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Vibration of a bridge beam due to highway traffic

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DYNAMIC response of a beam to the passage of a train of concentrated forces with random amplitudes is considered. Arrivals of forces at the beam are assumed to constitute a random (Poisson or, more general, correlated) process of events. Thus the excitation process idealizes the vehicular traffic load on a bridge. Based upon the introduction of two influence functions, the analytical technique is developed to determine the response of the beam. The explicit expressions for the expected value and the variance of the beam deflection are provided. As an example, the response of the beam to the stationary Poisson stream of forces is determined and discussed for some practical situations. The extension of the presented approach to the case of multi-axle vehicles is also outlined.

Rozważane są drgania belki pod wpływem przejazdu serii sił skupionych o losowych amplitudach. Przyjęto, że wjazdy sił na belkę stanowią losowy (poissonowski lub ogólniejszy — skorelowany) proces zdarzeń. Przyjęty proces wymuszenia jest zatem modelem obciążenia mostu ruchem drogowym. Reakcję belki wyznaczono w sposób analityczny, posługując się dwiema wprowadzonymi w pracy funkcjami wpływu. Wyprowadzono wzory na wartość oczekiwaną i wariancję ugięcia belki. Jako przykład zanalizowano, dla różnych praktycznych sytuacji, drgania belki pod działaniem stacjonarnego poissonowskiego strumienia sił. Omówiono także sposób rozszerzenia przedstawionej metody rozwiązania na przypadek pojazdów wieloosiowych.

Рассматриваются колебания балки под влиянием переезда серий сосредоточенных сил со случайными амплитудами. Принято, что въезды сил на балку составляют случайный (пуассоновский или более общий-коррелированный) процесс событий. Итак принятый процесс вынуждения является моделью нагружения моста дорожным движением. Реакция балки определена аналитическим образом, послуживаясь двумя, введенными в работе, функциями влияния. Выведены формулы для математического ожидания и варианции прогиба балки. Как пример анализируются, для разных практических ситуаций, колебания балки под действием стационарного пуассоновского потока сил. Обсужден также способ расширения представленного метода решения на случай многоосных транспортных средств.

1. Introduction

VIBRATIONS of bridge structures produced by travelling loads constitute one of the most mportant problems of structural dynamics. The problem has been investigated for many years and ample literature is listed, for example, in the book by FRYBA [1].

In most papers moving loads are treated as deterministic processes. Since, however, moving forces acting on a highway bridge (wheel pressure) have random magnitudes and appear at random instants, the traffic loading of a bridge should be treated as a stochastic process. Such an approach to the problem has been applied by few authors only. FRÝBA [1] considered the vibrations of a beam provoked by a single travelling concentrated force stochastically variable in time. KNOWLES [3] tackled the problem of vibrations of an infinitely long beam subject to a travelling concentrated force, its position on the beam being described by a strictly stationary process, the Gaussian stationary process and the

Wiener process. Vibrations of a beam under the action of travelling continuous loads were investigated by ROBSON [4], BOLOTIN [5] and FRÝBA [1].

A fundamental contribution to the problem of vibration of bridge beams due to random travelling loads was made by TUNG [6, 7, 8]. The author assumes each vehicle to be represented by a single concentrated force, the vehicles move at equal and constant velocities, have the same weight and appear on the bridge at random instants. The traffic is thus modelled in this case by a random stream of concentrated forces of equal amplitudes. Application of numerical techniques makes it possible to determine the one-dimensional probability density function of response of the system and the expected rate of threshold crossings. However, in the papers mentioned above, only random streams of independent arrivals are considered.

The present paper deals with beam vibrations due to the passage of a random series of concentrated forces having random amplitudes. Variability of the forces in time may be described by a certain nonrandom function. The time intervals between the instants of arrivals of the individual forces are treated as random variables (generally correlated). The series of forces constitutes a model of a random stream of vehicles of random weights which move at constant velocities; it represents a simplified model of loading of a bridge by traffic, the effects of inertia of the vertical motion of the vehicles being disregarded. The beam response is determined analytically. The formulae for the expected value and variance of the beam deflection are given; the possibility of extending the solution to the case of multi-axle vehicles is also discussed. The results presented may be useful in the analysis of dynamics of highway bridges and in estimating the reliability of civil engineering structures.

2. Formulation and general solution of the problem

Let us consider damped vibrations of a beam of length l produced by a series of forces moving in the same direction at a constant velocity v (Fig. 1). Assume that the forces arrive at the beam at random instants t_k and form the stream of force arrivals; the stream is assumed to be inhomogeneous and of intensity $\lambda(t)$.

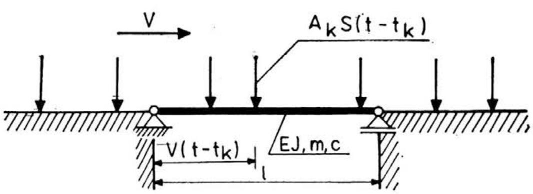

FIG. 1.

Let $N(t_i, t_j)$ and $dN(t)$ denote the number of force arrivals in the time intervals (t_i, t_j) and $(t, t+dt)$, respectively, and the symbols $P\{\}$ and $E[\,]$ denote the probability and the expected value of the magnitude in brackets.

Let us assume the probability of appearing of a force in an infinitesimal time interval

dt to be proportional to dt, and the probability of appearing of a larger number of forces to be negligibly small as a higher order vanishing value, so that

$$P\{dN(t) = 1\} = \lambda(t)dt + o(dt), \tag{2.1}$$

$$P\{dN(t) > 1\} = o(dt), \tag{2.2}$$

$$P\{dN(t) = 0\} = 1 - \lambda(t)dt + o(dt), \tag{2.3}$$

whence the relations follow:

$$E[dN(t)] = \lambda(t)dt + o(dt), \tag{2.4}$$

$$E[dN^2(t)] = \lambda(t)dt + o(dt). \tag{2.5}$$

Correlation between two arrival instants t_1 and t_2 is described by the product density function of second degree $\varphi(t_1, t_2)$ (cf. SRINIVASAN [9]) defined as

$$E[dN(t_1)dN(t_2)] = \varphi(t_1, t_2)dt_1 dt_2 \tag{2.6}$$

or, equivalently, by the second-order correlation function

$$f(t_1, t_2) = \varphi(t_1, t_2) - \lambda(t_1)\lambda(t_2). \tag{2.7}$$

In the case of independent arrivals (uncorrelated), $f(t_1, t_2) = 0$, the stream is a nonhomogeneous Poisson stream and the probability of appearing of n forces within the time interval (t_i, t_j) is given by the formula

$$P\{N(t_i, t_j) = n\} = \frac{\Lambda^n(t_i, t_j)}{n!} e^{-\Lambda(t_i, t_j)}, \tag{2.8}$$

where

$$\Lambda(t_i, t_j) = \int_{t_i}^{t_j} \lambda(\tau)d\tau.$$

In such a case the average number of forces acting on the beam at an arbitrary time $t > l/v$ equals

$$E\left[N\left(t - \frac{l}{v}, t\right)\right] = \int_{t-l/v}^{t} \lambda(\tau)d\tau. \tag{2.9}$$

In the case of the stationary Poisson process of arrivals, i.e. when $\lambda(t) = \lambda = \text{const}$, we obtain

$$E\left[N\left(t - \frac{l}{v}, t\right)\right] = \lambda l/v. \tag{2.10}$$

It is evident that the mean number of forces acting on the beam is in this case inversely proportional to their velocity.

Let the damped vibration of a beam with bending rigidity EI, mass density m and damping coefficient c be described by the equation

$$EIw^{IV}(x, t) + c\dot{w}(x, t) + m\ddot{w}(x, t) = \sum_{k \in N\left(t - \frac{l}{v}, t\right)} A_k S(t - t_k)\delta[x - v(t - t_k)] \tag{2.11}$$

and by the corresponding initial conditions at instant $t-l/v$; here δ is Dirac's delta, $w(x, t)$ — deflection of the beam, $(\dot{\ }) = \partial/\partial t$, $(\ )^{IV} = \partial^4/\partial x^4$. Amplitudes A_k are independent random variables of identical probabilistic characteristics independent of the times of arrivals t_k; assume the values $E[A_k] = E[A] = \text{const}$ and $E[A_k^2] = E[A^2] = \text{const}$ to be known. The deterministic function $S(t-t_k)$ describes the time-dependence of the force.

The state of displacement of the beam acted on by a series of travelling forces is the sum of the displacements produced by individual forces. At an arbitrary instant $t > l/v$ the beam performs the vibrations provoked by the forces which are actually present on the beam (their times of arrivals $t_k \in (t-l/v, t)$, and free vibrations provoked by forces which have already left the beam (i.e. $t_k \in (0, t-l/v)$. It is advantageous to introduce two influence functions $H_1(x, t-t_k)$ and $H_2(x, t-t_k-l/v)$, the first of which represents the beam deflection at time t produced by force $S(t-t_k)$ present on the beam $\big(t_k \in (t-l/v, t)\big)$, and the second one describes the free vibration due to the force which has already left the beam $(t_k \in (0, t-l/v))$. The functions satisfy the equations

$$EIH_1^{IV}(x, t-t_k)+c\dot{H}_1(x, t-t_k)+m\ddot{H}_1(x, t-t_k) = S(t-t_k)\,\delta[x-v(t-t_k)], \tag{2.12}$$

$$EIH_2^{IV}\left(x, t-t_k-\frac{l}{v}\right)+c\dot{H}_2\left(x, t-t_k-\frac{l}{v}\right)+m\ddot{H}_2\left(x, t-t_k-\frac{l}{v}\right) = 0, \tag{2.13}$$

together with the corresponding boundary and initial conditions (for $t = t_k$ and $t = t_k+l/v$)

$$H_1(x, 0) = 0, \quad \dot{H}_1(x, 0) = 0, \tag{2.14}$$

$$H_2(x, 0) = H_1\left(x, \frac{l}{v}\right), \quad \dot{H}_2(x, 0) = \dot{H}_1\left(x, \frac{l}{v}\right). \tag{2.15}$$

Deflection of the beam $w(x, t)$ at an arbitrary time $t > l/v$ is the sum of two effects, i.e. of forced and free vibrations,

$$w(x, t) = w_1(x, t)+w_2(x, t) = \sum_{k\in N\left(t-\frac{l}{v}, t\right)} A_k H_1(x, t-t_k) + \sum_{k\in N\left(0, t-\frac{l}{v}\right)} A_k H_2\left(x, t-t_k-\frac{l}{v}\right). \tag{2.16}$$

This formula may also be written in an integral form (Stieltjes integral of the $N(t)$ process) as

$$w(x, t) = w_1(x, t)+w_2(x, t) = \int_{t-l/v}^{t} A(\tau)H_1(x, t-\tau)dN(\tau)+\int_0^{t-l/v} A(\tau) \times H_2\left(x, t-\tau-\frac{l}{v}\right)dN(\tau). \tag{2.17}$$

Equation (2.17) will be used to determine the expected value and the variance of the deflection of the beam.

Performing the expected value operation on the formula (2.17) and taking into account Eq. (2.4), we obtain

$$E[w(x,t)] = E[w_1]+E[w_2] = E[A]\int_{t-l/v}^{t} H_1(x,t-\tau)\lambda(\tau)d\tau + E[A]\int_{0}^{t-l/v} H_2\left((x,t-\tau-\frac{l}{v}\right)\lambda(\tau)d\tau. \tag{2.18}$$

The formula for the variance of the deflection

$$\sigma_w^2(x,t) = E[w^2(x,t)]-E^2[w(x,t)] \tag{2.19}$$

is derived on the basis of Eq. (2.17) and Eqs. (2.5)–(2.7) to yield

$$\begin{aligned}\sigma_w^2(x,t) = {} & \mathrm{E}[A^2]\int_{t-l/v}^{t} H_1^2(x,t-\tau)\lambda(\tau)d\tau + E[A^2]\int_{0}^{t-l/v} H_2^2(x,t-\tau-l/v)\lambda(\tau)d\tau \\ & + E^2[A]\int_{t-l/v}^{t}\int_{t-l/v}^{t} H_1(x,t-\tau_1)H_1(x,t-\tau_2)f(\tau_1,\tau_2)d\tau_1 d\tau_2 \\ & + 2E^2[A]\int_{t-l/v}^{t}\int_{0}^{t-l/v} H_1(x,t-\tau_1)H_2(x,t-\tau_2-l/v)f(\tau_1,\tau_2)d\tau_1 d\tau_2 \\ & + E^2[A]\int_{0}^{t-l/v}\int_{0}^{t-l/v} H_2(x,t-\tau_1-l/v)H_2(x,t-\tau_2-l/v)f(\tau_1,\tau_2)d\tau_1 d\tau_2.\end{aligned} \tag{2.20}$$

In the particular case of an uncorrelated stream of arrivals (i.e. $f(\tau_1,\tau_2) = 0$), Eq. (2.20) is reduced to the form

$$\sigma_w^2(x,t) = \sigma_{w_1}^2+\sigma_{w_2}^2 = E[A^2]\int_{t-l/v}^{t} H_1^2(x,t-\tau)\lambda(\tau)d\tau + E[A^2]\int_{0}^{t-l/v} H_2^2(x,t-\tau-l/v)\lambda(\tau)d\tau. \tag{2.21}$$

Let us expand the influence functions $H_1(x,t-t_k)$ and $H_2(x,t-t_k-l/v)$ into the series of eigenfunctions $W_n(x)$

$$H_1(x,t-t_k) = \sum_{n=1}^{\infty} q_n(t-t_k)W_n(x), \tag{2.22}$$

$$H_2(x,t-t_k-l/v) = \sum_{n=1}^{\infty} f_n(t-t_k-l/v)W_n(x). \tag{2.23}$$

Substitution of expresions (2.22) and (2.23) into the respective formulae (2.12) and (2.13) and application of the orthogonality properties of eigenfunctions yields

$$\ddot{q}_n(t-t_k)+2\alpha\dot{q}_n(t-t_k)+\omega_n^2 q_n(t-t_k) = \frac{1}{\gamma_n^2}S(t-t_k)W_n[v(t-t_k)], \tag{2.24}$$

$$\ddot{f}_n(t-t_k-l/v)+2\alpha\dot{f}_n(t-t_k-l/v)+\omega_n^2 f_n(t-t_k-l/v) = 0, \tag{2.25}$$

where $2\alpha = c/m$, $\gamma_n^2 = \int_0^l mW_n^2(x)dx$, and ω_n is the free vibration frequency. From Eqs. (2.24) and (2.25) we obtain

$$q_n(t-t_k) = \frac{1}{\gamma_n^2} \int_{t_k}^{t} h_n(t-\tau) S(\tau-t_k) W_n[v(\tau-t_k)] d\tau. \tag{2.26}$$

$$f_n(t-t_k-l/v) = \frac{1}{\gamma_n^2} \int_{t_k}^{t_k+l/v} h_n(t-\tau) S(\tau-t_k) W_n[v(\tau-t_k)] d\tau, \tag{2.27}$$

Here $h_n(t-\tau)$ denotes the impulse response function

$$h_n(t-\tau) = \Omega_n^{-1} e^{-\alpha(t-\tau)} \sin\Omega_n(t-\tau), \tag{2.28}$$

where $\Omega_n^2 = \omega_n^2 - \alpha^2$.

Using the eigenfunction expansions (2.22) and (2.23), the formulae (2.18) and (2.20) are written in the form

$$E[w(x,t)] = E[w_1]+E[w_2] = E[A] \sum_{n=1}^{\infty} [\tilde{q}_n(t)+\tilde{f}_n(t)] W_n(x), \tag{2.29}$$

where

$$\begin{aligned} \tilde{q}_n(t) &= \int_{t-l/v}^{t} q_n(t-\tau)\lambda(\tau)d\tau, \\ \tilde{f}_n(t) &= \int_{0}^{t-l/v} f_n(t-\tau-l/v)\lambda(\tau)d\tau \end{aligned} \tag{2.30}$$

and

$$\sigma_w^2(x,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} [\mathrm{cov}_{q_iq_j}(t)+\mathrm{cov}_{f_if_j}(t)+\mathrm{cov}_{q_if_j}(t)] W_i(x) W_j(x). \tag{2.31}$$

Here

$$\begin{aligned} \mathrm{cov}_{q_iq_j}(t) &= E[A^2] \int_{t-l/v}^{t} q_i(t-\tau)q_j(t-\tau)\lambda(\tau)d\tau \\ &\quad + E^2[A] \int_{t-l/v}^{t}\int_{t-l/v}^{t} q_i(t-\tau_1)q_j(t-\tau_2)f(\tau_1,\tau_2)d\tau_1 d\tau_2, \\ \mathrm{cov}_{f_if_j}(t) &= E[A^2] \int_0^{t-l/v} f_i(t-\tau-l/v)f_j(t-\tau-l/v)\lambda(\tau)d\tau \\ &\quad + E^2[A] \int_0^{t-l/v}\int_0^{t-l/v} f_i(t-\tau_1-l/v)f_j(t-\tau_2-l/v)f(\tau_1,\tau_2)d\tau_1 d\tau_2, \\ \mathrm{cov}_{q_if_j}(t) &= E^2[A] \int_{t-l/v}^{t}\int_0^{t-l/v} q_i(t-\tau_1)f_j(t-\tau_2-l/v)f(\tau_1,\tau_2)d\tau_1 d\tau_2. \end{aligned} \tag{2.32}$$

When the stream of forces is uncorrelated ($f(t_1, t_2) = 0$), Eq. (2.31) is reduced to the form

$$(2.33)\qquad \sigma_w^2(x, t) = \sigma_{w_1}^2 + \sigma_{w_2}^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathrm{cov}_{q_i q_j}(t)\, W_i(x)\, W_j(x) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathrm{cov}_{f_i f_j}(t)\, W_i(x)\, W_j(x).$$

Here the expressions $\mathrm{cov}_{q_i q_j}(t)$ and $\mathrm{cov}_{f_i f_j}(t)$ are expressed solely in terms of single integrals given in Eq. (2.32).

3. Simply supported beam under stationary Poisson force stream

On the basis of the general solution given in Sect. 2, let us determine the probabilistic characteristics of vibrations of a simply supported beam subject to a series of constant forces $(S(t-t_k) = 1)$ under the assumption that the stream of arrivals is uncorrelated and stationary, i.e. $f(t_1, t_2) = 0$ and $\lambda(t) = \lambda = \mathrm{const}$. Hence $W_n(x) = \sin n\pi x/l$, and the generalized (normal) coordinates $q_n(t-t_k)$ and $f_n(t_k - t - l/v)$ are derived from the formulae (2.26) and (2.27),

$$(3.1)\qquad q_n(t-t_k) = \frac{2}{mlM_n}\left[a_{1n}\sin\beta_n(t-t_k) + a_{2n}\cos\beta_n(t-t_k) + a_{3n}e^{-\alpha(t-t_k)}\sin\Omega_n(t-t_k) - a_{2n}e^{-\alpha(t-t_k)}\cos\Omega_n(t-t_k)\right],$$

$$(3.2)\qquad f_n\left(t-t_k-\frac{l}{v}\right) = \frac{2}{mlM_n}e^{-\alpha\left(t-t_k-\frac{l}{v}\right)}\left[b_{1n}\sin\Omega_n\left(t-t_k-\frac{l}{v}\right) + b_{2n}\cos\Omega_n\left(t-t_k-\frac{l}{v}\right)\right].$$

Here

$$\omega_n = \left(\frac{n\pi}{l}\right)^2\sqrt{\frac{EI}{m}},$$

$$\beta_n = \frac{n\pi v}{l},$$

$$M_n = (\omega_n^2 - \beta_n^2)^2 + 4\alpha^2\beta_n^2,$$

$$(3.3)\qquad a_{1n} = \omega_n^2 - \beta_n^2, \qquad a_{2n} = -2\alpha\beta_n,$$

$$a_{3n} = \frac{\beta_n}{\Omega_n}[2\alpha^2 - (\omega_n^2 - \beta_n^2)],$$

$$b_{1n} = \frac{(-1)^n}{\Omega_n}(a_1\beta_n + a_{2n}\alpha) + a_{3n}e^{-\alpha\frac{l}{v}}\cos\Omega_n\frac{l}{v} + a_{2n}e^{-\alpha\frac{l}{v}}\sin\Omega_n\frac{l}{v},$$

$$b_{2n} = (-1)^n a_{2n} + a_{3n}e^{-\alpha\frac{l}{v}}\sin\Omega_n\frac{l}{v}\, a_{2n}e^{-\alpha\frac{l}{v}}\cos\Omega_n\frac{l}{v}.$$

Substitution of Eqs. (3.1) and (3.2) into Eq. (2.30) yields

$$(3.4)\qquad E[w_1(x,t)] = \frac{2\lambda E[A]}{ml}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi x}{l}}{M_n}\left\{\frac{a_{1n}}{\beta_n}[1-(-1)^n]+\frac{a_{3n}}{\omega_n^2}\left[\Omega_n-(\alpha\sin\Omega_n\frac{l}{v}+\Omega_n\cos\Omega_n\frac{l}{v}e^{-\alpha\frac{l}{v}}\right]-\frac{a_{2n}}{\omega_n^2}\left[\alpha+\left(\Omega_n\sin\Omega_n\frac{l}{v}-\alpha\cos\Omega_n\frac{l}{v}\right)e^{-\alpha\frac{l}{v}}\right]\right\},$$

$$(3.5)\qquad E[w_2(x,t)] = \frac{2\lambda E[A]}{ml}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi x}{l}}{M_n}[b_{1n}D_1(\alpha,\Omega_n)+b_{2n}D_2(\alpha,\Omega_n)].$$

Here D_1 and D_2 are integrals given in the Appendix.

The mutual covariance functions of generalized coordinates appearing in Eqs. (2.32) assume the form

$$(3.6)\qquad \mathrm{cov}_{q_nq_n}(t) = \sigma_{q_n}^2(t) = \frac{4\lambda}{(ml)^2M_n^2}\left\{\frac{1}{2}\frac{l}{v}M_n+a_{3n}^2I_3(2\alpha,\Omega_n)+a_{2n}^2I_4(2\alpha,\Omega_n)-a_{2n}a_{3n}I_1(\alpha,2\Omega_n)+2a_{1n}a_{3n}I_5(\alpha,\beta_n,\Omega_n)-2a_{1n}a_{2n}I_6(\alpha,\beta_n,\Omega_n)+2a_{2n}a_{3n}I_6(\alpha,\beta_n,\Omega_n)-2a_{2n}^2I_7(\alpha,\beta_n,\Omega_n)\right\},$$

$$(3.7)\qquad \mathrm{cov}_{q_iq_j} = \frac{4\lambda}{(ml)^2M_iM_j}\{a_{1i}a_{3j}I_5(\alpha,\beta_i,\Omega_j)-a_{1i}a_{2j}I_6(\alpha,\beta_i,\Omega_j)+a_{2i}a_{3j}I_6(\alpha,\Omega_j,\beta_i)-a_{2i}a_{2j}I_7(\alpha,\beta_i,\Omega_j)+a_{3i}a_{1j}I_5(\alpha,\beta_i,\beta_j)+a_{3i}a_{2j}I_6(\alpha,\Omega_i,\beta_j)+a_{3i}a_{3j}I_5(2\alpha,\Omega_i,\Omega_j)-a_{3i}a_{2j}I_6(2\alpha,\Omega_i,\Omega_j)-a_{2i}a_{1j}I_6(\alpha,\beta_j,\Omega_i)-a_{2i}a_{2j}I_7(\alpha,\Omega_i,\beta_j)-a_{2i}a_{3j}I_6(2\alpha,\Omega_j,\Omega_i)+a_{2i}a_{2j}I_7(2\alpha,\Omega_i,\Omega_j)\},$$

$$(3.8)\qquad \mathrm{cov}_{f_nf_n}(t) = \sigma_{f_n}^2(t) = \frac{4\lambda}{(ml)^2M_n^2}\{b_{1n}^2D_3(2\alpha,\Omega_n)+2b_{1n}b_{2n}D_4(2\alpha,\Omega_n)+b_{2n}^2D_5(2\alpha,\Omega_n)\},$$

$$(3.9)\qquad \mathrm{cov}_{f_if_j}(t) = \frac{4\lambda}{(ml)^2M_iM_j}\{b_{1i}b_{1j}D_6(2\alpha,\Omega_i,\Omega_j)+b_{1i}b_{2j}D_7(2\alpha,\Omega_i,\Omega_j)+b_{1j}b_{2i}D_7(2\alpha,\Omega_j,\Omega_i)+b_{2i}b_{2j}D_8(2\alpha,\Omega_i,\Omega_j)\},$$

$$(3.10)\qquad \mathrm{cov}_{q_if_j}(t) = 0,$$

where D_k and I_k are expressed in terms of the integrals given in the Appendix.

The solutions derived indicate that the expected value and the variance of the deflection $w_1(x,t)$ are time-independent. As far as the function $w_2(x,t)$ is concerned, it proves to be sufficient to know its asymptotic behaviour for $t\to\infty$, what takes place in the case when the effect of the transient process (initial perturbation) becomes negligible at the time of observation. From Eqs. (3.5), (3.8) and (3.9) for $t\to\infty$, it follows that

$$(3.11)\qquad E[w_2(x,\infty)] = \frac{2\lambda E[A]}{ml}\sum_{n=1}^{\infty}\frac{\Omega_nb_{1n}+\alpha b_{2n}}{M_n\omega_n^2}\sin\frac{n\pi x}{l},$$

$$\sigma^2_{f_n}(\infty) = \frac{\lambda}{(ml)^2 M_2^2 \omega_n^2}\,[b_{1n}^2\Omega_n^2 + 2b_{1n}b_{2n}\alpha\Omega_n + b_{2n}^2(2\alpha^2+\Omega_n^2)], \tag{3.12}$$

$$\begin{aligned}\mathrm{cov}_{f_i f_j}(\infty) = \frac{4\lambda}{(ml)^2 M_i M_j}\Bigg\{ & b_{1i}b_{1j}\frac{\alpha\Omega_i\Omega_j}{[4\alpha^2+(\Omega_i-\Omega_j)^2][4\alpha^2+(\Omega_i+\Omega_j)^2]} \\ & + \frac{1}{2}b_{1i}b_{2j}\left[\frac{\Omega_i-\Omega_j}{4\alpha^2+(\Omega_i-\Omega_j)^2} - \frac{\Omega_i+\Omega_j}{4\alpha^2+(\Omega_i+\Omega_j)^2}\right] \\ & + \frac{1}{2}b_{2i}b_{1j}\left[\frac{\Omega_j-\Omega_i}{4\alpha^2+(\Omega_i-\Omega_j)^2} - \frac{\Omega_i-+\Omega_j}{4\alpha^2+(\Omega_i+\Omega_j)^2}\right] \\ & + b_{2i}b_{2j}\left[\frac{\alpha}{4\alpha^2+(\Omega_i-\Omega_j)^2} + \frac{\alpha}{4\alpha^2+(\Omega_i+\Omega_j)^2}\right]\Bigg\}. \end{aligned} \tag{3.13}$$

In order to perform the quantitative analysis, a numerical example has been prepared by assuming that $\lambda = 0.5\ \mathrm{s}^{-1}$, what corresponds to the traffic capacity of 1800 vehicles/hour, $\alpha = 0.01\omega_1$, $\omega_1 = 10\ \mathrm{s}^{-1}$. The solution is found for the first eigenfunction $W_1(x) = \sin \pi x/l$. The dimensionless magnitudes are evaluated: $E[w]ml\omega_1^2/E[A]$, $\sigma_w^2 m^2 l^2 \omega_1^4/E[A^2]$ and $\sigma_w E[A]/E[w]\sqrt{E[A^2]}$ which correspond to the expected value $E[w]$, variance σ_w^2 and the variability coefficient $\sigma_w/E[w]$ of deflection of the beam. The results are shown in Fig. 2,3 and 4 as functions of the travel velocity. The dashed line denotes the solution for w_1, and the thin solid line — for $w = w_1 + w_2$. It is easily seen (Fig. 2) that the average deflection decreases with increasing force travel velocity in spite of the fact that, as it is known, response of the beam to a single passage of a travelling force increases with increasing travel velocity. This is only an apparent contradiction since, in the case of a random series of forces, the average number of forces acting on the beam (mean load) decreases with the velocity, see Eq. (2.10).

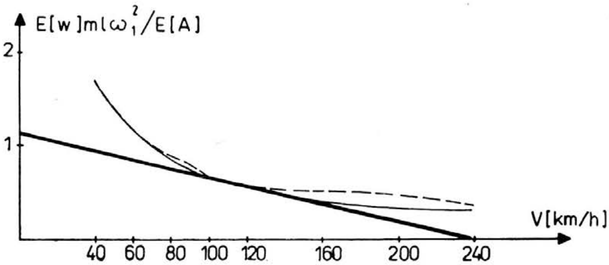

FIG. 2.

The effect of free vibrations, that is the difference $E[w_2] = E[w_1+w_2] - E[w_1]$, is negligibly small for velocities less than 130 km/h. $E[w_2]$ assumes negative values, what follows form the fact that at the instant when the force leaves the beam, the process of exponentially decaying free vibrations begins (starting from the negative value of deflection). The variance of the function w_1 (Fig. 3) decreases with the velocity. Contribution of the component w_2 in the total deflection variance cannot be disregarded, in particular when the velocity exceeds 120 km/h. Both the variance and the variability coefficient (Fig. 4) increase rapidly starting with the velocity 120 km/h.

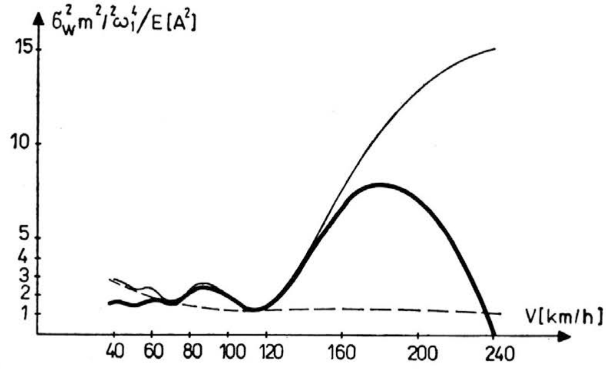

FIG. 3.

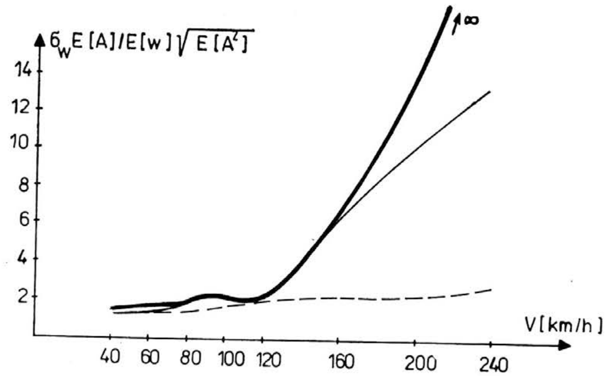

FIG. 4.

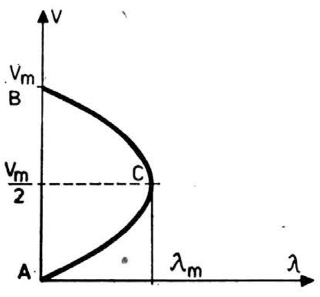

FIG. 5.

The assumption that the average arrival rate of forces λ is independent of the travel velocity is not accurate if the moving forces are asumed to model the traffic loads. In traffic engineering [10] it is known that the mean velocity of travel depends on the mean arrival rate. This relation is illustrated by Fig. 5.

The dimensionless magnitudes corresponding to $E[w]$, σ_w^2 and $\sigma_w/E[w]$ are found by means of the relation

$$\lambda(v) = 4\lambda_m \frac{v}{v_m}\left(1-\frac{v}{v_m}\right), \tag{3.14}$$

λ_m being the maximum mean arrival rate, and v_m — the maximum velocity; they are shown by heavy solid lines in Figs. 2, 3 and 4. It has been assumed that $\lambda_m = 0.5\ \text{s}^{-1}$ and $v_m = 240$ km/h.

In Figs. 6.7 and 8 the dimensionless quantities $E[w]$, σ_w^2 and $\sigma_w/E[w]$ are plotted against the mean arrival rate λ; dotted lines refer to the solution for w_1, solid lines — to the solution for $w = w_1 + w_2$. In this case the analysis is confined to the range of velocities represented by the portion BC of the diagram in Fig. 5, and so the velocity is expressed by the formula

$$v(\lambda) = \frac{1}{2} v_m \left(1 + \sqrt{1 - \frac{\lambda}{\lambda_m}}\right). \tag{3.15}$$

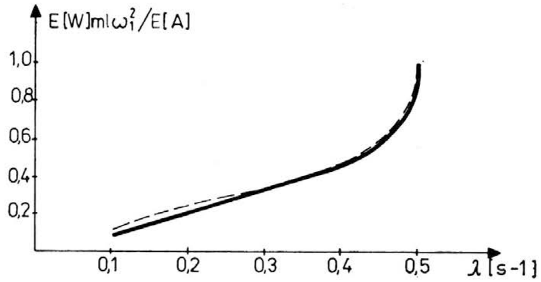

FIG. 6.

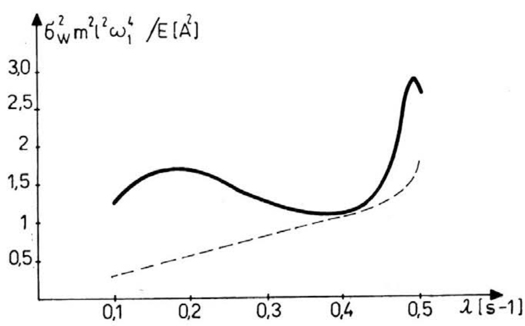

FIG. 7.

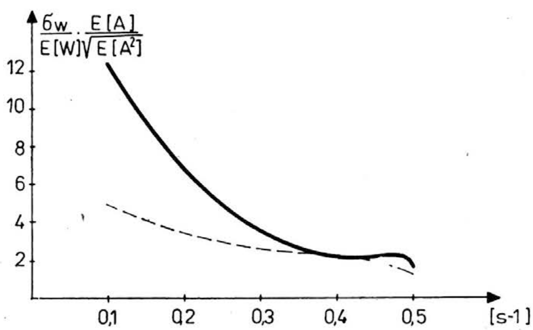

FIG. 8.

Here the values $v_m = 160$ km/h and $\lambda_m = 0.5$ s^{-1} have been assumed. The observation is confirmed that the contribution of w_2 to the expected value is small, contrary to the case of the variance and response variability coefficient where the contribution is considerable.

4. Discussion

The solutions and results presented here may be utilized in the analysis of vibrations of bridges induced by traffic. It follows from the investigations on the traffic [11, 12] that the Poisson process describes fairly well the motion of vehicles in the range of small and moderate arrival rate and traffic density. In such a case the relation between the mean velocity and arrival rate is illustrated by the curve BC in Fig. 5. In the case when the traffic arrival rate or the density are small (curve AC), the stream of vehicles should be treated as correlated. In [13] the effect of correlation of the excitation process on the system response is analysed in the case of loading by a random series of impulses. It may be expected that the effect of correlation for a moving series of forces has a similar character. Assumption of nonstationarity in the arrival process makes it possible to account for the periodic changes in the traffic density occuring during the day and night or following from the light signalling.

The solution obtained under the asumption that each vehicle is represented by a single concentrated force may be extended to the case of multi-axle vehicles represented by several forces. The influence functions H_1 and H_2 must then be modified. Let us illustrate this on the example of a biaxle vehicle represented by two concentrated forces of random amplitudes A_k, B_k and distance Δ between them. The influence $H_1(x, t)$, $H_2(x, t)$ have the form

$$
\begin{gathered}
H_1(x, t-t_k) = H_1^*(x, t-t_k)+H_1^{**}\left(x, t-t_k-\frac{\Delta}{v}\right),\\
H_2\left(x, t-t_k-\frac{l}{v}\right) = H_2^{**}\left(x, t-t_k-\frac{l}{v}\right)+H_2^{**}\left(x, t-t_k-\frac{l+\Delta}{v}\right),
\end{gathered}
\tag{4.1}
$$

where the functions H_1^* and H_1^{**} satisfy Eq. (2.12), while the functions H_2^* and H_2^* — Eq. (2.13) (with suitably shifted arguments).

By analogy with Eq. (2.17) we obtain

$$
\begin{aligned}
w(x, t) = & \int\limits_{t-\frac{l}{v}}^{t} A(\tau)H_1^*(x, t-\tau)dN(\tau)+\int\limits_{t-\frac{l+\Delta}{v}}^{t-\frac{\Delta}{v}} B(\tau)H_1^{**}\left(x, t-\tau-\frac{\Delta}{v}\right)dN(\tau)\\
& +\int\limits_{0}^{t-\frac{l}{v}} A(\tau)H_2^*\left(x, t-\tau-\frac{l}{v}\right)dN(\tau)+\int\limits_{0}^{t-\frac{l+\Delta}{v}} B(\tau)H_2^{**}\left(x, t-\tau-\frac{l+\Delta}{v}\right)dN(\tau).
\end{aligned}
\tag{4.2}
$$

The expected value and variance of the random function $w(x, t)$ defined by this formula are obtained similarly to Eqs. (2.18) and (2.20).

Deflection of the beam $w(x, t)$ given by Eq. (2.16) is, in the case of a Poisson process, the sum of independent random variables of identical distributions. In such case it may be shown, in similar manner like in the case of a series of impulses [14, 15], that when the arrival rate tends to infinity ($\lambda \to \infty$) and the amplitudes fulfil the relations $\bigwedge_{k>1} E[A^k] = \text{const}$, the distribution of the function of deflection $w(x, t)$ tends to normal distribution with the expected value $m_w = E[w]$ and the variance σ_w^2.

Appendix

$$I_1(a, b) = \int_{t-\frac{l}{v}}^{t} e^{-a(t-\tau)} \sin b(t-\tau)\, d\tau = \frac{b}{a^2+b^2} - \frac{a\sin b\frac{l}{v} + b\cos b\frac{l}{v}}{a^2+b^2} e^{-a\frac{l}{v}},$$

$$I_2(a, b) = \int_{t-\frac{l}{v}}^{t} e^{-a(t-\tau)} \cos b(t-\tau)\, d\tau = \frac{a}{a^2+b^2} + \frac{b\sin b\frac{l}{v} - a\cos b\frac{l}{v}}{a^2+b^2} e^{-a\frac{l}{v}},$$

$$I_3(a, b) = \int_{t-\frac{l}{v}}^{t} e^{-a(t-\tau)} \sin^2 b(t-\tau)\, d\tau = \frac{2b^2}{a(a^2+4b^2)}$$

$$-\frac{1}{2} e^{-a\frac{l}{v}} \left[\frac{1}{a} + \frac{1}{a^2+b^2}\left(-a\cos 2b\frac{l}{v} + 2b\sin 2b\frac{l}{v}\right)\right],$$

$$I_4(a, b) = \int_{t-\frac{l}{v}}^{t} e^{-a(t-\tau)} \cos^2 b(t-\tau)\, d\tau = \frac{a^2+2b^2}{a(a^2+4b^2)}$$

$$+\frac{1}{2} e^{-a\frac{l}{v}} \left[-\frac{1}{a} + \frac{1}{a^2+4b^2}\left(-a\cos 2b\frac{l}{v} + 2b\sin 2b\frac{l}{v}\right)\right],$$

$$I_5(a, b, c) = \int_{t-\frac{l}{v}}^{t} e^{-a(t-\tau)} \sin b(t-\tau) \sin c(t-\tau)\, d\tau$$

$$= \frac{1}{2} I_2(a, b-c) - \frac{1}{2} I_2(a, b+c) = \frac{2abc}{[a^2+(b-c)^2][a^2+(b+c)^2]}$$

$$+\frac{1}{2} \frac{e^{-a\frac{l}{v}}}{a^2+(b-c)^2} \left[-a\cos(b-c)\frac{l}{v} + (b-c)\sin(b-c)\frac{l}{v}\right]$$

$$-\frac{1}{2} \frac{e^{-a\frac{l}{v}}}{a^2+(b+c)^2} \left[-a\cos(b+c)\frac{l}{v} + (b+c)\sin(b+c)\frac{l}{v}\right],$$

$$I_6(a,b,c) = \int\limits_{t-\frac{l}{v}}^{t} e^{-a(t-\tau)} \sin b(t-\tau)\cos c(t-\tau)\,d\tau$$

$$= \frac{1}{2} I_1(a, b-c) + \frac{1}{2} I_1(a, b+c) = \frac{1}{2}\left[\frac{b-c}{a^2+(b-c)^2} + \frac{b+c}{a^2+(b+c)^2}\right]$$

$$-\frac{1}{2}\frac{e^{-a\frac{l}{v}}}{a^2+(b-c)^2}\left[a\sin(b-c)\frac{l}{v} + (b-c)\cos(b-c)\frac{l}{v}\right]$$

$$-\frac{1}{2}\frac{e^{-a\frac{l}{v}}}{a^2+(b+c)^2}\left[a\sin(b+c)\frac{l}{v} + (b+c)\cos(b+c)\frac{l}{v}\right],$$

$$I_7(a,b,c) = \int\limits_{t-\frac{l}{v}}^{t} e^{-a(t-\tau)} \cos b(t-\tau)\cos c(t-\tau)\,d\tau$$

$$= \frac{1}{2} I_2(a, b-c) + \frac{1}{2} I_2(a, b+c) = \frac{a(a^2+b^2+c^2)}{[a^2+(b-c)^2][a^2+(b+c)^2]}$$

$$+\frac{1}{2}\frac{e^{-a\frac{l}{v}}}{a^2+(b-c)^2}\left[-a\cos(b-c)\frac{l}{v} + (b-c)\sin(b-c)\frac{l}{v}\right]$$

$$+\frac{1}{2}\frac{e^{-a\frac{l}{v}}}{a^2+(b+c)^2}\left[-a\cos(b+c)\frac{l}{v} + (b+c)\sin(b+c)\frac{l}{v}\right],$$

$$D_1(a,b) = \int\limits_{0}^{t-\frac{l}{v}} e^{-a\left(t-\tau-\frac{l}{v}\right)} \sin b\left(t-\tau-\frac{l}{v}\right) d\tau$$

$$= \frac{b}{a^2+b^2} - \frac{a\sin b\left(t-\frac{l}{v}\right) + b\cos b\left(t-\frac{l}{v}\right)}{a^2+b^2} e^{-a\left(t-\frac{l}{v}\right)},$$

$$D_2(a,b) = \int\limits_{0}^{t-\frac{l}{v}} e^{-a\left(t-\tau-\frac{l}{v}\right)} \cos b\left(t-\tau-\frac{l}{v}\right) d\tau$$

$$= \frac{a}{a^2+b^2} + \frac{b\sin b\left(t-\frac{l}{v}\right) - a\cos b\left(t-\frac{l}{v}\right)}{a^2+b^2} e^{-a\left(t-\frac{l}{v}\right)},$$

$$D_3(a,b) = \int\limits_{0}^{t-\frac{l}{v}} e^{-a\left(t-\tau-\frac{l}{v}\right)} \sin^2 b\left(t-\tau-\frac{l}{v}\right) d\tau = \frac{2b^2}{a(a^2+4b^2)}$$

$$-\frac{1}{2} e^{-a\left(t-\frac{l}{v}\right)}\left\{\frac{1}{a} + \frac{1}{a^2+4b^2}\left[-a\cos 2b\left(t-\frac{l}{v}\right) + 2b\sin 2b\left(t-\frac{l}{v}\right)\right]\right\},$$

$$D_4(a, b) = \int_0^{t-\frac{l}{v}} e^{-a\left(t-\tau-\frac{l}{v}\right)} \sin b\left(t-\tau-\frac{l}{v}\right)\cos b\left(t-\tau-\frac{l}{v}\right) d\tau$$

$$= \frac{1}{2} D_1(a, 2b) = \frac{b}{a^2+4b^2} - \frac{\frac{1}{2} a \sin 2b\left(t-\frac{l}{v}\right) + b\cos 2b\left(t-\frac{l}{v}\right)}{a^2+4b^2} e^{-a\left(t-\frac{l}{v}\right)},$$

$$D_5(a, b) = \int_0^{t-\frac{l}{v}} e^{-a\left(t-\tau-\frac{l}{v}\right)} \cos^2 b\left(t-\tau-\frac{l}{v}\right) d\tau = \frac{a^2+2b^2}{a(a^2+4b^2)}$$

$$+ \frac{e^{-a\left(t-\frac{l}{v}\right)}}{2}\left\{-\frac{1}{a} + \frac{1}{a^2+4b^2}\left[-a\cos 2b\left(t-\frac{l}{v}\right) + 2b\sin 2b\left(t-\frac{l}{v}\right)\right]\right\},$$

$$D_6(a, b, c) = \int_0^{t-\frac{l}{v}} e^{-a\left(t-\tau-\frac{l}{v}\right)} \sin b\left(t-\tau-\frac{l}{v}\right)\sin c\left(t-\tau-\frac{l}{v}\right) d\tau$$

$$= \frac{1}{2}]D_2(a, b-c) - D_2(a, b+c)],$$

$$D_7(a, b, c) = \int_0^{t-\frac{l}{v}} e^{-a\left(t-\tau-\frac{l}{v}\right)} \sin b\left(t-\tau-\frac{l}{v}\right)\cos c\left(t-\tau-\frac{l}{v}\right) d\tau$$

$$= \frac{1}{2}[D_1(a, b-c) - D_1(a, b+c)],$$

$$D_8(a, b, c) = \int_0^{t-\frac{l}{v}} e^{-a\left(t-\tau-\frac{l}{v}\right)} \cos b\left(t-\tau-\frac{l}{v}\right)\cos c\left(t-\tau-\frac{l}{v}\right) d\tau$$

$$= \frac{1}{2}[D_2(a, b-c) + D_2(a, b+c)].$$

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