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A stability postulate for quasi-static processes of plastic deformation(*)

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A DEFINITION of stability of a quasi-static plastic deformation process is formulated. In the definition the persistent disturbances represented by time-dependent body forces and surface tractions are considered. The proposed energy measure for a disturbance constitutes an essential part of the stability postulate.

1. Introduction

VARIOUS theoretical approaches to instability phenomena in plastic solids have been used in the literature (see, for example, [1–4]). However, none of the used concepts of stability seems to be sufficiently universal. In the present paper a more general definition of stability of a plastic deformation *process* is formulated. The ideas of stability of motion for persistent disturbances (see [5]) and of stability of a process with respect to two metrics [6, 7] are employed. The proposed energy measure for a disturbance constitutes an essential part of the stability postulate.

2. Deformation process

We are concerned with a continuous plastic body subjected to an isothermal quasi-static process of deformation in some time interval $\mathscr{T} = [0, T]$. The properties of the material need not be precised here — the following considerations are formally valid for any type of constitutive relations which do not contain a natural time. Denote by V and S the body volume and surface, respectively, in some reference configuration. Introduce the Cartesian spatial coordinates (x_j) and the material coordinates (ξ_K), $j, K = 1, 2, 3$. The *deformation process* $\boldsymbol{\chi}$ is defined by

(1)
$$x_j = \chi_j(\boldsymbol{\xi}, t), \quad \boldsymbol{\xi} \in V, \quad t \in \mathscr{T},$$

where t denotes a scalar time-like parameter rather than a natural time, and $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$.

The body surface S is split into two parts. On the part S_u the displacement history is prescribed by the condition

(2)
$$\chi_j(\boldsymbol{\xi}, t) = \bar{x}_j(\boldsymbol{\xi}, t), \quad \boldsymbol{\xi} \in S_u, \quad t \in \mathscr{T},$$

where $\bar{x}_j$ are given functions. On the remaining part S_T the (position- and time-dependent)

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nominal tractions T_j referred to the reference configuration are assumed to be derivable from a given potential ω,

(3) $$T_j = -\frac{\partial\omega(\boldsymbol{\xi}, t, \mathbf{x})}{\partial x_j}, \quad \boldsymbol{\xi} \in S_T, \quad t \in \mathscr{T},$$

where $\mathbf{x} = (x_1, x_2, x_3)$ defines the position of a surface point in space. The loading device corresponding to the boundary conditions (2) and (3) has a potential energy

(4) $$W^e = \int_{S_T} \omega\, dS.$$

We assume that there exists a deformation process $\boldsymbol{\chi}^0$ which corresponds to a theoretical quasi-static solution to the problem. That process, the stability of which is to be studied, is called *fundamental process*, and the corresponding functions or quantities are distinguished by the superscript "0". The stresses in the fundamental process, related to the deformation history by constitutive relations, are at any instant in equilibrium with the surface tractions T_j and zero body forces. The stability of a fixed equilibrium configuration is considered as the stability of a particular fundamental process in which the functions in Eqs. (1), (2) and (3) do not depend on time.

The *kinematically admissible process* is defined as a process (1) which starts from the same initial configuration at $t = 0$ as the fundamental process and satisfies the condition (2) and appropriate regularity conditions (which are not precised here).

3. Persistent disturbances and energy measure

It is assumed that the fundamental process can be perturbed by some additional set of (nominal) body forces $F_j^*(\boldsymbol{\xi}, t)$ and surface tractions $T_j^*(\boldsymbol{\xi}, t)$ acting in the whole time interval $\mathscr{T}$. The forces F_j^* and T_j^* are thought to represent an overall disturbance of the idealized fundamental process. The *perturbed process* $\boldsymbol{\chi}^*$ is, by definition, a kinematically admissible process such that the stresses calculated from constitutive relations are at any moment in equilibrium with the corresponding set of body forces F_j^* and surface tractions $T_j + T_j^*$. The perturbed process is considered as a quasi-static one, possible inertia forces being included into perturbing forces. The quantities in the perturbed process are distinguished by the superscript "*".

To define the strength of a disturbance we use an energy measure. The *disturbance measure* ϱ is defined as

(5) $$\varrho = \varrho(\boldsymbol{\chi}^*, \boldsymbol{\chi}^0, t) = \sup_{\tau\in[0,t]} E^*(\boldsymbol{\chi}^*, \boldsymbol{\chi}^0, \tau),$$

where $E^*(\boldsymbol{\chi}^*, \boldsymbol{\chi}^0, \tau)$ is the energy supplied by a disturbance during the perturbed process $\boldsymbol{\chi}^*$ in the time interval $[0, \tau]$. This energy is assumed in the form

(6) $$E^* = W^* + (W^*_{\mathrm{ext}} - W^0_{\mathrm{ext}}),$$

where

(7) $$W^* = W^*(\boldsymbol{\chi}^*, \tau) = \int_0^\tau \left\{ \int_V F_j^* v_j^*\, dV + \int_S T_j^* v_j^*\, dS \right\} dt$$

is the work done directly by perturbing forces in the interval $[0, \tau]$, and the term in parentheses is the supplement of the energy

$$W_{\text{ext}} = W_{\text{ext}}(\chi, \tau) = \int_0^\tau \left\{ \int_S T_j v_j dS + \frac{dW^e}{dt} \right\} dt \tag{8}$$

taken in the interval $[0, \tau]$ from external sources by the *loading device* in the process χ^* with respect to that in the fundamental process χ^0. Here, $v_j = \partial\chi_j/\partial t$ denotes velocity, and the summation convention holds. Since $E^* = 0$ for $\tau = 0$, ϱ is nonnegative. In the case of examining stability of a fixed equilibrium configuration we have $W^*_{\text{ext}} = W^0_{\text{ext}} = 0$, and ϱ reduces to the maximum value of the work W^* done by perturbing forces.

The choice of the *distance d* between the perturbed and fundamental processes may depend on the particular problem under consideration. For instance, we can assume this distance in the form

$$d = d(\chi^*, \chi^0, t) = \int_V (x_j^* - x_j^0)(x_j^* - x_j^0) dV. \tag{9}$$

4. Stability postulate

DEFINITION. *The process χ^0 is stable in the time interval $\mathcal{T}$ if and only if for every number $\varepsilon > 0$ there is another number $\delta > 0$ such that for every perturbed process χ^* and each time $t \in \mathcal{T}$*

$$\varrho(\chi^*, \chi^0, t) < \delta \quad \textit{implies} \quad d(\chi^*, \chi^0, t) < \varepsilon.$$

The process χ^0 is called unstable if it is not stable.

There is a close analogy between this definition and the well-known definition of uniform stability of a process with respect to two metrics, due to MOVCHAN [6] (see also [7]). However, the definitions differ since the *persistent* disturbances are considered here rather than the initial ones. As a result, for given processes χ^0 and χ^* the disturbance measure ϱ is a *function* of time rather than a constant quantity. For this reason the usual definition had to be modified.

The above definition with the accompanying definitions of χ^*, ϱ and d constitutes a stability postulate. Implications of the postulate and the connections with the other approaches to stability problems will be discussed elsewhere.

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