theta^2-b^2\sin^2\eta)}-b\cos\eta}{\sqrt{(\theta^2-c^2\sin^2\zeta)}-c\cos\zeta}.$$

We have thence

$$\beta''\sqrt{(\theta^2-c^2\sin^2\zeta)}-\gamma''\sqrt{(\theta^2-b^2\sin^2\eta)}=\beta''c\cos\zeta-\gamma''b\cos\eta,$$

or, squaring and reducing

$$\beta''^2(\theta^2-c^2)+\gamma''^2(\theta^2-b^2)+2\beta''\gamma''\{-\sqrt{(\theta^2-c^2\sin^2\zeta)}\sqrt{(\theta^2-b^2\sin^2\eta)}+bc\cos\eta\cos\zeta\}=0,$$

that is,

$$\beta''^2(\theta^2-c^2)+\gamma''^2\ (\theta^2-b^2)+2\beta''\gamma''\,.\,bc\cos\alpha\ =0\,;$$

and, similarly,

$$\gamma''^2(\theta^2-a^2)+\alpha''^2\ (\theta^2-c^2)+2\gamma''\alpha''\ .\,ca\cos\beta=0,$$

$$\alpha''^2(\theta^2-b^2)+\beta''^2(\theta^2-a^2)+2\alpha''\beta''\,.\,ab\cos\gamma=0,$$

or, what is the same thing,

$$\frac{\beta''^2}{b^2-\theta^2}+\frac{\gamma''^2}{c^2-\theta^2}-\frac{2bc\cos\alpha}{b^2-\theta^2\,.\,c^2-\theta^2}\beta''\gamma''=0,$$

$$\frac{\gamma''^2}{c^2-\theta^2}+\frac{\alpha''^2}{a^2-\theta^2}-\frac{2ca\cos\beta}{c^2-\theta^2\,.\,a^2-\theta^2}\gamma''\alpha''=0,$$

$$\frac{\alpha''^2}{a^2-\theta^2}+\frac{\beta''^2}{b^2-\theta^2}-\frac{2ab\cos\gamma}{a^2-\theta^2\,.\,b^2-\theta^2}\alpha''\beta''=0\,;$$

writing

$$\alpha'',\ \beta'',\ \gamma'' = X\sqrt{(a^2-\theta^2)},\quad Y\sqrt{(b^2-\theta^2)},\quad Z\sqrt{(c^2-\theta^2)},$$

and

$$\frac{bc\cos\alpha}{\sqrt{(b^2-\theta^2 \,.\, c^2-\theta^2)}},\ \frac{ca\cos\beta}{\sqrt{(c^2-\theta^2 \,.\, a^2-\theta^2)}},\ \frac{ab\cos\gamma}{\sqrt{(a^2-\theta^2 \,.\, b^2-\theta^2)}} = f,\ g,\ h,$$

the equations are

$$\begin{aligned} Y'^2 + Z'^2 - 2fY'Z' &= 0, \\ Z'^2 + X'^2 - 2gZ'X' &= 0, \\ X'^2 + Y'^2 - 2hX'Y' &= 0. \end{aligned}$$

Writing the last two under the form

$$\begin{aligned} X'^2 - 2gZ'X' + Z'^2 &= 0, \\ X'^2 - 2hY'X' + Y'^2 &= 0, \end{aligned}$$

and eliminating X', we have

$$-4(1-g^2)(1-h^2)\,Y'^2Z'^2 + (Y'^2 + Z'^2 - 2ghY'Z')^2 = 0,$$

which, in virtue of the first equation, is

$$-4(1-g^2)(1-h^2)\,Y'^2Z'^2 + 4(gh-f)^2\,Y'^2Z'^2 = 0,$$

that is,

$$(1-g^2)(1-h^2) - (gh-f)^2 = 0;$$

or, what is the same thing,

$$1 - f^2 - g^2 - h^2 + 2fgh = 0.$$

I remark that we may write

$$\begin{aligned} gh - f &= \sqrt{(1-g^2)}\sqrt{(1-h^2)}, \\ hf - g &= \sqrt{(1-h^2)}\sqrt{(1-f^2)}, \\ fg - h &= \sqrt{(1-f^2)}\sqrt{(1-g^2)}, \end{aligned}$$

the signs on the right-hand side being either all $+$, or else one $+$ and two $-$, so that the product is $+$. In fact, multiplying the assumed equations, we have

$$f^2g^2h^2 - fgh(f^2+g^2+h^2) + g^2h^2 + h^2f^2 + f^2g^2 - fgh = 1 - f^2 - g^2 - h^2 + g^2h^2 + h^2f^2 + f^2g^2 - f^2g^2h^2,$$

that is,

$$1 - f^2 - g^2 - h^2 + fgh(1 + f^2 + g^2 + h^2) - 2f^2g^2h^2 = 0,$$

or,

$$(1 - f^2 - g^2 - h^2 + 2fgh)(1 - fgh) = 0,$$

which is right; but with a different combination of signs the result would not have been obtained.

Substituting for f, g, h their values, we have

$$(a^2-\theta^2)(b^2-\theta^2)(c^2-\theta^2) - b^2c^2(a^2-\theta^2)\cos^2\alpha - c^2a^2(b^2-\theta^2)\cos^2\beta$$
$$- a^2b^2(c^2-\theta^2)\cos^2\gamma + 2a^2b^2c^2\cos\alpha\cos\beta\cos\gamma = 0,$$

where the term independent of θ is

$$a^2b^2c^2(1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma),$$

which is $=0$ in virtue of $\alpha+\beta+\gamma=2\pi$. We have, therefore, for θ^2 the quadric equation

$$b^2c^2\sin^2\alpha+c^2a^2\sin^2\beta+a^2b^2\sin^2\gamma-(a^2+b^2+c^2)\,\theta^2+\theta^4=0,$$

giving for θ^2 the two real positive values

$$\theta^2=\tfrac{1}{2}\{a^2+b^2+c^2\pm\sqrt{(\Omega)}\},$$

where

$$\begin{aligned}\Omega^2&=(a^2+b^2+c^2)^2-4(b^2c^2\sin^2\alpha+c^2a^2\sin^2\beta+a^2b^2\sin^2\gamma)\\&=a^4+b^4+c^4+2b^2c^2\cos 2\alpha+2c^2a^2\cos 2\beta+2a^2b^2\cos 2\gamma\\&=(a^2+b^2\cos 2\gamma+c^2\cos 2\beta)^2+(b^2\sin 2\gamma-c^2\sin 2\beta)^2.\end{aligned}$$

I write now

$$\frac{a\cos\xi}{\sqrt{(a^2-\theta^2)}},\ \frac{b\cos\eta}{\sqrt{(b^2-\theta^2)}},\ \frac{c\cos\zeta}{\sqrt{(c^2-\theta^2)}}=X,\ Y,\ Z,$$

and also

$$\frac{\sqrt{a^2-\theta^2}}{a}\sin\alpha,\ \frac{\sqrt{b^2-\theta^2}}{b}\sin\beta,\ \frac{\sqrt{c^2-\theta^2}}{c}\sin\gamma=A,\ B,\ C.$$

The equations for $\cos\xi$, $\cos\eta$, $\cos\zeta$ become

$$\begin{aligned}Y^2+Z^2-2fYZ-(1-f^2)&=0,\\Z^2+X^2-2gZX-(1-g^2)&=0,\\X^2+Y^2-2hXY-(1-h^2)&=0,\end{aligned}$$

and

$$AX+BY+CZ=0,$$

in virtue of the relation between f, g, h. The first three equations are satisfied by a two-fold relation between X, Y, Z; viz. treating these as coordinates, the equations represent three quadric cylinders having a common conic.

To prove this, I write

$$1-f^2,\ 1-g^2,\ 1-h^2,\ gh-f,\ hf-g,\ fg-h=\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{f},\ \mathrm{g},\ \mathrm{h}.$$

We have, as usual,

$$\mathrm{bc}-\mathrm{f}^2,\ \mathrm{ca}-\mathrm{g}^2,\ \mathrm{ab}-\mathrm{h}^2,\ \mathrm{gh}-\mathrm{af},\ \mathrm{hf}-\mathrm{bg},\ \mathrm{fg}-\mathrm{ch},\ \text{each }=0:$$

the equations

$$\mathrm{a}X+\mathrm{h}Y+\mathrm{g}Z=0,\quad \mathrm{h}X+\mathrm{b}Y+\mathrm{f}Z=0,\quad \mathrm{g}X+\mathrm{f}Y+\mathrm{c}Z=0,$$

represent each of them one and the same plane, which I say is that of the conic in question.

The three given equations are

$$Y^2 + Z^2 - 2fYZ \ \ - \mathrm{a} = 0,$$
$$Z^2 + X^2 - 2gZX \ \ - \mathrm{b} = 0,$$
$$X^2 + Y^2 - 2hXY - \mathrm{c} = 0,$$

say these are $U = 0$, $V = 0$, $W = 0$; it is to be shown that $\mathrm{c}V - \mathrm{b}W$, $\mathrm{a}W - \mathrm{c}U$, $\mathrm{b}U - \mathrm{a}V$, each contain the linear factor in question. We have

$$\mathrm{c}V - \mathrm{b}W = (\mathrm{c} - \mathrm{b})X^2 - \mathrm{b}Y^2 + \mathrm{c}Z^2 - 2\mathrm{c}gZX + 2\mathrm{b}hXY;$$

or, what is the same thing,

$$\mathrm{a}(\mathrm{c}V - \mathrm{b}W) = \mathrm{a}(\mathrm{c} - \mathrm{b})X^2 - \mathrm{h}^2Y^2 + \mathrm{g}^2Z^2 - 2g\mathrm{g}^2ZX + 2h\mathrm{h}^2XY.$$

Assuming this

$$= (\mathrm{a}X + \mathrm{h}Y + \mathrm{g}Z)(\lambda X - \mathrm{h}Y + \mathrm{g}Z),$$

we have

$$\mathrm{a}\lambda = \ \ \mathrm{a}(\mathrm{c} - \mathrm{b}),$$
$$\mathrm{g}(\ \ \mathrm{a} + \lambda) = -2g\mathrm{g}^2,$$
$$\mathrm{h}(-\mathrm{a} + \lambda) = \ \ 2h\mathrm{h}^2,$$

that is,

$$\lambda = \mathrm{c} - \mathrm{b}, \quad \mathrm{a} + \lambda = -2g\mathrm{g}, \quad -\mathrm{a} + \lambda = 2h\mathrm{h}:$$

but $\lambda = \mathrm{c} - \mathrm{b}, = -\mathrm{h}^2 + \mathrm{g}^2$, and the other two equations are $\mathrm{a} + \mathrm{c} - \mathrm{b} + 2g\mathrm{g} = 0$, $\mathrm{a} + \mathrm{b} - \mathrm{c} + 2h\mathrm{h} = 0$, which are identically true.

The values of X, Y, Z are thus determined as the coordinates of the intersection of the conic with the plane $AX + BY + CZ = 0$; or, what is the same thing, of the line

$$AX + BY + CZ = 0,$$
$$\mathrm{a}X + \mathrm{h}Y + \mathrm{g}Z = 0,$$

with any one of the three cylinders.

We may, however, complete the analytical solution in a different manner as follows:

Assuming as above $\sqrt{(\mathrm{bc})} = \mathrm{f}$, $\sqrt{(\mathrm{ca})} = \mathrm{g}$, $\sqrt{(\mathrm{ab})} = \mathrm{h}$, and thence $\mathrm{h}\sqrt{(\mathrm{c})} - \mathrm{g}\sqrt{(\mathrm{b})} = 0$, we obtain from the second and the third equations

$$Y = hX + \sqrt{(\mathrm{c})}\sqrt{(1 - X^2)}, \quad Z = gX - \sqrt{(\mathrm{b})}\sqrt{(1 - X^2)},$$

(the signs are one $+$ the other $-$, in order that this may consist with the equation $\mathrm{a}X + \mathrm{h}Y + \mathrm{g}Z = 0$). Substituting in $AX + BY + CZ = 0$, we have

$$(A + Bh + Cg)X + \{B\sqrt{(\mathrm{c})} - C\sqrt{(\mathrm{b})}\}\sqrt{(1 - X^2)} = 0,$$

that is,

$$(A + Bh + Cg)^2X^2 - (B^2\mathrm{c} + C^2\mathrm{b} - 2BC\mathrm{f})(1 - X^2) = 0,$$

or say

$$(A+Bh+Cg)^2 X^2+\{B^2(1-h^2)+C^2(1-g^2)-2BC(gh-f)\}(X^2-1)=0,$$

that is,

$$(A^2+B^2+C^2+2BCf+2CAg+2ABh)X^2=\{B^2+C^2+2BCf-(Bh+Cg)^2\},$$

or writing

$$A^2+B^2+C^2+2BCf+2CAg+2ABh=\Delta,$$

say we have

$$\Delta X^2=B^2+C^2+2BCf-(Bh+Cg)^2,$$

$$\Delta Y^2=C^2+A^2+2CAg-(Cf+Ah)^2,$$

$$\Delta Z^2=A^2+B^2+2ABh-(Ag+Bf)^2.$$

Now attending to the values of A, B, C, f, g, h, we have

$$BCf,\ CAg,\ ABh=\sin\beta\sin\gamma\cos\alpha,\quad \sin\gamma\sin\alpha\cos\beta,\quad \sin\alpha\sin\beta\cos\gamma,$$

and thence

$$\Delta=\sin^2\alpha\left(1-\frac{\theta^2}{a^2}\right)+\sin^2\beta\left(1-\frac{\theta^2}{b^2}\right)+\sin^2\gamma\left(1-\frac{\theta^2}{c^2}\right)$$
$$+2(\sin\beta\sin\gamma\cos\alpha+\sin\gamma\sin\alpha\cos\beta+\sin\alpha\sin\beta\cos\gamma);$$

in virtue of $\alpha+\beta+\gamma=2\pi$, the last term is

$$=2(\cos\alpha\cos\beta\cos\gamma-1),$$

whence

$$\Delta=-\theta^2\left(\frac{\sin^2\alpha}{a^2}+\frac{\sin^2\beta}{b^2}+\frac{\sin^2\gamma}{c^2}\right),\text{ say this is }=-\theta^2\Lambda.$$

Moreover $Bh+Cg=\dfrac{a\sin\alpha}{\sqrt{(a^2-\theta^2)}}$, whence the value of ΔX^2 is

$$=\sin^2\beta\left(1-\frac{\theta^2}{b^2}\right)+\sin^2\gamma\left(1-\frac{\theta^2}{c^2}\right)+2\sin\beta\sin\gamma\cos\alpha-\left(1-\frac{\theta^2}{a^2}\right)\sin^2\alpha.$$

Here the constant term is

$$=\sin^2\beta+\sin^2\gamma+2\sin\beta\sin\gamma\cos\alpha,$$

that is,

$$=1-(1-\sin^2\beta)(1-\sin^2\gamma)+\sin^2\beta\sin^2\gamma+2\sin\beta\sin\gamma\cos\alpha$$
$$=1-\cos^2\beta\cos^2\gamma-\cos^2\alpha+(\cos\alpha+\sin\beta\sin\gamma)^2$$
$$=1-\cos^2\alpha,\ =\sin^2\alpha,$$

or the whole is

$$\sin^2\alpha\left(1-\frac{a^2}{a^2-\theta^2}\right)-\theta^2\left(\frac{\sin^2\beta}{b^2}+\frac{\sin^2\gamma}{c^2}\right),$$

which is

$$= -\theta^2\left(\frac{\sin^2\alpha}{a^2-\theta^2}+\frac{\sin^2\beta}{b^2}+\frac{\sin^2\gamma}{c^2}\right),$$

so that we have

$$\Lambda X^2 = \left(\frac{\sin^2\alpha}{a^2-\theta^2}+\frac{\sin^2\beta}{b^2}+\frac{\sin^2\gamma}{c^2}\right).$$

Similarly,

$$\Lambda Y^2 = \left(\frac{\sin^2\alpha}{a^2}+\frac{\sin^2\beta}{b^2-\theta^2}+\frac{\sin^2\gamma}{c^2}\right),$$

$$\Lambda Z^2 = \left(\frac{\sin^2\alpha}{a^2}+\frac{\sin^2\beta}{b^2}+\frac{\sin^2\gamma}{c^2-\theta^2}\right);$$

and hence also

$$\Lambda(1-X^2) = \frac{-\theta^2\sin^2\alpha}{a^2(a^2-\theta^2)},\quad \Lambda(1-Y^2) = \frac{-\theta^2\sin^2\beta}{b^2(b^2-\theta^2)},\quad \Lambda(1-Z^2) = \frac{-\theta^2\sin^2\gamma}{c^2(c^2-\theta^2)},$$

where

$$\Lambda = \frac{\sin^2\alpha}{a^2}+\frac{\sin^2\beta}{b^2}+\frac{\sin^2\gamma}{c^2}.$$

The equation in X is

$$\Lambda\left(1-\frac{a^2\cos^2\xi}{a^2-\theta^2}\right) = \frac{-\theta^2\sin^2\alpha}{a^2(a^2-\theta^2)},$$

that is,

$$\Lambda(a^2\sin^2\xi-\theta^2) = -\theta^2\frac{\sin^2\alpha}{a^2},$$

or

$$a^2\sin^2\xi = \theta^2\left(1-\frac{\sin^2\alpha}{a^2\Lambda}\right),$$

and the like for η, ζ. Writing for greater convenience $\frac{\sin^2\alpha}{a^2}$, $\frac{\sin^2\beta}{b^2}$, $\frac{\sin^2\gamma}{c^2} = p, q, r$, then $\Lambda = p+q+r$, and we have

$$\sin^2\xi = \frac{\theta^2}{a^2}\,\frac{q+r}{p+q+r},\quad \sin^2\eta = \frac{\theta^2}{b^2}\,\frac{r+p}{p+q+r},\quad \sin^2\zeta = \frac{\theta^2}{c^2}\,\frac{p+q}{p+q+r},$$

(whence also $a^2\sin^2\xi + b^2\sin^2\eta + c^2\sin^2\zeta = 2\theta^2$: as a simple verification, observe that, if the projection is rectangular, the axes being all equally inclined to the plane of projection, then $\xi=\eta=\zeta=90°$, $a=b=c=\theta\sin s$, and the equation is $3\sin^2 s = 2$; s, s are here the sides of an isosceles quadrantal triangle, the included angle being 120°, that is, we have $\cos 120° \,(= -\tfrac{1}{2}) = -\cot^2 s$, that is, $\cot^2 s = \tfrac{1}{2}$, or $\sin^2 s = \tfrac{2}{3}$, which is right).

I remark, that a geometrical solution may be obtained upon very different principles. We have on a sphere the trirectangular triangle XYZ, which by parallel lines is projected into ABC. Every great circle of the sphere is projected into an ellipse having double contact at the extremities of a diameter with the ellipse which is the apparent contour of the sphere. Moreover, if the arc of great circle XY is a quadrant, then the radius through X and the tangent at Y are parallel to each other, whence, if Ω be the projection of the centre, and AB the projection of the arc XY, then in the projection the line ΩA and the tangent at B are parallel to each other. It is now easy to derive a construction: with centre Ω, and conjugate semi-axes $(\Omega B, \Omega C)$, $(\Omega C, \Omega A)$, $(\Omega A, \Omega B)$ respectively, describe three ellipses; and find a concentric ellipse having double contact with each of these (there are in fact two such ellipses, one touching the three ellipses internally, and giving an imaginary solution; the other touching them externally, which is the ellipse intended). Drawing then through the ellipse a right cylinder (there are two such cylinders, but only one of them is real), and inscribing in it a sphere, and projecting on to the surface of the sphere by lines parallel to the axis of the cylinder, the three ellipses are projected into three great circles cutting at right angles, or, say, the elliptic arcs BC, CA, AB are projected into the trirectangular triangle XYZ.