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On some mathematical problems of the nonlinear Boltzmann equation

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THE PRESENT state of mathematical investigations on the Boltzmann equation is briefly outlined. In particular problems of existence and asymptotics, concerning space homogeneous and global solutions with different boundary conditions, are discussed. The existence of local solutions with external forces and the application of mollifiers are also briefly discussed. The asymptotic expansions resulting from the Boltzmann equation and concerning the bulk and the initial layer solutions are considered. Some of the many still open problems are mentioned.

Przedstawiono krótko obecny stan badań matematycznych nad równaniem Boltzmanna. W szczególności przedyskutowano zagadnienie istnienia i asymptotyki dla przestrzennie jednorodnych i globalnych rozwiązań z różnymi warunkami brzegowymi. Przedyskutowano również istnienie lokalnych rozwiązań z zewnętrznymi siłami oraz zastosowanie czynników wygładzających. Rozważono asymptotyczne rozwinięcia wynikające z równania Boltzmanna dla rozwiązań wewnętrznych i warstwy początkowej. Wymieniono niektóre z licznych, jeszcze otwartych problemów.

Кратко представлено настоящее состояние математических исследований уравнения Больцмана. В частности обсуждена проблема существования и асимптотики для пространственно однородных и глобальных решений с разными граничными условиями. Обсуждено тоже существование локальных решений с внешними силами, а также применение сглаживающих факторов. Рассмотрено асимптотическое разложение, следующие из уравнения Больцмана для внутренного решения и начального слоя. Представлены некоторые из многочисленных, открытых еще проблем.

1. Introduction

AT THE 100 th ANNIVERSARY celebrations of the Boltzmann equation in Vienna about 10 years ago, Professor G. E. Uhlenbeck, one of the outstanding contributors to Boltzmann's research field, said: "The Boltzmann equation has become such a generally accepted and central part of statistical mechanics, that it almost seems blasphemy to question its validity and to seek out its limitations", and he was right in stating further that many developments originated just from these questions which generated a remarkable revival of interest in it, say in the last third of this century.

We would like to turn our attention now to some of those problems. Let us recall, to start with, the classical formulation of Boltzmann's equation concerning the evolution of the one particle distribution function, $f = f(x, \xi, t)$ of a monoatomic dilute gas:

$$\frac{\partial f}{\partial t} + \xi \cdot \mathrm{grad}_x f + X \cdot \mathrm{grad}_\xi f = J(f, f), \tag{1.2}$$

where x, ξ are the position and velocity vectors, t time, $f(x, \xi, t)\,dxd\xi$ is the number of

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particles with velocities between ξ and $(\xi+d\xi)$ in the physical space between x and $(x+$ $+dx)$ and X is an external field force.

$$J(f,f) = \int d\xi_1 \int d\chi d\varepsilon g I(g,\chi)[f(x,\xi',t)f(x,\xi_1',t)-f(x,\xi,t)f(x,\xi_1,t)] \tag{1.2}$$

is the collision integral, where $I(g,\chi)$ is the collision cross section, $g = |\xi-\xi_1| = |\xi'-\xi_1'|$ is the relative velocity of the particles without an index and with index 1, whose post-collisional parameters are primed, and χ is the scattering angle of the binary collisions. The collision process depends strongly on the particle interaction potential $U(r)$ which, for the spherically symmetric particles considered, depends only on their distance apart, r. For interparticle potentials of the form $U(r) = \frac{1}{r^s}$ the collision rate is: $gI(g,\chi) \sim$ $\sim g^{1-\frac{2(d-1)}{s}}\sigma(\chi)$ where d is the physical space dimensionality and $\sigma(\chi)$ the differential collision cross section. It can be seen that for $s = 2(d-1)$ the collision rate becomes independent of the relative velocity g and then this interaction law corresponds to Maxwell's molecules. Particle interaction potentials with $s \leqslant 2(d-1)$ are called "soft" and for $s > 2(d-1)$, which includes hard spheres $(s \to \infty)$, are called "hard" interaction potentials.

The Boltzmann equation has some intrinsic properties which correspond to the fundamental mechanical conservation laws for the: number of particles $n = \int f d\xi dx$, momentum $P = \int \xi f d\xi dx$ and kinetic energy $E = \frac{1}{2}\int \xi^2 f d\xi dx$. The thermodynamic principle concerning the growth of entropy is included in the so-called H-theorem: $\frac{\partial}{\partial t}\int f \log f d\xi \leqslant 0$, (see e.g. C. CERCIGNANI [11]).

This survey of mathematical problems of the Boltzmann equation is mainly devoted to the existence and properties of solutions of Eq. (1.1). In what follows we shall use several definitions of solutions of this equation. By a classical solution of Eq. (1.1) we shall understand a function $f(x,\xi,t)$ which is continuously differentiable with respect to x, ξ and t variables and which fulfills this equation on the classical sense. We shall consider also Eq. (1.1) in Banach spaces of functions of x and ξ variables (or ξ variables only). Let B be one of such spaces. A distribution function $f(x,\xi,t)$ can be considered as a trajectory $f(t)$ in B and the term $\xi\cdot\mathrm{grad}_x + X\cdot\mathrm{grad}_\xi$ is just an unbounded operator in B. We shall call $f(t)$ a strong solution of Eq. (1.1) in B if $f(t)$ is a strongly differentiable trajectory in B and fulfills Eq. (1.1) in the norm of B.

We shall also use the notion of a mild solution. To define it let us write Eq. (1.1) in a simplified version:

$$\frac{\partial f}{\partial t} + \xi\cdot\mathrm{grad}_x f = J(f,f) \tag{1.3}$$

and define the transformation

$$T_\tau(x,\xi,t) = (x+\tau\cdot\xi, \xi, t+\tau).$$

The function $\varphi(\tau) = f(T_\tau(x,\xi,t))$ is called a mild solution of Eq. (1.3) if $\varphi(\tau)$ is differentiable with respect to τ and satisfies the following equation:

$$\frac{d\varphi(\tau)}{d\tau} = J(f,f)T_\tau(x,\xi,t).$$

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Numerous problems of rarefied gas dynamics of modern technology which could not be properly described by the continuous hydrodynamic models were successfully solved using different approximations to the Boltzmann equation and experimentally verified. Despite the persisting difficulties of applying the Boltzmann equation to describe the motion of dense gases, these properties were overwhelming. In addition, the possibility of using the Boltzmann equation as the starting step for deriving a hierarchy of hydrodynamic models, complete, with their coefficients based only on the knowledge of the particle interaction potential, has greatly increased the renewed interest in a deeper understanding and wider knowledge of their fundamental mathematical properties to contribute to a better description and understanding of the mechanics of fluids and the limitations of currently used theories.

2. The spatially uniform Boltzmann equation

In the case of spatially uniform problems the distribution function is independent of the space variables and the Boltzmann equation is then greatly simplified:

$$\frac{\partial f}{\partial t} = J(f,f), \qquad f(\xi, 0) = f_0(\xi), \tag{2.1}$$

where f_0 is the initial distribution function required in this case.

The first successful attempt to prove the existence and uniqueness for this equation was made by CARLEMAN [10] for the case of rigid spherical molecules and an axially symmetric distribution function in velocity space. Carleman proved that if f_0 is nonnegative, continuous, axially symmetric and satisfies the estimate

$$\sup_{\xi}(1+|\xi|^2)^{s/2}f_0(\xi) < +\infty \quad \text{for} \quad s > 6$$

then there exists, for all $t \geqslant 0$, a classical solution of Eq. (2.1) which is nonnegative, and satisfies $\sup_{t,\xi}(1+|\xi|^2)^{s/2}f(t,\xi) < +\infty$. This solution is unique among all solutions of this class. An essential improvement of this result was obtained by MORGENSTERN [36] who considered the equation in $L^1(R^3)$ space and proved for Maxwell molecules that there exists a unique solution of Eq. (2.1) global in time in L^1, provided that the initial data are nonnegative and belong to $L^1(R^3)$ too. POVZNER [43] also investigated solutions in $L^1(R^3)$ for the case of hard potentials including rigid spheres and proved that if $f_0(1+|\xi|^2) \in L^1$, then there exists a mild solution of Eq. (2.1), for $f_0(1+|\xi|^2)^2 \in L^1$ this solution is unique. ARKERYD [1] improved the last result showing that if $f_0(1+|\xi|^2)^2 \in L^1$, there is a unique, strong solution of Eq. (2.1) in $L^1(R^3)$ for all $t \geqslant 0$.

Having a solution for all times we can analyse the time evolution of the solution and the relaxation of the time dependent solution to a stationary one. In the spatially uniform case there is a good candidate for the stationary solution, namely the Maxwell distribution. It is easy to check that this is really a solution, and the only remaining question is its uniqueness. CARLEMAN [10] has shown that this is the case for his continuous, polynomially decreasing distribution function and ARKERYD [1] proved the uniqueness of the

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Maxwellian distribution as a stationary solution among all L^1 solutions, provided that the collision kernel is a positive function almost everywhere.

Also convergence to the equilibrium distribution was established in the Carleman case, i.e. a uniform convergence to the Maxwellian with the same hydrodynamic moments as the initial distribution and in the L^1 case ARKERYD [1] proved that if $f_0(1+|\xi|^2)^2 \in L^1$, then the solution converges weakly in $L^1(R^3)$ towards the Maxwellian with the same hydrodynamic moments as f_0.

Although the mathematical theory of the spatially uniform Boltzmann equation is nearly complete, there are still several open questions. The most important one concerns relaxation to equilibrium. A great effort in solving this problem has been made in connection with the so called "exact solutions" or "BKW-modes" and the Krook–Wu conjecture. For more details in this field, still in rapid progress, the excellent paper by ERNST [16] is recommended.

Despite this effort the question whether the Arkeryd result concerning weak L^1 convergence is the strongest possible one remains open. Further progress was achieved by DI BLASIO [13] who proved that for initial data which are close enough to equilibrium the difference between the equilibrium Maxwellian and the actual solution decreases exponentially in time. Let us mention, however, that this result is a by product of the theory developed for weakly nonlinear spatially nonuniform equations which will be discussed later.

Another open problem is connected with the potential of molecular interactions. All previously mentioned results are valid for hard potentials, i.e. inverse power law potentials with exponents $s \geqq 4$ and a cut-off. The problem of soft potentials, i.e. with $2 < s < 4$, was for a long time unsolved and, what is more important, this was also the case for inverse power law potentials without a cut-off. This last problem is of great physical importance as most calculations made in the kinetic theory refer to intermolecular forces of infinite range. The existence problem for an intermolecular potential without a cut-off was solved by ARKERYD [3]. Arkeryd proved that for inverse power intermolecular potentials with $s > 2$ and nonnegative initial data f_0 such that:

$$(1+|\xi|^2)f_0 \in L^1(R^3) \quad \text{and} \quad f_0 \log f_0 \in L^1(R^3)$$

there exists a mild solution of Eq. (2.1).

The last result covers of course the existence problem for soft potentials but the question of uniqueness remains open in both cases of: soft potentials and potentials without a cut-off.

3. Weakly nonlinear spatially nonuniform Boltzmann equation

If we are interested in solutions of the Boltzmann equation which are close to the Maxwell distribution, we can introduce, following GRAD [21], a function $F(x, \xi, t)$:

$$f = \omega + \omega^{1/2} F, \tag{3.1}$$

where f is a solution of the Boltzmann equation and ω is the Maxwell distribution

$$\omega(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{|\xi|^2}{2}\right). \tag{3.2}$$

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Then the equation satisfied by F is:

$$\frac{\partial F}{\partial t}+\xi \operatorname{grad}_x F = LF+\nu\Gamma(F, F), \tag{3.3}$$

where

$$LF = \omega^{-1/2}[J(\omega, \omega^{1/2}F)+J(\omega^{1/2}f, \omega)],$$

$$\nu\Gamma(F, F) = \omega^{-1/2}J(\omega^{1/2}F, \omega^{1/2}F).$$

If f is close to equilibrium (which means that F is small enough), the term $\nu\Gamma(F, F)$ is only a small perturbation of the linear part of Eq. (3.3), thus Eq. (3.3) is a weakly nonlinear equation.

Let us consider Eq. (3.3) for a gas contained in a region $\Omega \subset R^3$. If $\Omega = R^3$, we obtain an initial value problem (a Cauchy problem) for Eq. (3.3):

$$\frac{\partial F}{\partial t}+\xi \operatorname{grad}_x F = LF+\nu\Gamma(F, F), \quad x \in R^3, \quad \xi \in R^3, \quad t > 0, \tag{3.4}$$

$$F(x, \xi, 0) = F_0(x, \xi).$$

But if $\Omega \neq R^3$, we have to supplement Eq. (3.4) with a boundary condition on $\partial\Omega$. For this purpose let us assume that Ω is an open domain in R^3 with a smooth boundary $\partial\Omega$ in the sense that a Lyapounoff condition holds. Let n be the unit normal to $\partial\Omega$ pointed towards the interior of Ω. Then, along the boundary, we can split the distribution function into two parts:

$$f = f^+ + f^-, \tag{3.5}$$

where

$$f^+(x, \xi, t) = \begin{cases} f(x, \xi, t), & x \in \partial\Omega, \quad \xi \cdot n(x) \geqq 0, \quad t \geqq 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$f^-(x, \xi, t) = \begin{cases} 0, & \text{otherwise}, \\ f(x, \xi, t), & x \in \partial\Omega, \quad \xi \cdot n(x) < 0, \quad t \geqq 0. \end{cases}$$

Following GUIRAUD [23] the boundary conditions can be written in the form:

$$f^+ = Kf^-. \tag{3.6}$$

The operator K is assumed to be linear and of local type, i.e. for every $x \in \partial\Omega$ and every $t > 0$ there is a linear operator which operates on functions of ξ alone. Let us assume that along the boundary the temperature T_w and the mean velocity u_w are known. The Maxwellian corresponding to these parameters is

$$\omega_w = \frac{1}{(2\pi T_w)^{3/2}} \exp\left(-\frac{|\xi-u_w|^2}{2T_w}\right).$$

We assume that the operator K satisfies the following relations:

$$f^- \geqq 0 \Rightarrow Kf^- \geqq 0,$$

$$\int_{R^3} f^-|\xi \cdot n|\, d\xi = \int_{R^2} Kf^-|\xi \cdot n|\, d\xi, \tag{3.7}$$

$$\omega_w^+ = K\omega_w^-.$$

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The operator K can be written as follows:

$$K = K_0 + K_1.$$

Where K_0 is such that Eq. (3.7) holds with $T_w = 1$, $u_w = 0$ and K_1 depends continuously on $|T_w - 1|$ and $|u_w|$.

To find the boundary conditions in terms of F we should apply Eq. (3.1) to Eq. (3.6). Let us set

$$\mathcal{G} = \omega^{-1} K_0 \omega, \quad \mathcal{G}_1 = \omega^{-1} K_1 \omega \tag{3.8}$$

then the boundary conditions for F can be written in the form:

$$F^+ = \mathcal{G} F^- + \mathcal{G}_1 (1+F)^-.$$

Thus a boundary value problem for Eq. (3.3) can be stated as:

$$\begin{aligned} &\frac{\partial F}{\partial t} + \xi \cdot \mathrm{grad}_x F = LF + \nu \Gamma(F, F), && x \in \Omega, \quad \xi \in R^3, \quad t > 0, \\ &F^+ = \mathcal{G} F^- + \mathcal{G}_1 (1+F)^-, && x \in \partial\Omega, \quad \xi \in R^3, \quad t \geqq 0, \\ &F(x, \xi, 0) = F_0(x, \xi), && x \in \Omega, \quad \xi \in R^3, \quad t = 0. \end{aligned} \tag{3.9}$$

A very particular case of boundary value problems is the specular reflection case in a rectangular domain. The boundary condition is then

$$f(x, \xi) = f\big(x, \xi - 2n(n \cdot \xi)\big).$$

By reflection of the fundamental domain Ω with respect to each of three coordinate planes we obtain a domain Ω^* consisting of eight replicas of Ω. In Ω^* the function f satisfies a periodic boundary condition, hence by periodicity it can be extended to the whole R^3-space as a periodic function. The same is true for F and a boundary value problem for Eq. (3.3) in the case of specular reflection in a rectangular domain can be formulated as an initial value problem in a subspace of periodic functions:

$$\begin{aligned} &\frac{\partial F}{\partial t} + \xi \cdot \mathrm{grad}_x F = LF + \nu \Gamma(F, F), \quad x \in R^3, \quad \xi \in R^3, \quad t > 0, \\ &F(x, \xi, 0) = F_0(x, \xi), \\ &F \text{ — periodic in } x, \end{aligned} \tag{3.10}$$

Let us introduce now functional spaces in which the problems (3.4), (3.9), (3.10) will be solved.

Let $W^l{}_p(\Omega)$ be the usual Sobolev space. We shall consider functions which are in $W^l{}_p$ with respect to the x-variable, and in $L^r(R^3)$ with respect to the ξ-variable. Denote this space $B^l_{r,p}$, i.e.

$$B^l_{r,p} = \{F(x, \xi) : x \in \Omega, \xi \in R^3, N^l_{r,p}\{F\} < \infty\},$$

where $N^l_{r,p}$ is the norm in $B^l_{r,p}$ given by

$$N^1_{r,p}\{F\} = \left(\int\limits_{R^3} d\xi \left(\sum_{|k|=0}^{l} \int\limits_{\Omega} |D^k_x F|^p dx \right)^{r/p} \right)^{1/r}$$

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and

$$D_x^k = \frac{\partial^{|k|}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}}, \qquad |k| = k_1+k_2+k_3$$

(extension to $r = \infty$ or $p = \infty$ is obvious).

In some cases we shall need more restrictions on the initial data. To formulate the restrictions let us note that the operator $-\xi \operatorname{grad}_x F + LF$ has in $B^l_{r,p}$ a fivefold degenerated eigenvalue $\lambda = 0$ and let us denote by P the projection of $B^l_{r,p}$ on the eigenspace corresponding to this eigenvalue.

The problem (3.4) was solved globally in time by MASLOVA and FIRSOV [34]. They proved that for initial data F_0 such that the sum

$$N^2_{\infty,\infty}\{\varphi F_0\}+N^2_{\infty,1}\{\varphi F_0\}+N^5_{\infty,2}\{\varphi F_0\} \quad \text{with} \quad \varphi = (1+|\xi|^2)^{3/2}$$

is small enough, there exists a unique solution $F(t)$ to Eq. (3.4) such that:

$$(1+|\xi|^2)^{3/2}F(t) \in B^2_{\infty,\infty} \cap B^5_{\infty,2}.$$

Another proof was given by NISHIDA and IMAI [38]. They have chosen such initial data that the sum

$$N^0_{1,2}\{F_0\}+N^3_{\infty,2}\{\varphi F_0\} \quad \text{with} \quad \varphi = (1+|\xi|^2)^{3/2}$$

is small enough and have proved that a solution $F(t)$ exists globally in time and

$$(1+|\xi|^2)^{3/2}F(t) \in B^3_{\infty,2}.$$

It is easy to see that the Maxwell distribution ω given by Eq. (3.2) is the stationary solution corresponding to the Cauchy problem (3.4) and the rapidity of decay to equilibrium is an interesting problem. Maslova and Firsov give two following estimates: in the space $B^2_{\infty,\infty}$ it is like $(1+t)^{-9/8}$ and in the space $B^2_{\infty,2}$ like $(1+t)^{-3/8}$. Nishida and Imai obtained the following estimates: the decay in the space $B^3_{\infty,2}$ is like $(1+t)^{-3/4}$ but with the additional conditions on the initial data that: $xF_0 \in B^0_{1,2}$ and $F_0 \in \operatorname{Ker} P$, we have the improved decay of order $(1+t)^{-5/4}$.

The problem (3.10) was first solved by UKAI [46] who proved that if $F_0 \in \operatorname{Ker} P$ and $N^2_{\infty,2}\{(1+|\xi|^2)F_0\}$ is small enough, then there exists a global unique solution $F(t)$ such that $(1+|\xi|^2)F(t) \in B^2_{\infty,2}$. FIRSOV [17] has partially extended this result, removing the restriction $F_0 \in \operatorname{Ker} P$ and proving that if the sum

$$N^3_{\infty,\infty}\{\varphi F_0\}+N^7_{\infty,2}\{\varphi F_0\} \quad \text{with} \quad \varphi = (1+|\xi|^2)^{r/2}, \quad r > 5$$

is small enough, then there exists a unique solution $F(t)$ of (3.10) and $(1+|\xi|^2)^{5/2}F(t) \in B^3_{\infty,\infty} \cap B^3_{\infty,2} \cap B^7_{\infty,2}$. The result of Firsov is however restricted to hard intermolecular potentials with an exponent $s > 8$.

As in the case of the Cauchy problem (3.4), also for Eq. (3.10) the corresponding stationary solution is the Maxwell distribution ω, but the decay to equilibrium is exponential. Strictly speaking it was shown in both papers that there exists a positive constant γ such that the decay is of order $e^{-\gamma t}$ in $B^2_{\infty,2}$(Ukai) and in $B^3_{\infty,\infty} \cap B^3_{\infty,2}$(Firsov).

The problem (3.9) is much more complicated than (3.4) and (3.10). It has been partly solved by GUIRAUD [24] under several restrictions. Guiraud considered a gas consisting

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of rigid spheres in a convex domain Ω, with a smooth boundary $\partial\Omega$ whose principal curvatures are bounded from below. It was also assumed that the state of the gas is such that $\mathscr{G}_1 = 0$ and several restrictions of analytical character were imposed on $\mathscr{G}$. In consequence, specular reflection and diffusive boundary conditions were excluded from consideration. Under these assumptions it was proved that if $N^0_{\infty,\infty}\{(1+|\xi|^2)^{r/2}\omega^{1/2}F_0\}$ is small enough for $r > 3$, then there exists a unique mild solution $F(t)$ to Eq. (3.9) such that $(1+|\xi|^2)^{r/2}\omega^{1/2}F(t) \in B^0_{\infty,\infty}$. This result was partly extended by SHIZUTA and ASANO [44]. They considered a gas of hard-potential molecules, with a cut-off, in a convex domain with a three-times continuously differentiable boundary $\partial\Omega$ with positive principal curvatures. The boundary condition is restricted to specular reflection. They proved that if all these assumptions are fulfilled and $N^0_{\infty,\infty}\{(1+|\xi|^2)^{r/2}F_0\}$ is small enough with $r \geqq 1$, then there exists a unique mild solution $F(t)$ to Eq. (3.9) such that $(1+|\xi|^2)^{r/2}F(t) \in B^0_{\infty,\infty}$. A similar result was proved recently by MASLOVA [33]. She assumed that a gas consists of molecules interacting by hard potentials, with a cut-off, no additional assumptions concerning the boundary $\partial\Omega$ of the bounded domain Ω are necessary and that the very simple boundary condition

$$F^+ = 0 \tag{3.11}$$

holds. Under these assumptions she proved that if the sum

$$N^0_{\infty,2}\{\varphi F_0\}+N^0_{\infty,\infty}\{\varphi F_0\} \quad \text{with} \quad \varphi = (1+|\xi|^2)^{r/2}, \quad r \geqq 3$$

is small enough, then there exists a unique, mild solution $F(t)$ to Eq. (3.9) with the boundary condition (3.11) and $(1+|\xi|^2)^{r/2}F(t) \in B^2_{\infty,\infty}$.

It can be seen that in the case $\mathscr{G}_1 = 0$ the Maxwell distribution is a stationary solution to the problem (3.9). Hence there remains only to find the speed of decay to equilibrium. This problem was solved in all the above mentioned papers and it was shown that in all cases the decay to equilibrium is exponential.

The existence of stationary solutions can be proved for a wider class of boundary conditions than for time-dependent problems. GUIRAUD [23] has shown, that under the same assumptions as in the time-dependent case, except that the condition $\mathscr{G}_1 = 0$ is replaced by a continuous function $\mathscr{G}_1$ of $|T_w - 1|$ and $|u_w|$ such that the sum $|T_w - 1| + |u_w|$ is small enough, there exists a unique stationary solution F of the problem (3.9) such that

$$(1+|\xi|^2)^{r/2}\omega^{1/2}F \in B^0_{\infty,\infty}, \quad r > 3.$$

HEINTZ [25] extended this result of Guiraud, removing the restrictions imposed on the character of the boundary requiring only that Ω should be a bounded domain, and extended the range of molecular interaction potentials. He also assumed less restrictive properties of the operator $\mathscr{G}$. In particular, his boundary conditions include specular reflection and diffusive boundary conditions.

The number of unsolved problems for the spatially nonuniform Boltzmann equation is very large. The most important one is connected with the smallness of the initial data. Several attempts have been made to avoid this restriction (see: PALCZEWSKI [40]. UKAI and ASANO [47]); but the existence was proved only locally in time.

Another open problem is to solve Eq. (3.9) for a wider class of boundary conditions. The first step in this direction would be to fill in the gap between the results obtained for

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the stationary and the nonstationary case. It seems to the authors that an extension of Guiraud's nonstationary results to the case $\mathscr{G}_1 \neq 0$ is rather a technical problem. More involved but of greater interest is the extension of Heintz's result to the nonstationary problem Eq. (3.9).

A very important and interesting physically problem is the case of external flows around a body, which corresponds to Eq. (3.9) with Ω unbounded, or the internal flow in an infinitely long tube. For such problems Eq. (3.9) has to be supplemented by the condition:

$$F \to 0 \quad \text{for} \quad |x| \to \infty .$$

Maslova [32] considered the stationary case in which Ω is the exterior of a bounded domain and proved that for diffusive boundary conditions there exists a unique solution F such that $N^0_{\infty,p}\{(1+|\xi|^2)^3\omega^{1/2}F\} < +\infty$ provided that $\sup|\omega^{-1}(K\omega^- - \omega^+)|$ is small enough. Ukai and Asano [49] considered both stationary and nonstationary solutions in exterior of a bounded, convex domain, for dissipative boundary conditions and regular reflexion law, and flows with small velocity at infinity. They proved that if

$$N^0_{\infty,\infty}\{\varphi^\beta F_0\} + N^0_{\infty,p}\{\varphi^{\beta-1/p}F_0\} + N^0_{2,2}\{F_0\} + N^0_{2,q}\{F_0\}$$

with

$$\varphi = (1+|\xi|), \qquad \beta > 3, \qquad p \in [2,4], \qquad q \in [1,2]$$

is small enough then Eq. (3.9) supplemented by the above condition at infinity possess a unique solution globally in time. They proved also that the stationary problem has a unique solution F such that

$$N^0_{\infty,\infty}\{\varphi^\beta F\} + N^0_{\infty,p}\{\varphi^{\beta-1/p}F\} < +\infty .$$

Let us note that all results mentioned in this section hold for hard potentials only. This is due to the fact that the problem is weakly nonlinear and has been solved using a solution of the linear problem. The rapid decay of the solution of the linear problem necessary for the proof of the existence of a solution of the nonlinear problem, can easily be obtained only if the continuous spectrum of the linear problem is bounded away from zero, which is the case only for hard potentials and is not true for soft ones.

Although soft potentials are more difficult to deal with, some attempts have been made to treat this case. In particular, Caflish [6] has solved the problem Eq. (3.10) under the assumption that $F_0 \in \operatorname{Ker} P$ and $N^4_{\infty,2}\{\exp \alpha\xi^2 F_0\}$ is small with $0 < \alpha < 1/4$. An essential part of this paper is the solution of the linear problem and the proof that the function, which solves this linear problem, decays like $\exp(-\lambda t^\beta)$ with $\beta < 1$. This result is used to show the existence of solutions of the nonlinear problem and the decay to equilibrium of these solutions, which is also like $\exp(-\lambda t^\beta)$.

Another approach to the Boltzmann equation leading also to the weakly nonlinear problem is possible. For this purpose let us introduce a nondimensional parameter $\varepsilon = \mathrm{Kn}^{-1}$, where Kn is the Knudsen number, in front of the collision term

$$\frac{\partial f}{\partial t} + \xi \cdot \operatorname{grad}_x f = \varepsilon J(f,f).$$

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For very large mean free paths ε is small and the equation considered becomes again weakly nonlinear. Only the boundary value problem was considered in this case:

$$\begin{aligned} &\frac{\partial f}{\partial t}+\xi \operatorname{grad}_x f = \varepsilon J(f,f), && x\in\Omega, && \xi\in R^3, && t>0,\\ (3.12)\qquad &f^+ = Kf^-, && x\in\partial\Omega, && \xi\in R^3, && t\geqq 0,\\ &f(x,\xi,0)=f_0(x,\xi), && x\in\bar{\Omega}, && \xi\in R^3. \end{aligned}$$

Maslova [28, 29, 30] solved the time-independent problem for Eq. (3.12). She considered a gas consisting of rigid spheres and boundary conditions of the diffusive type. For the case of a bounded domain Ω in R^3 she proved that if ε is small enough, then the stationary problem corresponding to Eq. (3.12) has a solution in $B^0_{1,\infty}$ provided $N^0_{1,\infty}\{(1+|\xi|^2)^{r/2}f_0\} < +\infty$ for $r \geqq 1$. Existence was also proved for the Couette problem (Ω-interval in R^1) provided $N^0_{1,\infty}\{(1+|\xi|^2)^{r/2}e^{\alpha|\xi|^2}f_0\} < +\infty$ for $r \geqq 2, \alpha > 0$. Generally there is no uniqueness for these solutions, but it can be proved that if the stationary problem with $\varepsilon = 0$ has a unique solution, the same is true for $\varepsilon > 0$. The problem of existence of global solutions for the nonstationary Eq. (3.12) was unsolved for a long time. Lately, Babovsky [5] has partially solved it showing that for a bounded domain in R^3, small initial data and special stochastic boundary conditions there exists a global solution to Eq. (3.12).

4. Existence problem for the full nonlinear Boltzmann equation; mollifiers, local solutions

The existence of solutions to the Boltzmann equation for all times $t \geqq 0$ is one of the fundamental mathematical problems of the kinetic theory. In the previous sections we have shown how this problem has been solved in several particular cases, but a global solution to the full nonlinear Boltzmann equation remains still unknown. The usual way to construct global solutions to nonlinear equations is the following one: First we construct a local solution, then using *a priori* estimates, we show that the solution does not grow too rapidly and thus can be extended in time. Applying this procedure to the Boltzmann equation we can prove, with greater or smaller effort, the local existence of solutions. But the only a priori estimates at our disposal are the conservation laws of mass, momentum and energy (equalities) and the H-theorem (an inequality). However, we have four a priori estimates, they all hold in the $L^1(R^6)$ space only. Hence we have to operate in this space with the quadratic term $J(f,f)$. Since generally for $f \in L^1$, $J(f,f)$ is not in L^1, the procedure breaks down. To avoid this difficulty, several modifications of the term $J(f,f)$ have been introduced. The general aim of all these modifications was the same: to ensure that for $f\in L^1(R^6)$, $J(f,f)$ is also an element of the same space.

The modifications of Morgenstern [36] and of Povzner [43] were the first ones. They multiply the collision kernel in $J(f,f)$ by a position function $h(x,y)$. This modified collision operator, through integration in position space, acts as a mollifier. In this case a solution exists for all times. Arkeryd [2] introduced another modification. He truncated the function f in $J(f,f)$ if the result was greater than a given constant N; and this again allowed a global solution.

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A very interesting modification was proposed by CERCIGNANI, GREENBERG and ZWEIFEL [12]. They replaced the configuration space by a lattice and the streaming term $\xi \cdot \mathrm{grad}_x f$ by its finite-difference approximation. The space $L^1(R^6)$ is then replaced by $B = L^1(R^3, l_1)$ and for $f \in B$ we have $J(f, f) \in B$ (this due to the estimate $\sup\limits_x |f| \leqq ||f||_{l_1}$). This again gives the global existence of a solution. This approach was widely used by different authors (see: SPOHN [45], GREENBERG, VOIGT and ZWEIFEL [22], PALCZEWSKI [42]).

The global existence can also be proved, if we apply a typical nonlinear partial differential equation modification. Namely the term $J(f, f)$ can be replaced by $J(f^*, f)$, where $f^* = f * \varphi$, and φ is a usual smoothing function, i.e.

$$\varphi \in C_0^\infty, \quad \int \varphi = 1, \quad \mathrm{supp}\varphi \subset K(0, 1).$$

An interesting result in this direction was obtained by WIESER [50] by smoothing the solution in adding a smoothing term Δf to the left hand side of the equation. This led also to the global existence proof.

Having the global solutions to modified equations we can analyse in what sense they approximate solutions to the original Boltzmann equation. Usually in modified equations we have a parameter whose convergence to an extreme limit, zero or infinity, corresponds to a convergence of the modified equation to the original Boltzmann equation. Hence the analysis is in two steps. First we look for a limiting solution of the modified equation as the parameter tends to the limit; then we must check whether the limiting function fulfills the original equation.

This first step was realized by GREENBERG, VOIGT and ZWEIFEL [22], who proved that there is a sub-sequence of solutions on a lattice, which converges weakly to a limit as the lattice spacing tends to zero. A similar behaviour can be proved for Arkeryd's modification and perhaps for some others. However, the second step, i.e. the fulfillment in the limit of the Boltzmann equation, remains still an unsolved problem.

In connection with these unsuccessful attempts let us mention papers dedicated to local existence, which shed some light on the problem of global existence. KANIEL and SHINBROT [26] developed a method of successive approximations which give a solution on the time interval on which we can find a proper upper bound for a solution. The problem is that we can only find this upper bound in a finite time interval. PALCZEWSKI [41] has proved local existence in $L^1(R^6)$. This solution can be extended to an infinite time interval provided the particle density remains finite. These results show that if a global solution does not exist, it is due to the blowing up of the solution or of its moments.

We end this section by calling attention to another unsolved mathematical problem of great practical interest, the global existence problem for the complete Boltzmann equation including external forces, as formulated in Eq. (1.1). The only known results are the local existence theorems of GLIKSON [18, 19] for small initial data and of ASANO [4] for arbitrary initial data and hard or soft intermolecular potentials. The linear problem in the presence of an external force or any other simple case remain also unsolved.

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5. The asymptotic limit of solutions of the Boltzmann equation

The Boltzmann equation will now be taken in the following form:

$$\frac{\partial f}{\partial t}+\xi \operatorname{grad}_x f = \frac{1}{\varepsilon} J(f,f), \tag{5.1}$$

where ε is the mean free path of the gas. The fluid-dynamical variables of the gas are related to the distribution function in the classical way as follows:

$$\begin{cases} \varrho = \int f(x,\xi,t)d\xi & \text{mass density,} \\ u_i = \varrho^{-1}\int \xi_i f(x,\xi,t)d\xi & i=1,2,3, \text{ hydrodynamic velocity,} \\ e = \varrho^{-1}\int |c|^2 f(x,\xi,t)d\xi & \text{internal energy,} \end{cases} \tag{5.2}$$

where

$$c_i = \xi_i - u_i.$$

The conservation laws, which hold for Eq. (5.1), can be written in terms of the fluid-dynamic variables, as is well known:

$$\begin{gathered} \frac{\partial}{\partial t}+\operatorname{div}(\varrho\mathbf{u}) = 0, \\ \frac{\partial}{\partial t}(\varrho\mathbf{u})+\operatorname{div}(\varrho\mathbf{u}\cdot\mathbf{u}+\mathbf{P}) = 0, \\ \frac{\partial}{\partial t}\varrho\left(e+\frac{1}{2}u^2\right)+\operatorname{div}\left[\varrho\left(e+\frac{1}{2}u^2\right)\mathbf{u}+\mathbf{P}\mathbf{u}+\mathbf{q}\right] = 0, \end{gathered} \tag{5.3}$$

where the stress tensor $\mathbf{P}$ and the heat flow vector $\mathbf{q}$ are given by the following relations:

$$P_{ij} = \int c_i c_j f(x,\xi,t)d\xi = p_{ij}+p\delta_{ij},$$

$$q_i = \frac{1}{2}\int c_i|c|^2 f(x,\xi,t)d\xi$$

and p is the scalar pressure.

The temperature of a gas is defined by the equation of state:

$$3/2kT = e.$$

In order to close the system of fluid-dynamic equations (5.3), $\mathbf{P}$ and $\mathbf{q}$ must be expressed in terms of ϱ, u, e. This can be done using the Hilbert or the Chapman–Enskog procedures. Using the Hilbert procedure, we expand the distribution function f into a power series of ε:

$$f = \sum_{n=0}^{\infty}\varepsilon^n f^{(n)}. \tag{5.4}$$

Substituting Eq. (5.4) into Eq. (5.1) we obtain in the zeroth order for $f^{(0)}$ the Maxwellian

$$f^{(0)}(x,\xi,t) = \varrho(2\pi kT)^{-3/2}\exp(-|\xi-u|^2(2kT)^{-1}) \tag{5.5}$$

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and the corresponding ϱ, u, T fulfill the system of the familiar compressible Euler equations.

$$
\begin{aligned}
&\frac{\partial \varrho}{\partial t}+\operatorname{div}(\varrho \mathbf{u}) = 0,\\
(5.6)\qquad &\frac{\partial}{\partial t}(\varrho \mathbf{u})+\operatorname{div}(\varrho \mathbf{u}\mathbf{u})+\operatorname{grad} p = 0,\\
&\frac{\partial}{\partial t}\varrho\left(e+\frac{1}{2}u^2\right)+\operatorname{div}\left[\varrho \mathbf{u}\left(e+\frac{1}{2}u^2\right)+pu\right] = 0.
\end{aligned}
$$

In higher orders we obtain nonhomogeneous Euler equations.

The Chapman–Enskog expansion, which is a little more involved than the Hilbert one, gives in the zeroth order the same solution as the Hilbert expansion, but in higher orders results in the more involved partial differential hydrodynamic equations, called Navier–Stokes, Burnett etc.

An important physical problem is, whether and in what range of the relevant parameters do the fluid dynamical solutions approximate the solutions of the Boltzmann equation. This can be formulated mathematically as follows: Is the finite expansion corresponding to Eq. (5.4) an approximate solution to the Eq. (5.1)? This problem was partially solved by GRAD [21]; but the first general solution was given by NISHIDA [37]. He considered the initial value problem for Eq. (5.1) with periodic initial data (actually Nishida refers to nonperiodic initial data; but one of his compactness arguments is relevant only for periodic solutions). Denoting by f^ε a solution of Eq. (5.1) for $\varepsilon > 0$, it was shown that f^ε exists on the time interval $[0, \tau]$, independent of ε, provided the initial distribution is sufficiently close to an absolute Maxwellian and is analytic in the space variable. The convergence of f^ε on $[0, \tau]$ to f^0 as $\varepsilon \to 0$ was shown, where f^0 for $t \in [0, \tau]$ is a local Maxwellian given by Eq. (5.5) with fluid dynamical quantities ϱ, u, e, which solve the compressible Euler equations (5.6). In general, the convergence is not uniform near $t = 0$, and the limit f^0 is discontinuous at $t = 0$.

The limiting function f^0 plays the role of the outer (or bulk) solution in the theory of singular perturbations (see O'MALLEY [39], ECKHAUS [14]). The singular behaviour at $\varepsilon = 0 = t$ is due to the presence of an initial layer. This initial layer was discussed in the paper of UKAI and ASANO [48], who proved that a necessary and sufficient condition for the uniform convergence of f^ε near $t = 0$ and the continuity of f^0 on $[0, \tau]$ is that the initial distribution is itself a local Maxwellian. CAFLISH [7] has reversed and partially extended the above results, assuming that solutions of the compressible Euler equations (5.6) are given in the time interval $[0, \tau]$. He then constructed the local Maxwellian equation (5.5), and taking the initial distribution $f_0 = f^0(0)$ proved that Eq. (5.1), with these initial data, has a unique solution f^ε for $\varepsilon > 0$ and

$$||f^\varepsilon - f^0|| \leqslant C\varepsilon \quad \text{on} \quad [0, \tau].$$

In agreement with the spirit of the asymptotic expansion method, we can expect that fluid-dynamical solutions are asymptotic to solutions of the Boltzmann equation, except in some regions of space and time. These regions are the initial layers, the boundary layers and the shock layers, where the gradients of the fluid-dynamical quantities are large.

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Let us note that problems arising in the analysis of all these layers have been omitted in previously discussed papers; since an initial distribution in the form of a local Maxwellian eliminated the initial layer, periodicity of solutions excluded the boundary layer, and smoothness of solutions of the Euler equations exclude the appearance of shocks.

In order to discuss initial layers, we have to introduce a stretched variable $\tau = t/\varepsilon$. The initial layer equation becomes then:

$$\frac{\partial f}{\partial \tau} + \varepsilon \xi \operatorname{grad}_x f = J(f, f). \tag{5.7}$$

Expanding f as in Eq. (5.4) and substituting in Eq. (5.7) we obtain the following zeroth order equation:

$$\frac{\partial f^{(0)}}{\partial \tau} = J(f^{(0)}, f^{(0)}).$$

This is the spatially uniform Boltzmann equation, discussed in Sect. 2. What is needed now for a physically relevant solution in this case is not only the existence of solutions, but also a rapid decay to equilibrium. In the linear case the decay is exponential and GRAD [20] was able to construct the initial layer solution for the linearised Boltzmann equation. However, for the nonlinear equation this problem remains still unsolved.

In the boundary layer case the stretched variable is $\eta = x/\varepsilon$ and the equation of zero-order approximation reads

$$\xi \operatorname{grad} f^{(0)} = J(f^{(0)}, f^{(0)}). \tag{5.8}$$

This equation has to be solved in the half-space $\eta_1 > 0$ with a boundary condition on the plane $\eta_1 = 0$. Equation (5.8) is the stationary Boltzmann equation and the boundary problems for it have been discussed in Sect. 3. As can be seen there, no existence results for the half-space problem are available and nothing is known about the decay to equilibrium. Even for the linear case we know only one rigorous result of MASLOVA [31], which solved the linear problem in the one-dimensional case.

For the shock-layer problem, which is most interesting from the physical point of view, the known mathematical results are rather limited. Strong mathematical results were obtained by CAFLISCH and NICOLAENKO [9]. They considered stationary plane weak shock waves. They used the difference between the speed of sound c_0 and the shock wave speed s, as the small parameter $\varepsilon = c_0 - s$ and proved the existence of shock-wave-structure solutions of the Boltzmann equation for sufficiently small ε. They showed also that the first two terms in the expansion in power series of ε are exactly the same as the corresponding Taylor solution of the Navier–Stokes equations.

All we said above, except about the shock-wave structure, concerned the Hilbert expansion only. Although this expansion, as we have seen, is directly related to the Euler equations, it is also possible to include the Navier-Stokes equations in this scheme. This can be done following ELLIS' and PINSKY'S [15] idea of rearrangement of terms in the expansion (see CAFLISCH [8]). Let us mention also that KAWASHIMA, MATSUMURA and NISHIDA [27] have proved that, if the Boltzmann equation and the compressible Navier–Stokes equations have global solutions, then, as $t \to \infty$, they have the same asymptotics:

$$||f(t) - f^{(0)}(t)|| \leqslant c(1+t)^{-5/4},$$

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when $f^{(0)}$ is a local Maxwellian, given by Eq. (5.5) with the fluid-dynamical parameters, which are solutions of the corresponding compressible Navier–Stokes equations. This result is restricted to the case when global solutions exist, which is the case only if the initial data are close enought to equilibrium.

However, this is not a justification for the Chapman–Enskog procedure. The difficulties concerning the last mentioned procedure are due to the fact that it has not been formulated as a normal asymptotic procedure. Some effort in formulating the Chapman–Enskog procedure as an asymptotic one has been made by Mika and Palczewski [35]; but their results are strictly applicable to systems of ordinary differential equations. Thus the problem whether the Chapman–Enskog procedure gives an asymptotic solution to the Boltzmann equation on the level of the Navier–Stokes equation remains one of the crucial unanswered questions of the kinetic and continuum theories of fluid dynamics.

6. Closing remarks

In this rather selective presentation of the mathematical problems of the Boltzmann equation we have by far not covered the great wealth of problems, questions and existing results connected with it. The continuing scientific effort, aimed at a better understanding of the important mathematical aspects of this equation, which we think gives the fullest description of the behaviour of not very dense media, composed of a very large number of particles, is still lively.

We omitted altogether the many useful models, trying to replace the stumbling block of the equation considered, which is the collision term. This was very ably covered in a recent monograph of Ernst [16]. We have also very briefly considered the relation between the Boltzmann equation and the continuum fluid-dynamic equation, which was so capably and nicely treated very recently by Caflisch [8].

We hope, however, that we called your attention to the many open, rather difficult existing problems and the progress, achieved lately, concerning the theoretical side of this important equation.

To end, we would like to emphasize that, although the distribution function, which is the dependent variable of the Boltzmann equation, bears a wealth of data much beyond the interest and needs of the physicists, a fuller rigorous understanding of its mathematical properties would provide the sound background necessary for a fuller assessment of the existing experimental and approximate theoretical results, and guide the necessary developments to improve our grip on the field of science, connected with gases and liquids and other related fields.

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