A class of exact solutions for the flow of a viscoelastic fluid

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THE FLOW of the BKZ fluid is studied in the case of two infinite paralled plates rotating about a common axis with the same angular velocity. It is shown that an infinity of exact solutions exist for a nontrivial subclass of these fluids.

1. Introduction

BERKER [1] established the existence of an infinite set of nontrivial solutions for the flow of an incompressible linearly viscous fluid between two infinite parallel flat plates rotating with the same constant angular velocity about the same axis. The trivial rigid body motion is a special case of the infinite set.

The form of the velocity field assumed by Berker was

$$(1) \qquad u = -\Omega\big(y - g(z)\big), \quad v = \Omega\big(x - f(z)\big), \quad w = 0,$$

where, u, v, w are the x, y and z components of the velocity, respectively. Equation (1) belongs to the class of "pseudo-plane" motions (cf. BERKER [2], [3]) and represents a motion wherein in any $z = $ constant plane, the streamlines are concentric circles, the locus of the centers being in general a curve in space, $x = f(z)$, and $y = g(z)$ determining the equation of the locus.

RAJAGOPAL [4] has shown that the motion represented by Eq. (1) is one with constant stretch history. He showed that this then implies that the equation of motion of a simple fluid is of the same order as the Navier–Stokes equation and thus the adherence boundary condition is sufficient for determinacy. In the specific case of an incompressible and homogeneous fluid of second grade, RAJAGOPAL and GUPTA [5] have established an exact solution for the equations of motion.

In this brief note we study the flow of the fluid model introduced by BERNSTEIN, KEARSLEY and ZAPAS [6] between two infinite parallel plates rotating about a common axis with the same angular velocity. The fluid model, referred to as the BKZ fluid model, has proved useful in modeling the non-Newtonian behavior exhibited by certain fluids. In the case of an interesting but nontrivial subclass of these fluids, we find that an infinity of exact solutions exists, a result similar to Berker's analysis. In the case of the more general fluid model, the problem reduces to solving an integro-differential system with the appropriate boundary conditions.

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2. Equations of motion

For the sake of brevity, we refer the reader to [4] for the detailed kinematics of the motion represented by Eq. (1). It has been shown in [4] that this motion is one of constant stretch history. It follows that the stress $\mathbf{T}$ in the fluid can be represented by

(2) $$\mathbf{T} = -p\mathbf{I} + \mathbf{f}(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3),$$

where $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ are first three Rivlin–Ericksen tensors defined through

$$\mathbf{A}_1 = \operatorname{grad}\mathbf{v} + (\operatorname{grad}\mathbf{v})^T$$

and

$$\mathbf{A}_n = \frac{d}{dt}(\mathbf{A}_{n-1}) + \mathbf{A}_{n-1}(\operatorname{grad}\mathbf{v}) + (\operatorname{grad}\mathbf{v})^T\mathbf{A}_{n-1}, \quad n = 2, 3.$$

In the above equation, $\dfrac{d}{dt}$ denotes the material time derivative.

For the motion under consideration, a simple and straightforward computation yields

(3) $$\mathbf{F}_t(\tau) = \begin{pmatrix} \cos\Omega(t-\tau) & \sin\Omega(t-\tau) & -g'(z)\sin\Omega(t-\tau) + f'(z)[1-\cos\Omega(t-\tau)] \\ -\sin\Omega(t-\tau) & \cos\Omega(t-\tau) & g'(z)[1-\cos\Omega(t-\tau)] + f'(z)\sin\Omega(t-\tau) \\ 0 & 0 & 1 \end{pmatrix},$$

(4) $$\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & \Omega g'(z) \\ 0 & 0 & -\Omega f'(z) \\ \Omega g'(z) & -\Omega f'(z) & 0 \end{pmatrix},$$

(5) $$\mathbf{A}_2 = \begin{pmatrix} 0 & 0 & -\Omega^2 f'(z) \\ 0 & 0 & -\Omega^2 g'(z) \\ -\Omega^2 f'(z) & -\Omega^2 g'(z) & 2\Omega^2[f'^2(z) + g'^2(z)] \end{pmatrix}$$

and

(6) $$\mathbf{A}_3 = -\Omega^2\mathbf{A}_1.$$

Thus it follows from Eqs. (2) and (6) that

(7) $$\mathbf{T} = -p\mathbf{I} + \hat{\mathbf{f}}(\mathbf{A}_1, \mathbf{A}_2).$$

In view of Eqs. (4) and (5) the constitutive expression in Eq. (7) reduces to the form

(8) $$\mathbf{T} = -p\mathbf{I} + \tilde{\mathbf{f}}(f'(z)g'(z)).$$

From the balance of linear momentum and the assumed form of the motion in Eq. (1), the equations for $f(z)$, $g(z)$ and p are found to be

(9) $$\frac{d\tilde{f}_{13}}{dz} - \frac{\partial}{\partial x}(p + \varrho\phi) = -\varrho\Omega^2(x - f(z)),$$

$$\frac{d\tilde{f}_{23}}{dz} - \frac{\partial}{\partial y}(p + \varrho\phi) = -\varrho\Omega^2(y - g(z)),$$

$$\frac{d\tilde{f}_{33}}{dz} - \frac{\partial}{\partial z}(p + \varrho\phi) = 0,$$

where ϕ is the potential from which the specific body force is derived, i.e. $b = -\operatorname{grad}\phi$.

The equations for $f(z)$ and $g(z)$, obtained by eliminating $p+\varrho\phi$ from Eqs. (9) are

(10)

$$\frac{d\tilde{f}_{13}}{dz} - \varrho\Omega^2 f(z) = q,$$

$$\frac{d\tilde{f}_{23}}{dz} - \varrho\Omega^2 g(z) = s,$$

where q and s are constants.

The indeterminate scalar field p is then given by the expression

(11)
$$p+\varrho\phi = \frac{\varrho\Omega^2}{2}(x^2+y^2)+(qx+sy)+\tilde{f}_{33}+c.$$

The appropriate boundary conditions due to the adherence of the fluid at the top and bottom plates are

(12)
$$f(h) = f(-h) = 0$$

and

(13)
$$g(h) = g(-h) = 0.$$

If the locus of the centers intersects the $z = 0$ plane at $(l, 0, 0)$, (the $x-y$ axes can always be aligned in a manner that such is indeed the case), it follows that

(14)
$$f(0) = l, \quad g(0) = 0.$$

3. Fluid model

The stress $\mathbf{T}$ in a BKZ fluid is given by

(15)
$$T = -p\mathbf{I}+2 \int\limits_{-\infty}^{t} \{U_1\mathbf{C}_t^{-1}(\tau)-U_2\mathbf{C}_t(\tau)\}d\tau,$$

where

(16)
$$\mathbf{C}_t(\tau) = \mathbf{F}_t(\tau)^T\mathbf{F}_t(\tau).$$

In this constitutive equation, U is the strain energy potential which is a function of the principal invariants of $\mathbf{C}_t(\tau)$ and $t-\tau$, i.e.

(17)
$$U = U(I_1, I_2, t-\tau),$$

with

(18)
$$I_1 = \operatorname{Tr}\mathbf{C}_t^{-1}(\tau), \quad I_2 = \operatorname{Tr}\mathbf{C}_t(\tau)$$

and

(19)
$$U_i = \partial U/\partial I_i, \quad i = 1, 2.$$

A simple computation yields

(20)
$$\mathbf{C}_t(\tau) = 1-\frac{(t-\tau)}{\Omega}\mathbf{A}_1 + \frac{(1-C)}{\Omega^2}\mathbf{A}_2,$$

and

$$(21) \qquad \mathbf{C}_t^{-1}(\tau) = \mathbf{1} + \frac{(t-\tau)}{\Omega}[1+2(1-C)(f'^2+g'^2)]\mathbf{A}_1$$

$$-\frac{(1-C)}{\Omega^2}[1+2(1-C)(f'^2+g'^2)]\mathbf{A}_2 + \frac{(t-\tau)^2}{\Omega^2}\mathbf{A}_1^2$$

$$+\frac{(1-C)^2}{\Omega^4}\mathbf{A}_2^2 - \frac{S(1-C)}{\Omega^3}(\mathbf{A}_1\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1),$$

where $S = \sin\Omega(t-\tau)$, $C = \cos\Omega(t-\tau)$.

Also,

$$(22) \qquad I_1(t,\tau) = I_2(t,\tau) = 3+2[1-\cos\Omega(t-\tau)](f'^2+g'^2) \equiv I(\Omega(t-\tau),z).$$

It follows from Eqs. (4), (5), (15), (20)–(22) that

$$(23) \qquad \begin{aligned} \tilde{f}_{13}(f'(z),g'(z),\Omega) &= f'(z)B(z,\Omega)+g'(z)A(z,\Omega), \\ \tilde{f}_{23}(f'(z),g'(z),\Omega) &= -f'(z)A(z,\Omega)+g'(z)B(z,\Omega), \\ \tilde{f}_{33}(f'(z),g'(z),\Omega) &= P(z,\Omega)-(f'^2+g'^2)Q(z,\Omega), \end{aligned}$$

where

$$(24) \qquad \begin{aligned} A(z,\Omega) &= 2\int_0^\infty \tilde{U}\big(I(\Omega\alpha,z);\alpha\big)\sin\Omega\alpha\,d\alpha, \\ B(z,\Omega) &= 2\int_0^\infty \tilde{U}\big(I(\Omega\alpha,z);\alpha\big)(1-\cos\Omega\alpha)\,d\alpha, \end{aligned}$$

in which

$$(25) \qquad \tilde{U}(I,\alpha) = U_1(I,I,\alpha)+U_2(I,I,\alpha),$$

and

$$(26) \qquad \begin{aligned} P(z,\Omega) &= 2\int_0^\infty \tilde{U}_{12}\big(I(\Omega\alpha,z);\alpha\big)d\alpha, \\ Q(z,\Omega) &= 4\int_0^\infty \tilde{U}_2\big(I(\Omega\alpha,z);\alpha\big)(1-\cos\Omega\alpha)\,d\alpha, \end{aligned}$$

in which

$$(27) \qquad \begin{aligned} \tilde{U}_{12} &= U_1(I,I,\alpha)-U_2(I,I,\alpha), \\ \tilde{U}_2 &= U_2(I,I,\alpha). \end{aligned}$$

4. Exact solution

Let us consider the class of BKZ fluids wherein U_1 and U_2 are independent of I_1 and I_2. Thus

$$(28) \qquad U_i = U_i(t-\tau), \quad i = 1, 2$$

and hence the strain energy potential U has the form

(29) $$U = I_1 U_1(t-\tau) + I_2 U_2(t-\tau) + U_3(t-\tau).$$

From the definitions (24)

(30) $$A = G_2(\Omega), \quad B = G_1(\Omega),$$

where $G_1(\Omega)$ and $G_2(\Omega)$ denote the real and imaginary parts of the complex shear modulus of linear viscoelasticity. Also, from Eqs. (26) and (27), $P(z, \Omega)$ and $Q(z, \Omega)$ become independent of z.

It follows from (10), and $(23)_{1,2}$ and (30) that

(31)
$$f''G_1(\Omega) + g''G_2(\Omega) - \varrho\Omega^2 f = q,$$
$$-f''G_2(\Omega) + g''G_1(\Omega) - \varrho\Omega^2 g = s.$$

Let us introduce a complex valued function $F(z)$ defined through

$$F(z) = f(z) + ig(z),$$

where $i = \sqrt{-1}$. Equations (31) imply that

(33) $$F'' - (m+in)^2 F = \frac{(q+is)}{(\overline{G}(\Omega))^2}(G_1 + iG_2),$$

where m and n are defined through

$(34)_1$ $$m^2 = \frac{\varrho\Omega^2}{2(\overline{G}(\Omega))^2} \cdot [\overline{G}(\Omega) + G_1(\Omega)]$$

and

$(34)_2$ $$n^2 = \frac{\varrho\Omega^2}{2(\overline{G}(\Omega))^2} \cdot [\overline{G}(\Omega) - G_1(\Omega)],$$

where

(35) $$\overline{G}(\Omega) = [G_1^2(\Omega) + G_2^2(\Omega)]^{1/2}.$$

The boundary conditions (12), (13) and (14) become

(36) $$F(-h) = F(h) = 0$$

and

(37) $$F(0) = l.$$

The system (33)–(37) can be solved exactly to yield

$(38)_1$ $$f(z) = \frac{l}{\Delta}\{(\cosh mh \cos nh - \cosh mz \cos nz)(\cosh mh \cos nh - 1)$$
$$+ (\sinh mh \sinh nh - \sinh mz \sin nz)\sinh mh \sin nh\},$$

and

$(38)_2$ $$g(z) = \frac{l}{\Delta}\{(\sinh mh \sin nh - \sinh mz \sin nz)(\cosh mh \cos nh - 1)$$
$$- (\cosh mh \cos nh - \cosh mz \cos nz)\sinh mh \sin nh\},$$

where

(39) $$\Delta = (\cosh mh \cos nh - 1)^2 + (\sinh mh \sin nh)^2,$$

m and n being defined as in Eqs. $(34)_1$ and $(34)_2$, respectively.

From (11) and (23)$_3$, the indeterminate scalar field $p + \varrho\phi$ becomes

(40) $$p + \varrho\phi = \frac{\varrho\Omega^2}{2}(x^2 + y^2) + (qx + sy) + P(\Omega) - (f'^2 + g'^2)Q(\Omega) + c.$$

in which

(41)
$$q = \frac{l\varrho\Omega^2}{\Delta}(\Delta - 1 + \cosh mh \cos nh),$$

$$s = -\frac{l\varrho\Omega^2}{\Delta}\sinh mh \sin nh.$$

Note that when $l = 0$, that is the solution corresponding to the rigid body motion, q and s are zero. The shear stresses at the upper and lower fluid layers $z = \pm h$, as calculated from Eqs. (8), (23)$_{1,2}$ and (30) are given by

(42)
$$T_{13} = \frac{l}{\Delta}(\cosh mh - \cos nh)[(mG_2(\Omega) - nG_1(\Omega))\sin nh - (mG_1(\Omega)$$
$$+ nG_2(\Omega))\sinh mh)],$$

$$T_{23} = \frac{l}{\Delta}(\cosh mh - \cos nh)[(mG_2(\Omega) - nG_1(\Omega))\sinh mh + (mG_1(\Omega)$$
$$+ nG_2(\Omega))\sin nh].$$

It is easily seen from Eqs. (30) that $f(z)$ and $g(z)$ are even functions of z. Consequently, Eqs. (11) and (23) show that the shear stresses are odd functions of z while T_{33} is an even function of z. It follows that the normal tractions are either tensile on both the upper and lower fluid layers, or they are compressive on both layers. The tangential components of the traction vector point in the same sense on the upper and lower fluid layers.

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