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On generalized solutions of a nonlinear boundary value problem of elasticity
Resonance case

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THE DIRICHLET problem for the general Lamé equations with a nonlinear right hand side is considered. It is assumed that the homogeneous problem has p linearly independent weak solutions (resonance case). Necessary and sufficient conditions of the Landesman–Lazer type are proved for the existence of the weak solution to the nonlinear problem.

Rozważono zagadnienie Dirichleta dla ogólnego układu równań Lamégo z nieliniowymi prawymi stronami. Założono, że zagadnienie jednorodne ma p liniowo-niezależnych słabych rozwiązań (przypadek rezonansowy). Wyprowadzono warunki konieczne i dostateczne typu Landesmana–Lazera dla istnienia słabych rozwiązań zagadnienia nieliniowego.

Рассматривается задача Дирихле для общей системы уравнений Ламе с нелинейными правыми сторонами. Предпалагается, что задача однородная имеет p линейно-независимых слабых решений (резонансный случай). Выведены необходимые и достаточные условия типа Ландесмана–Лазера для существования решений нелинейной задачи.

1. Introduction

LET $D \subset R^3$ be a domain and ∂D its Lipschitz boundary. The following Dirichlet Problem of the Theory of Elasticity will be considered: Find the displacement (vector-) function u satisfying the general Lamé equations

$$(1.1)\qquad \frac{\partial}{\partial x_i}(\lambda \operatorname{div} u)+\sum_{j=1}^{3}\left[\frac{\partial}{\partial x_j}\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right]+\omega^2 u_i = -g_i(x, u(x)), \qquad x \in D,$$

and the boundary conditions

$$(1.2)\qquad u_i(x) = 0, \quad x \in \partial D, \quad (i = 1, 2, 3),$$

where λ, μ are Lamé coefficients such that $\lambda, \mu \in L^\infty(D)$, $\lambda(x) \geqslant 0$, $\mu(x) \geqslant \mu_0 > 0$ for all $x \in \overline{D}$, where μ_0 is a constant, ω is a real constant, $g\colon D \times R^3 \to R^3$ is a given vector function, components $g_i(x, u_1, u_2, u_3)$, $(i = 1, 2, 3)$, satisfy the Carathéodory continuity condition, i.e. g_i are measurable with respect to x for fixed (u_1, u_2, u_3), $-\infty < u_i < +\infty$, and continuous in (u_1, u_2, u_3) for almost all $x \in D$.

Let us introduce the following spaces (notation as in [6]) $W = [L^2(D)]^3$, $\mathring{W} = [\mathring{H}_1]^3$, the respective norms

$$||u||_W = \Big(\sum_{i=1}^{3} ||u_i||^2_{L^2}\Big)^{1/2}, \qquad ||u||_{\mathring{W}} = \Big(\sum_{i=1}^{3} ||u_i||^2_{\mathring{H}_1}\Big)^{1/2}$$

and inner products

$$(u,v)_W = \sum_{i=1}^{3} (u_i, v_i)_{L^2}, \qquad (u,v)_{\mathring{W}} = \sum_{i=1}^{3} (u_i, v_i)_{\mathring{H}_1},$$

($\mathring{H}_1$ is the well-known Sobolev space). W and $\mathring{W}$ are Hilbert spaces.

In consistency with the definition 1.1 of the paper [3], we define the weak (generalized) solution to the problem (1.1), (1.2).

DEFINITION 1.1. *A function u is called the weak solution of the Dirichlet problem* (1.1), (1.2) *if*

(1.3) $$u \in \mathring{W},$$

(1.4) $$\int_D \left[\lambda(\operatorname{div} u)(\operatorname{div} v) + \frac{\mu}{2} \sum_{i,j=1}^{3} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)\right] dx$$

$$-\omega^2 \int_D \left(\sum_{i=1}^{3} u_i v_i\right) dx = \int_D \sum_{i=1}^{3} g_i(x, u(x)) v_i(x)\, dx \quad \textit{for all} \quad v \in \mathring{W}.$$

In the paper [3] we have considered the nonresonance case of the problem (1.1), (1.2), i.e. the problem (1.1), (1.2) under the condition that the linear homogeneous problem

(1.5) $$\frac{\partial}{\partial x_i}(\lambda \operatorname{div} u) + \sum_{j=1}^{3} \left[\frac{\partial}{\partial x_j} \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] + \omega^2 u_i = 0, \quad x \in D,$$

(1.6) $$u_i(x) = 0, \quad x \in \partial D \quad (i = 1, 2, 3),$$

has only the trivial weak solution.

For g_i satisfying the inequality

(1.7) $$|g_i(x, u_1, u_2, u_3)| \leqslant a_i(x) + b \sum_{k=1}^{3} |u_k|,$$

where $a_i(x) \in L^2(D)$, and b is a sufficiently small positive constant, we have proved the existence of a weak solution to the problem (1.1), (1.2). In the present paper we consider the resonance case, i.e. the problem (1.5), (1.6) may have now nontrivial weak solutions. Necessary and sufficient conditions will be proved for the existence of a weak solution to the problem (1.1), (1.2).

E. M. LANDESMAN, A. C. LAZER [5] have been the first to consider the similar problem

$$Lu + \alpha u + g(u) = h(x), \quad x \in D \subset R^n; \quad u(x) = 0, \quad x \in \partial D,$$

where L is a second-order, self-adjoint, uniformly elliptic on D, α is a positive constant, g is a real-valued function, bounded and continuous on the real line (satisfying some additional restrictions), h is a real function in $L^2(D)$.

These authors gave a necessary and sufficient condition for the existence in case $L + \alpha I$ has one-dimensional null space. S. A. WILLIAMS [7] extended the result to finite dimensional null space. The proofs of the sufficient conditions in [5] and [7] were based on Schauder's Fixed Point Theorem. P. HESS [4] contributed to the subject by a short and elegant method of proving the sufficient condition. Since then the matter has been developed in various directions by many authors. The most comprehensive paper so far is that written by H. BREZIS and L. NIRENBERG [2] (see further references there).

2. The necessary condition

Following Sect 2 of the paper [3] we denote

$$(2.1)\qquad \langle u, v\rangle_{\mathring{W}} = \int_D \left[\lambda(\operatorname{div} u)(\operatorname{div} v) + \frac{\mu}{2}\sum_{i,j=1}^{3}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)\right] dx.$$

The form $\langle u, v\rangle_{\mathring{W}}$ is the well-defined inner product in $\mathring{W}$. The norm $\|\cdot\|_{\tilde{\mathring{W}}}$ induced by it is equivalent to the norm $\|\cdot\|_{\mathring{W}}$. Furthermore, there exists a unique element $Au \in \mathring{W}$ (see (2.6) in [3]) such that

$$\langle Au, v\rangle_{\mathring{W}} = (u, v)_W \quad \text{for all} \quad v \in \mathring{W}.$$

The latter defines an operator $A\colon W \to \mathring{W}$ linear and continuous and such that its restriction to $\mathring{W}$ is compact.

The equation (1.4) can be written in the operator form

$$(2.2)\qquad u - \omega^2 Au = Ag(x, u), \quad u \in \mathring{W}.$$

The operator A is self-adjoint in $\mathring{W}$ and W. Indeed,

$$\langle Au, v\rangle_{\mathring{W}} = (u, v)_W = (v, u)_W = \langle Av, u\rangle_{\mathring{W}} = \langle u, Av\rangle_{\mathring{W}},$$

$$(Au, v)_W = \langle A^2u, v\rangle_{\mathring{W}} = \langle Au, Av\rangle_{\mathring{W}} = \langle u, A^2v\rangle_{\mathring{W}} = (Av, u)_W = (u, Av)_W.$$

Assume that the linear homogeneous problem (1.5), (1.6) has p linearly independent weak solutions. Hence the equation

$$(2.3)\qquad u - \omega^2 Au = 0, \quad u \in \mathring{W}$$

has p linearly independent solutions; they span a subspace of $\mathring{W}$.

Let $w^{(1)}, \ldots, w^{(p)}$ be a basis of the subspace. Every nonzero element z of that subspace may be decomposed as

$$z = \alpha_1 w^{(1)} + \ldots + \alpha_p w^{(p)}, \quad \text{where} \quad \alpha_1^2 + \ldots + \alpha_p^2 > 0.$$

Let us denote

$$D^+_{i,(\alpha_1,\ldots,\alpha_p)} = \{x \in D;\ \alpha_1 w_i^{(1)} + \ldots + \alpha_p w_i^{(p)} > 0\},$$

$$D^-_{i,(\alpha_1,\ldots,\alpha_p)} = \{x \in D;\ \alpha_1 w_i^{(1)} + \ldots + \alpha_p w_i^{(p)} < 0\}, \quad i = 1, 2, 3.$$

THEOREM 2.1. *Let g_i, $i = 1, 2, 3$, satisfy the Carathéodory continuity condition and inequalities*

$$(2.4)\qquad h_i(x) \leqslant g_i(x, u) \leqslant H_i(x) \quad \textit{for all} \quad (x, u) \in D \times R^3$$

$i = 1, 2, 3$, *where* $h(x)$, $H(x) \in W$. *If there exists a weak solution to the problem* (1.1), (1.2), *then for all real numbers* $\alpha_1, \ldots, \alpha_p$ *with* $\alpha_1^2 + \ldots + \alpha_p^2 = 1$

$$(2.5)\qquad \sum_{i=1}^{3}\left[\int_{D^+_{i,(\alpha_1,\ldots,\alpha_p)}} H_i(x)|z_i|\,dx - \int_{D^-_{i,(\alpha_1,\ldots,\alpha_p)}} h_i(x)|z_i|\,dx\right] \geqslant 0,$$

where

$$z_i = \alpha_i w_i^{(1)} + \ldots + \alpha_p w_i^{(p)}.$$

P r o o f. Let u be a weak solution to the problem (1.1), (1.2). Then

$$(u, w^{(j)})_W = \omega^2 (Au, w^{(j)})_W + (Ag(x, u), w^{(j)})_W = \omega^2 (u, Aw^{(j)})_W + (g(x, u), Aw^{(j)})_W$$
$$= (u, w^{(j)})_W + \frac{1}{\omega^2}(g(x, u), w^{(j)})_W \quad \text{for} \quad j = 1, \ldots, p.$$

Hence $(g(x, u), w^{(j)})_W = 0$ and $(g(x, u), z)_W = 0$, where $z = \alpha_1 w^{(1)} + \ldots + \alpha_p w^{(p)}$, $\alpha_1, \ldots, \alpha_p$ are arbitrary real numbers. Thus, by the assumptions (2.4) we have

$$0 = \int_D \sum_{i=1}^{3} g_i(x, u) z_i dx = \sum_{i=1}^{3} \Big[\int_{D_{i,(\alpha_1,\ldots,\alpha_p)}^+} g_i(x, u) z_i dx + \int_{D_{i,(\alpha_1,\ldots,\alpha_p)}^-} g_i(x, u) z_i dx \Big]$$
$$\leqslant \sum_{i=1}^{3} \Big[\int_{D_{i,(\alpha_1,\ldots,\alpha_p)}^+} H_i(x)|z_i| dx - \int_{D_{i,(\alpha_1,\ldots,\alpha_p)}^-} h_i(x)|z_i| dx \Big].$$

3. The sufficient condition

THEOREM 3.1. *Let* g_i, $i = 1, 2, 3$, *satisfy the Carathéodory continuity condition, the inequality* (1.7) *with* $b = 0$ *i.e.*

$$|g_i(x, u_1, u_2, u_3)| \leqslant a_i(x), \quad \text{where} \quad a_i(x) \in L^2(D),$$

and the relations

$$\lim_{u_i \to -\infty} g_i(x, u_1, u_2, u_3) = h_i(x), \qquad \lim_{u_i \to \infty} g_i(x, u_1, u_2, u_3) = H_i(x),$$

where limits are taken uniformly with respect to the remaining variables u_j *where* $j \neq i$, $h(x), H(x) \in W$. *If for all real numbers* $\alpha_1, \ldots, \alpha_p$ *with* $\alpha_1^2 + \ldots + \alpha_p^2 = 1$

$$\sum_{i=1}^{3} \Big[\int_{D_{i,(\alpha_1,\ldots,\alpha_p)}^+} H_i(x)|z_i| dx - \int_{D_{i,(\alpha_1,\ldots,\alpha_p)}^-} h_i(x)|z_i| dx \Big] > 0, \tag{3.1}$$

where $z_i = \alpha_1 w_i^{(1)} + \ldots + \alpha_p w_i^{(p)}$, *then there exists a weak solution to the problem* (1.1), (1.2).

P r o o f. Consider the equation

$$u - \omega^2 Au = 0, \quad u \in \mathring{W}.$$

The number $\frac{1}{\omega^2}$ is an isolated eigenvalue of the compact operator A. Therefore, for n sufficiently large the equation

$$u - \omega^2 Au = \frac{1}{n} Au \tag{3.2}$$

has only the trivial solution in $\mathring{W}$, and there exists $u_n \in \mathring{W}$ such that

$$u_n - \omega^2 Au_n - \frac{1}{n} Au_n = Ag(x, u_n) \tag{3.3}$$

(Theorem 3.1 of the paper [3]).

Equation (3.3) means that

$$(3.4) \qquad \langle u_n, v\rangle_{\mathring{W}} - \omega^2 (u_n, v)_W - \frac{1}{n}(u_n, v)_W = \big(g(x, u_n), v\big)_W$$

for all $v \in \mathring{W}$. We shall prove that $\|u_n\|_{\tilde{\mathring{W}}} \leqslant$ const. Suppose to the contrary that $\lim\limits_{n\to\infty} \|u_n\|_{\tilde{\mathring{W}}} = +\infty$.

Let

$$s_n = \frac{u_n}{\|u_n\|_{\tilde{\mathring{W}}}}, \qquad \|s_n\|_{\tilde{\mathring{W}}} = 1.$$

Then

$$(3.5) \qquad \langle s_n, v\rangle_{\mathring{W}} - \omega^2 (s_n, v)_W - \frac{1}{n}(s_n, v)_W = \frac{1}{\|u_n\|_{\tilde{\mathring{W}}}}\big(g(x, u_n), v\big)_W.$$

Since $\mathring{W}$ is a reflexive space, the unit ball is weakly compact and $s_n \rightharpoonup s$ in $\mathring{W}$.

The left hand side of Eq. (3.5) is a linear continuous functional on $\mathring{W}$. Therefore by assumptions on g we have from Eq. (3.5)

$$(3.6) \qquad \langle s, v\rangle_{\mathring{W}} - \omega^2 (s, v)_W = \lim_{n\to\infty} [\langle s_n, v\rangle_{\mathring{W}} - \omega^2 (s_n, v)_W] = 0$$

for all $v \in \mathring{W}$.

Hence $s = c(\alpha_1 w^{(1)} + \ldots + \alpha_p w^{(p)})$, where $\alpha_1^2 + \ldots + \alpha_p^2 = 1$ and $c \geqslant 0$. By (3.5) and (3.6) we have

$$(3.7) \qquad \langle s_n - s, s_n - s\rangle_{\mathring{W}} - \omega^2 (s_n - s, s_n - s)_W = \langle s_n, s_n - s\rangle_{\mathring{W}} - \omega^2 (s_n, s_n - s)_W$$
$$= \frac{1}{n}(s_n, s_n - s)_W + \frac{1}{\|u_n\|_{\tilde{\mathring{W}}}}\big(g(x, u_n), s_n - s\big)_W \to 0$$

as $n \to \infty$.

Further, by the inequality (2.4) of [3]

$$\langle s_n - s, s_n - s\rangle_{\mathring{W}} \geqslant c_2^2 \|s_n - s\|^2_{\mathring{W}}.$$

So we have

$$(3.8) \qquad c_2^2 \|s_n - s\|^2_{\mathring{W}} \leqslant \omega^2 (s_n - s, s_n - s)_W + [\langle s_n - s, s_n - s\rangle_{\mathring{W}} - \omega^2 (s_n - s, s_n - s)_W].$$

Using Eq. (3.7) and Rellich's Theorem ([1] p. 30), the last relation yields $s_n \to s$ in $\mathring{W}$. Furthermore, by $\|s_n\|_{\mathring{W}} = 1$ we have $c > 0$. The relation (3.6) implies

$$(3.9) \qquad \langle s, s_n\rangle_{\mathring{W}} - \omega^2 (s, s_n)_W = \langle s_n, s\rangle_{\mathring{W}} - \omega^2 (s_n, s)_W = 0 \qquad \text{for all} \quad n.$$

Moreover, $(s_n, s)_W \to \|s\|^2_W > 0$. Setting $v = s$ in Eq. (3.5) and applying Eq. (3.9), we obtain

$$-\frac{1}{n}(s_n, s)_W = \frac{1}{\|u_n\|_{\tilde{\mathring{W}}}}\big(g(x, u_n), s\big)_W$$

i.e. for sufficiently large n

$$(3.10) \qquad \int_D \sum_{i=1}^{3} g_i(x, u_n) s_i \, dx < 0.$$

By the definition of s_n: $u_n = ||u_n||_{\tilde{\mathring{W}}} s_n$, i.e. $(u_n)_i = ||u_n||_{\tilde{\mathring{W}}}(s_n)_i$. Moreover, $s_n \to s$ in W, i.e. $(s_n)_i \to s_i$ in $L^2(D)$ and, accordingly, $(s_{n_k})_i \to s_i$ a.e. in D for some subsequence. By virtue of the Lebesgue Convergence Theorem applied to Eq. (3.10), we have from

$$\limsup_{n\to\infty} \int_D \sum_{i=1}^{3} g_i(x, u_n) s_i \, dx \leqslant 0,$$

the following inequality:

$$\sum_{i=1}^{3} \Big[\int_{D^+_{i,(\alpha_1,\ldots,\alpha_p)}} \lim_{n\to\infty} g_i(x, ||u_n||_{\tilde{\mathring{W}}} s_n) s_i \, dx + \int_{D^-_{i,(\alpha_1,\ldots,\alpha_p)}} \lim_{n\to\infty} g_i(x, ||u_n||_{\tilde{\mathring{W}}} s_n) s_i \, dx \Big] \leqslant 0.$$

Hence

$$\sum_{i=1}^{3} \Big[\int_{D^+_{i,(\alpha_1,\ldots,\alpha_p)}} H_i(x) c |z_i| \, dx - \int_{D^-_{i,(\alpha_1,\ldots,\alpha_p)}} h_i(x) c |z_i| \, dx \Big] \leqslant 0, \tag{3.11}$$

where

$$z_i = \frac{s_i}{c} = \alpha_1 w_i^{(1)} + \ldots + \alpha_p w_i^{(p)}.$$

But $c > 0$ and the latter contradicts the assumption (3.1).

Thus $||u_n||_{\tilde{\mathring{W}}} \leqslant$ const for all n and $u_n \rightharpoonup u$ in $\mathring{W}$ as $n \to \infty$. Passing to the limit in Eq. (3.4) as $n \to \infty$ we obtain

$$\langle u, v\rangle_{\mathring{W}} - \omega^2 (u, v)_W = \big(g(x, u), v\big)_W \tag{3.12}$$

for every $v \in \mathring{W}$. This means that u is a weak solution to the problem (1.1), (1.2).

References

1. S. AGMON, *Lectures on elliptic boundary value problems*, D. Van Nostrand Company Inc., Princeton 1965.
2. H. BREZIS, L. NIRENBERG, *Characterizations of the ranges of some nonlinear operators and applications to boundary value problems*, Annali della Scuola Norm. Sup. di Pisa, 5, 225–326, 1978.
3. J. CHMAJ, M. MAJCHROWSKI, *On generalized solutions of a nonlinear boundary value problem of elasticity. Nonresonance case*, Arch. Mech., **35**, 5–6, 1983.
4. P. HESS, *On a theorem by Landesman and Lazer*, Ind. Univ. Math. J., **23**, 827–829, 1974.
5. E. M. LANDESMAN, A. C. LAZER, *Nonlinear perturbations of linear elliptic boundary value problems at resonance*, J. Mathematics and Mechanics, **19**, 7, 609–623, 1970.
6. J. NEČAS, I. HLAVÁČEK, *Mathematical theory of elastic and elastico-plastic bodies: an introduction*, Elsevier Scientific Publ. Comp., Amsterdam 1981.
7. S. A. WILLIAMS, *A sharp sufficient condition for solution of a nonlinear elliptic boundary value problem*, J. Differential Equations, **8**, 580–586, 1970.

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