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On some class of problems of linear elasticity with constraints for displacements and stresses

Z. NANIEWICZ (WARSZAWA)

BOUNDARY value problems of the linear elasticity theory with constraints for displacement fields and stress fields are discussed. Some necessary and sufficient conditions for the existence of solutions to the problems for a certain class of constraints are formulated.

Tematem pracy są problemy brzegowe liniowej sprężystości z więzami dla pól przemieszczeń i pól naprężenia. Sformułowano warunki konieczne i wystarczające istnienia rozwiązań dla pewnej klasy więzów.

В работе рассмотрены краевые задачи линейной упругости с некоторым классом связей для перемещений и напряжений. Сформулированы необходимые и достаточные условия существования решений этих задач.

Introduction

IN THE PAPER the weak formulation of the linear elasticity problems with convex constraints for displacements and stresses is presented. It has been shown that in general any problem of this kind leads to the system of two variational inequalities in which the basic unknowns are the displacement field and the stress field. To this system two other problems are assigned: one for the displacement field only and the second for the stress field only. The relationship between solutions of these three problems is examined. The obtained results are applied to formulate some necessary and sufficient conditions for the existence of solutions to the boundary value problems under consideration.

1. Basic concepts and assumptions

Consider an equilibrium state of a body which is identified with a bounded region B with the regular boundary([1]), in Euclidean three-space. The problem will be analysed under the basic assumptions of the linear elasticity.

As a displacement space we shall assume the Hilbert space $\mathbf{H}^1(B)$ of all vector functions square integrable together with their first partial derivatives in B, equipped with the norm

$$||\mathbf{u}||_1^2 = \int_B \left(\mathbf{u}^2 + \mathrm{tr}(\nabla\mathbf{u}\nabla\mathbf{u}^T)\right)dv,$$

where $\nabla\mathbf{u}$ is a gradient of $\mathbf{u}$.

([1]) In the sense of the definition given in [5].

We assume that in every problem under consideration there is a given nonempty closed convex subset $\mathscr{U}$ of $\mathbf{H}^1(B)$, called the set of all admissible displacement fields. If $\mathscr{U}$ is the proper subset of $\mathbf{H}^1(B)$, then we deal with certain restrictions which will be termed the displacement constraints([2]).

To characterize external forces which can act at the body, we introduce the test function space $\mathscr{V}$ which is assumed to have the following two properties:

(i) $\mathscr{V}$ is a closed linear subspace of $\mathbf{H}^1(B)$;

(ii) $\mathscr{V} \supset \mathscr{U}-\mathscr{U}$,

where $\mathscr{U}-\mathscr{U} = \{\mathbf{v} \in \mathbf{H}^1(B)\colon \mathbf{v} = \mathbf{w}-\mathbf{u}$ for some $\mathbf{w}, \mathbf{u} \in \mathscr{U}\}$. External forces will be represented by linear continuous functionals on $\mathscr{V}$, i.e. by elements of $\mathscr{V}^*$ where $\mathscr{V}^*$ denotes the dual of $\mathscr{V}$. The pairing over $\mathscr{V}^* \times \mathscr{V}$ will be denoted by $\langle \cdot, \cdot \rangle_1$. The value of work done by external forces represented by $\mathbf{f}^* \in \mathscr{V}^*$ over any $\mathbf{v} \in \mathscr{V}$ is equal to the value of $\mathbf{f}^*$ at $\mathbf{v}$, i.e. $\langle \mathbf{f}^*, \mathbf{v} \rangle_1$. If the body is subjected to the body forces $\mathbf{b}$ square integrable in B and to the surface tractions $\mathbf{p}$ square integrable on ∂B, then the functional representing these forces is given by

$$\langle \mathbf{f}^*, \mathbf{v} \rangle_1 = \int_B \mathbf{b} \cdot \mathbf{v}\, dv + \int_{\partial B} \mathbf{p} \cdot \mathbf{v}\, ds, \qquad \forall \mathbf{v} \in \mathscr{V} \text{([3])}.$$

It must be stressed that in general the test function space $\mathscr{V}$ to the contrary of $\mathscr{U}$ is not uniquely defined in the problem under consideration.

As a strain space we shall assume the Hilbert space $\mathscr{S}$ of all symmetric tensor functions square integrable in B, equipped with the norm:

$$\|\tilde{\mathbf{E}}\|_2 = \int_B \operatorname{tr}(\tilde{\mathbf{E}}\tilde{\mathbf{E}})\, dv, \qquad \forall \tilde{\mathbf{E}} \in \mathscr{S}.$$

The displacement-strain relation is described by the linear continuous operator $\mathbf{E}\colon \mathbf{H}^1(B) \to \mathscr{S}$, which assigns to any $\mathbf{u} \in \mathbf{H}^1(B)$ the symmetric part of the displacement gradient $\nabla \mathbf{u}$, i.e.

$$\mathbf{E}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T), \qquad \forall \mathbf{u} \in \mathbf{H}^1(B).$$

Any element $\tilde{E} \in \mathscr{S}$ for which there exists such $\mathbf{v} \in \mathbf{H}^1(B)$ that $\tilde{\mathbf{E}} = \mathbf{E}(\mathbf{v})$ is called a strain field.

As a stress space we shall assume the Hilbert space $\mathscr{T}$ of all symmetric tensor functions square integrable in B, equipped with the norm

$$\|\mathbf{T}\|_2^2 = \int_B \operatorname{tr}(\mathbf{T}\mathbf{T})\, dv, \qquad \forall \mathbf{T} \in \mathscr{T}.$$

Elements of $\mathscr{T}$ are called stress fields.

We assume that in every problem under consideration there is given the nonempty closed convex subset Σ of $\mathscr{T}$, which will be termed the set of all admissible stress fields.

([2]) In our approach the displacement boundary conditions are interpreted as displacement constraints and hence are included into the set $\mathscr{U}$.

([3]) If the test function space $\mathscr{V}$ coincides with $\mathbf{H}_0^1(B) = \{\mathbf{u} \in \mathbf{H}^1(B)\colon \mathbf{u}|_{\partial B} = 0\}$, then $\int_{\partial B} \mathbf{p} \cdot \mathbf{v}\, ds = 0$ for every $\mathbf{v} \in \mathscr{V}$.

If Σ is a proper subset of $\mathscr{T}$, then we deal with certain restrictions which will be called the stress constraints([4]).

Internal forces will be represented by linear continuous functionals on $\mathscr{S}$, i.e, by elements of $\mathscr{S}^*$, the dual of $\mathscr{S}$. The pairing over $\mathscr{S}^* \times \mathscr{S}$ will be denoted by $\langle \cdot\, , \cdot \rangle_2$. The value of work done by the internal forces represented by $\mathbf{T}^* \in \mathscr{S}^*$ over any $\tilde{\mathbf{E}} \in \mathscr{S}$ is given by $\langle \mathbf{T}^*, \tilde{\mathbf{E}} \rangle_2$.

To any stress field $\mathbf{T} \in \mathscr{T}$ there will be assigned the functional $\mathscr{T}^{(*)} \in \mathbf{T}^*$, defined by

$$\langle \mathbf{T}^{(*)}, \tilde{\mathbf{E}} \rangle_2 \overset{\mathrm{df}}{=} \int_B \operatorname{tr}(\mathbf{T}\tilde{\mathbf{E}})\, dv, \qquad \forall \tilde{\mathbf{E}} \in \mathscr{S}.$$

The functional $\mathbf{T}^{(*)}$ represents internal forces corresponding to the state of stress equal to $\mathbf{T}$ (from the Riesz Representation Theorem it follows that the operator "$(*)$" is a linear isomorphism between $\mathbf{T}$ and $\mathscr{S}^*$).

To obtain the equations of equilibrium we introduce the linear continuous operator $\mathbf{E}_1$ being the restriction of $\mathbf{E}$ to $\mathscr{V}$, i.e. $\mathbf{E}_1 : \mathscr{V} \to \mathscr{S}$, $\mathbf{E}_1 = \mathbf{E}|_{\mathscr{V}}$. By $\mathbf{E}_1^* : \mathscr{S}^* \to \mathscr{V}^*$ there will be denoted the adjoint of $\mathbf{E}_1$ defined for any $\mathbf{T}^* \in \mathscr{S}^*$ as follows:

$$\langle \mathbf{E}_1^* \mathbf{T}^*, \mathbf{v} \rangle_1 \overset{\mathrm{df}}{=} \langle \mathbf{T}^*, \mathbf{E}_1(\mathbf{v}) \rangle_2, \qquad \mathbf{v} \in \mathscr{V}.$$

Let $\mathbf{f}^* \in \mathscr{V}^*$ represent the external forces acting at the body and let $\mathbf{T} \in \Sigma$ be the stress field. We assume the equations of equilibrium in the form

$$\mathbf{E}_1^* \mathbf{T}^{(*)} - \mathbf{f}^* = \mathbf{0}, \tag{1.1}$$

or, equivalently,

$$\int_B \operatorname{tr}\big(\mathbf{T}\mathbf{E}(\mathbf{v})\big)\, dv - \langle \mathbf{f}^*, \mathbf{v} \rangle_1 = 0, \qquad \forall \mathbf{v} \in \mathscr{V}.$$

The above equations assert that the work done by the external forces over any $\mathbf{v} \in \mathscr{V}$ is equal to the work done by the internal forces over the strain field corresponding to $\mathbf{v}$.

Let $\mathbf{K}$ be the compliance field tensor. We suppose that $\mathbf{K}$ determines the linear operator denoted again by $\mathbf{K}$, $\mathbf{K}: \mathscr{T} \to \mathscr{S}$ with the effective domain $D(\mathbf{K}) \supset \Sigma$. Let us define $\mathbf{K}^0 : \mathscr{S}^* \to \mathscr{S}$ by means of the formula

$$\mathbf{K}^0 \mathbf{T}^* \overset{\mathrm{df}}{=} \mathbf{K}\mathbf{T}, \quad \text{if} \quad \mathbf{T}^* = \mathbf{T}^{(*)} \quad \text{for some } \mathbf{T} \in \mathscr{T}. \tag{1.2}$$

From the above definition it follows that the effective domain of $\mathbf{K}^0$, denoted by $D(\mathbf{K}^0)$, contains $\Sigma^{(*)} = \{\boldsymbol{\sigma}^* \in \mathscr{S}^* : \boldsymbol{\sigma}^* = \boldsymbol{\sigma}^{(*)} \text{ for some } \boldsymbol{\sigma} \in \Sigma\}$ (since the operator "$(*)$" is bijective, it follows that $\Sigma^{(*)}$ is a nonempty closed convex subset of $\mathscr{S}^*$). In problems with constraints for stresses we introduce an extra strain field $\check{\mathbf{E}}$ which may be not compatible with admissible displacement fields, i.e. the equality $\check{\mathbf{E}} = \mathbf{E}(\mathbf{v})$ may not hold for any $\mathbf{v} \in \mathscr{U}$.

The constitutive relation will be given in the form

$$\mathbf{K}^0 \mathbf{T}^{(*)} - \check{\mathbf{E}} = \mathbf{0}, \tag{1.3}$$

([4]) The difference between a strain space and a stress space follows from the physical interpretation and the transformation rules in the curvilinear coordinate systems.

or, equivalently,

$$\int_B \operatorname{tr}\big((\mathbf{KT}-\check{\mathbf{E}})\boldsymbol{\sigma}\big)dv = 0, \qquad \forall \boldsymbol{\sigma} \in \mathscr{T}.$$

To obtain the full treatment of the elasticity problems we are to deal with, that is the problems with constraints for displacements and stresses, we have to introduce, apart from the equations of equilibrium (1.1) and constitutive relations (1.3), the defining relations for external forces represented by $\mathbf{f}^*$ and the defining relations for the extra strain field $\check{\mathbf{E}}$.

Let us pass to the defining relations for the external forces. We suppose that the total external forces acting at the body are the sum of loadings, constraint reactions and field reactions. By loadings we mean such external forces which are uniquely determined by the displacement field. By the constraint reactions we understand such external forces which, roughly speaking, can maintain the displacement constraints (or which are due to the displacement constraints). The external forces which cannot be treated as loadings and are not due to the displacement constraints are said to be field reactions (for example, the forces of friction, [5, 16]).

To any $\mathbf{v} \in \mathscr{U}$ there will be assigned the set

$$\mathscr{U}_{\mathbf{v}} = \{\mathbf{w} \in \mathscr{V} : \mathbf{w} = \mathbf{z}-\mathbf{v} \quad \text{for some} \quad \mathbf{z} \in \mathscr{U}\},$$

which will be called a set of all displacement increments related to $\mathbf{v}$ admissible by the constraints. The convex cone generated by $\mathscr{U}_{\mathbf{v}}$ and denoted by co $\mathscr{U}_{\mathbf{v}}$, will be termed the set of all virtual displacements related to $\mathbf{v}$.

ASSUMPTION 1. The set of all constraint reactions which can act at the body in a position described by $\mathbf{v} \in \mathbf{H}^1(B)$ is given by multivalued mapping $\Phi:\mathbf{H}^1(B) \to 2^{\mathscr{V}^*}$, defined by

$$\Phi(\mathbf{v}) = \begin{cases} \emptyset & \text{if} \quad \mathbf{v} \notin \mathscr{U}, \\ \{\mathbf{h}^* \in \mathscr{V}^*: \langle \mathbf{h}^*, \mathbf{w}\rangle_1 \geqslant 0, \quad \forall \mathbf{w} \in \operatorname{co} \mathscr{U}_v\} & \text{if} \quad \mathbf{v} \in \mathscr{U}. \end{cases}$$

This assumption states that the work done by the reaction forces over an arbitrary virtual displacement related to $\mathbf{v} \in \mathscr{U}$ is always nonnegative.

Let $f_1 : \mathbf{H}^1(B) \to \overline{R}$ be a convex lower-semicontinuous function with the property that the effective domain $D(\partial f_1)$ of its subdifferential $\partial f_1 :\mathbf{H}^1(B) \to 2^{\mathbf{H}^1(B)^*}$ contains $\mathscr{U}$, i.e.

$$D(\partial f_1) \supset \mathscr{U} \tag{1.4}$$

and

$$\partial(f_1 + \operatorname{ind}_{\mathscr{U}}) = \partial f_1 + \partial \operatorname{ind}_{\mathscr{U}}, \tag{1.5}$$

where $\operatorname{ind}_{\mathscr{U}}:\mathbf{H}^1(B) \to \overline{\mathbf{R}}$ is the indicator function of $\mathscr{U}$ and $\partial\operatorname{ind}_{\mathscr{U}}: \mathbf{H}^1(B) \to 2^{\mathbf{H}^1(B)^*}$ is its subdifferential, [6].

ASSUMPTION 2. The set of all field reactions which can act at the body, if it is subjected to $\mathbf{v} \in \mathbf{H}^1(B)$, is characterized by multivalued mapping $\Phi_1 : \mathbf{H}^1(B) \to 2^{\mathscr{V}^*}$, defined as follows:

$$\Phi_1(\mathbf{v}) = \begin{cases} \emptyset & \text{if} \quad \mathbf{v} \notin \mathscr{U}, \\ \{\mathbf{h}^* \in \mathscr{V}^*: \langle \mathbf{h}, \mathbf{w}-\mathbf{v}\rangle_1 \geqslant f_1(\mathbf{w})-f_1(\mathbf{v}), \quad \forall \mathbf{w} \in \mathscr{U}\} & \text{if} \quad \mathbf{v} \in \mathscr{U}. \end{cases}$$

This assumption states that the work done by the field reactions over an arbitrary displacement increments related to $\mathbf{v}$ admissible by the constraints is never smaller than the corresponding work done by the so-called "control forces" [21].

Let $f_2: \mathbf{H}^1(B) \to \bar{R}$ be a convex Gateaux differentiable function with a differential $\mathbf{f}_2': \mathbf{H}^1(B) \to \mathbf{H}^1(B)^*$ and let $\mathbf{f}_0^*$ be a given element of $\mathscr{V}^*$.

Assumption 3. The loadings which can act at the body in a position described by $\mathbf{v} \in \mathbf{H}^1(B)$ are given by mapping $\Phi_2: \mathbf{H}^1(B) \to 2^{\mathscr{V}^*}$, defined by

$$\Phi_2(\mathbf{v}) = \begin{cases} \emptyset & \text{if} \quad \mathbf{v} \notin \mathscr{U}, \\ \{-f_2'(\mathbf{v}) + \mathbf{f}_0^*\} & \text{if} \quad \mathbf{v} \in \mathscr{U}. \end{cases}$$

The above decomposition of loadings into the dead load $\mathbf{f}_0^*$ and the potential load $-f_2'(\mathbf{v})$ is not unique and is due to the character of the problem under consideration (this ambiguity does not affect the final results).

Now, let $\mathbf{u} \in \mathscr{U}$ denote the displacement field. The total external forces acting at the body will be assumed in the form of the following external force relation:

$$\mathbf{f}^* \in \Phi(\mathbf{u}) + \Phi_1(\mathbf{u}) - f_2'(\mathbf{u}) + \mathbf{f}_0^*. \tag{1.6}$$

Let us pass to the defining relations of an extra strain field. We shall postulate that the extra strain field $\check{\mathbf{E}}$ can be decomposed into the four parts which will be called: strain field (understood in the classical sense), constraint strain incompatibilities, field strain incompatibilities and initial an strain field. By the constrained strain incompatibilities we understand such elements of $\mathscr{S}$ which, roughly speaking, can maintain stress constraints (or which are due to stress constraints).

To any $\boldsymbol{\sigma} \in \mathscr{T}$ there will be assigned a set $\Sigma_{\boldsymbol{\sigma}} = \Sigma - \boldsymbol{\sigma}$, which will be called a set of all stress increments fields related to $\boldsymbol{\sigma}$ admissible by the constraints. The convex cone generated by $\Sigma_{\boldsymbol{\sigma}}$, denoted by co $\Sigma_{\boldsymbol{\sigma}}$, will be termed a set of all virtal stress fields related to $\boldsymbol{\sigma}$.

Assumption 4. The set of all constraint strain incompatibilities related to the stress field $\mathbf{T} \in \mathscr{T}$ is given by multivalued mapping $\Psi: T \to 2^S$, defined by

$$\Psi(\mathbf{T}) = \begin{cases} \emptyset & \text{if} \quad \mathbf{T} \notin \Sigma, \\ \{\tilde{\mathbf{E}} \in \mathscr{S}: \langle \boldsymbol{\sigma}^{(*)}, \tilde{\mathbf{E}} \rangle_2 \geqslant 0, \quad \forall \boldsymbol{\sigma} \in \mathrm{co}\Sigma_{\mathbf{T}}\} & \text{if} \quad \mathbf{T} \in \Sigma. \end{cases}$$

This assumption states that the work done by internal forces corresponding to an arbitrary virtual stress field related to $\mathbf{T} \in \Sigma$ over the constraint strain incompatibilities is always nonnegative.

Let $f_3: \mathbf{T} \to \bar{R}$ be a proper convex lower semicontinuous function. Let us define $\tilde{f}_3: \mathscr{S}^* \to \bar{R}$ by

$$\tilde{f}_3(\boldsymbol{\sigma}^*) \overset{\mathrm{df}}{=} f_3(\boldsymbol{\sigma}), \quad \text{if} \quad \boldsymbol{\sigma}^* = \boldsymbol{\sigma}^{(*)} \quad \text{for some } \boldsymbol{\sigma} \in \mathscr{T}$$

(since the operator "$(*)$" is bijective, it follows that $\tilde{f}_3$ is a proper convex lower semicontinuous function from $\mathscr{S}^*$ into $\bar{R}$). We suppose that the effective domain $D(\partial\tilde{f}_3)$ of its subdifferential $\partial\tilde{f}_3: \mathscr{S}^* \to 2^{\mathscr{S}}$ contains $\Sigma^{(*)}$, i.e.

$$D(\partial\tilde{f}_3) \supset \Sigma^{(*)} \tag{1.7}$$

and

$$\partial\tilde{f}_3 + \partial\,\mathrm{ind}_{\Sigma^{(*)}} = \partial(\tilde{f}_3 + \mathrm{ind}_{\Sigma^{(*)}}), \tag{1.8}$$

where $\text{ind}_{\Sigma^{(*)}}: \mathcal{S}^* \to \bar{R}$ is the indicator function of $\Sigma^{(*)}$ and $\partial \text{ind}_{\Sigma^{(*)}}: \mathcal{S}^* \to 2^{\mathcal{S}}$ is its subdifferential (for instance, the condition (1.8) is satisfied if $D(\partial \tilde{f}_3) = \mathcal{S}^*$).

ASSUMPTION 5. The set of all field strain incompatibilities related to $\mathbf{T} \in \mathcal{T}$ is given by multivalued mapping $\Psi_1: \mathcal{T} \to 2^{\mathcal{S}}$, defined by

$$\Psi_1(\mathbf{T}) = \begin{cases} \emptyset & \text{if} \quad \mathbf{T} \notin \Sigma, \\ \{\tilde{\mathbf{E}} \in \mathcal{S}: \langle \boldsymbol{\sigma}^{(*)} - \mathbf{T}^{(*)}, \tilde{\mathbf{E}} \rangle_2 \geqslant f_3(\boldsymbol{\sigma}) - f_3(\mathbf{T}), \forall \boldsymbol{\sigma} \in \Sigma\} & \text{if} \quad \mathbf{T} \in \Sigma. \end{cases}$$

The above assumption states that the work done by the internal forces corresponding to an arbitrary stress increment field related to $\mathbf{T} \in \Sigma$ admissible by the constraints over the field strain incompatibilities is never smaller than the corresponding work done by what will be called "control incompatibilities".

Now let $\mathbf{u} \in \mathcal{U}$ be the displacement field and let $\mathbf{T} \in \Sigma$ be the stress field. The extra strain field will be assumed in the form of the following extra strain relation:

$$\check{\mathbf{E}} \in \mathbf{E}(\mathbf{u}) + \Psi(\mathbf{T}) + \Psi_1(\mathbf{T}) + \tilde{\mathbf{E}}_0, \tag{1.9}$$

where $\tilde{\mathbf{E}}_0 \in \mathcal{S}$ will be interpreted as the known initial strain field.

The full treatment of the linear elasticity problems with convex constraints for displacements and stresses is given by equations of equilibrium (1.1), constitutive relations (1.3), external force relations (1.6) and extra strain relations (1.9).

For a detailed discussion concerning the foundations of solid mechanics with constraints imposed independently on deformations and stresses with the realization of stress constraints by strain incompatibilities the reader is referred to [15–22]. Some special classes of problems of this kind can be found in [7, 14].

2. Governing relations

In general, many different test function spaces can be introduced for the given set of all admissible displacement fields. We shall see in Sect. 4 that the proper choice of $\mathcal{V}$ is strictly related to the existence of solutions. From the point of view of mechanics this means that it is necessary to choose such a system of external forces which is able to maintain the constraints.

Taking into account the interrelations between the displacement constraints and the test function space we shall consider three following cases:

I. $\mathcal{V} = \mathbf{H}^1(B)$ and $\mathcal{U}$ is an arbitrary nonempty closed convex subset of $\mathbf{H}^1(B)$;

II. $\mathcal{V}$ is a such closed linear subspace of $\mathbf{H}^1(B)$ on which Korn's inequality holds, i.e.

$$\|\mathbf{E}(\mathbf{v})\|_2 \geqslant c\|\mathbf{v}\|_1, \quad \forall \mathbf{v} \in \mathcal{V}, \tag{K_1}$$

with some positive constant $c > 0$, and $\mathcal{U}$ is such a nonempty closed convex subset of $\mathbf{H}^1(B)$ that

$$\mathcal{U} \subset \mathcal{V} + \mathbf{u}_0$$

for a certain element $\mathbf{u}_0 \in \mathbf{H}^1(B)$;

III. $\mathscr{V} = \mathbf{H}^1(B)$ and the functions $\varphi \overset{\mathrm{df}}{=} \mathrm{ind}_{\mathscr{U}} + f_1 + f_2$ and $\langle \mathbf{f}_0^*, \cdot \rangle_1$ satisfy the following conditions:

$$\text{(2.1)} \qquad \begin{aligned} \varphi(\mathbf{v}) &= \varphi(\mathbf{v}+\boldsymbol{\rho}), \quad \forall \boldsymbol{\rho} \in \mathscr{R}, \quad \mathbf{v} \in \mathscr{U}, \\ \langle f_0^*, \boldsymbol{\rho} \rangle_1 &= 0, \quad \forall \boldsymbol{\rho} \in \mathscr{R}, \end{aligned}$$

where $\mathscr{R}$ is the set of all rigid displacements and $\mathrm{ind}_{\mathscr{U}}: \mathbf{H}^1(B) \to \bar{R}$ is the indicator function of $\mathscr{U}$.

2.1. Governing relations in Case I

To characterize the set of all constraint reactions we recall that the indicator function of $\mathscr{U}$, $\mathrm{ind}_{\mathscr{U}}: \mathbf{H}^1(B) \to \bar{R}$, is defined by

$$\mathrm{ind}_{\mathscr{U}}(\mathbf{v}) = \begin{cases} 0 & \text{if} \quad \mathbf{v} \in \mathscr{U}, \\ +\infty & \text{if} \quad \mathbf{v} \notin \mathscr{U} \end{cases}$$

and its subdifferential, denoted by $\partial \mathrm{ind}_{\mathscr{U}}$, is the multivalued mapping which assigns to any $\mathbf{v} \in \mathbf{H}^1(B)$ the empty set if $\mathbf{v} \notin \mathscr{U}$ and

$$\{\mathbf{h}^* \in \mathbf{H}^1(B)^* : \langle \mathbf{h}^*, \mathbf{w}-\mathbf{v} \rangle_1 \leqslant 0, \quad \forall \mathbf{w} \in \mathscr{U}\},$$

if $\mathbf{v} \in \mathscr{U}$, i.e.

$$\partial \mathrm{ind}_{\mathscr{U}}(\mathbf{v}) = \begin{cases} \emptyset & \text{if} \quad \mathbf{v} \notin \mathscr{U}, \\ \{\mathbf{h}^* \in \mathbf{H}^1(B)^* : \langle \mathbf{h}^*, \mathbf{w}-\mathbf{v} \rangle_1 \leqslant 0, \quad \forall \mathbf{w} \in \mathscr{U}\} & \text{if} \quad \mathbf{v} \in \mathscr{U}. \end{cases}$$

Note, that under the assumptions we have

$$\text{(2.2)} \qquad \Phi = -\partial \mathrm{ind}_{\mathscr{U}}.$$

The subdifferential of f_1, $\partial f_1 : \mathbf{H}^1(B) \to 2^{\mathbf{H}^1(B)^*}$ is the multivalued mapping defined by

$$\partial f_1(\mathbf{v}) = \{\mathbf{h}^* \in \mathbf{H}^1(B)^* : \langle \mathbf{h}^*, \mathbf{w}-\mathbf{v} \rangle_1 \leqslant f_1(\mathbf{w}) - f_1(\mathbf{v}), \quad \forall \mathbf{w} \in \mathbf{H}^1(B)\}.$$

From Assumption 2 it follows that Φ_1 can be expressed by

$$\text{(2.3)} \qquad \Phi_1 = -\partial(f_1 + \mathrm{ind}_{\mathscr{U}}).$$

By virtue of Eqs. (2.2) and (2.3) and Assumption 3 the external force relation (1.6) can be written as

$$\mathbf{f}^* \in -\partial \mathrm{ind}_{\mathscr{U}}(\mathbf{u}) - \partial(f_1 + \mathrm{ind}_{\mathscr{U}})(\mathbf{u}) - f_2'(\mathbf{u}) + \mathbf{f}_0^*.$$

Using Eq. (1.5) and taking into account the basic results concerning subdifferential calculus [6], we derive

$$\text{(2.4)} \qquad \mathbf{f}^* \in -\partial\varphi(\mathbf{u}) + \mathbf{f}_0^*,$$

where $\partial\varphi : \mathbf{H}^1(B) \to 2^{\mathbf{H}^1(B)^*}$ is the subdifferential of φ. From Eq. (2.4) it follows that the equations of equilibrium (1.1) can be written in the following form:

$$\text{(2.5)} \qquad E^* T^{(*)} - f_0^* \in \partial\varphi(\mathbf{u}).$$

Analogously, the following formulas for mappings Ψ and Ψ_1 can be obtained:

$$\text{(2.6)} \qquad \begin{cases} \Psi(\boldsymbol{\sigma}) \overset{\mathrm{df}}{=} -\partial \mathrm{ind}_{\Sigma^{(*)}}(\boldsymbol{\sigma}^{(*)}), & \forall \boldsymbol{\sigma} \in \mathscr{T}, \\ \Psi_1(\boldsymbol{\sigma}) \overset{\mathrm{df}}{=} -\partial(\tilde{f_3} + \mathrm{ind}_{\Sigma^{(*)}})(\boldsymbol{\sigma}^{(*)}), & \forall \boldsymbol{\sigma} \in \mathscr{T}. \end{cases}$$

From Eqs. (1.9) and (2.6) we have

(2.7) $$\check{\mathbf{E}} \in \mathbf{E}(\mathbf{u}) - \partial \operatorname{ind}_{\Sigma^{(*)}}(\mathbf{T}^{(*)}) - \partial(\tilde{f}_3 + \operatorname{ind}_{\Sigma^{(*)}})(\mathbf{T}^{(*)}) + \langle \mathbf{T}^{(*)}, \tilde{\mathbf{E}}_0 \rangle_2 .$$

Setting

$$\psi(\boldsymbol{\sigma}^*) \overset{\mathrm{df}}{=} \tilde{f}_3(\boldsymbol{\sigma}^*) + \operatorname{ind}_{\Sigma^{(*)}}(\boldsymbol{\sigma}^*) - \langle \boldsymbol{\sigma}^*, \tilde{\mathbf{E}}_0 \rangle_2, \qquad \forall\, \boldsymbol{\sigma}^* \in \mathscr{S}^*$$

and using (1.7) we obtain

(2.8) $$\check{\mathbf{E}} \in \mathbf{E}(\mathbf{u}) - \partial\psi(\mathbf{T}^{(*)}),$$

where $\partial\psi: \mathscr{S}^* \to 2^{\mathscr{S}}$ is the subdifferential of ψ (under our assumptions ψ is a proper convex lower semicontinuous function of $\mathscr{S}^*$ into $\bar{R}$). It yields the following form of the constitutive relation:

(2.9) $$\mathbf{K}^0\mathbf{T}^{(*)} - \mathbf{E}(\mathbf{u}) \in -\partial\psi(\mathbf{T}^{(*)}).$$

Finally, we arrive at the following form of the governing relations for the displacement field $\mathbf{u}$ and stress field $\mathbf{T}$:

(2.10) $$\begin{cases} \mathbf{E}^*\mathbf{T}^{(*)} - \mathbf{f}_0^* \in -\partial\varphi(\mathbf{u}), \\ \mathbf{K}^0\mathbf{T}^{(*)} - \mathbf{E}(\mathbf{u}) \in -\partial\varphi(\mathbf{T}^{(*)}), \end{cases}$$

where φ and ψ are proper convex lower semicontinuous functions. The relations (2.10) can be written as the system of the two following variational inequalities:

$$\int_B \operatorname{tr}\big(\mathbf{T}\mathbf{E}(\mathbf{v}-\mathbf{u})\big)dv - \langle \mathbf{f}_0^*, \mathbf{v}-\mathbf{u}\rangle_1 + \varphi(\mathbf{v}) - \varphi(\mathbf{u}) \geqslant 0, \qquad \forall\, \mathbf{v} \in \mathbf{H}^1(B),$$

$$\int_B \operatorname{tr}\big((\boldsymbol{\sigma}-\mathbf{T})(\mathbf{K}\mathbf{T}-\mathbf{E}(\mathbf{u}))\big)dv + \tilde{\psi}(\sigma) - \tilde{\psi}(\mathbf{T}) \geqslant 0, \qquad \forall\, \boldsymbol{\sigma} \in \mathscr{T},$$

where

$$\tilde{\psi}(\boldsymbol{\sigma}) \overset{\mathrm{df}}{=} \psi(\boldsymbol{\sigma}^{(*)}), \qquad \forall\, \boldsymbol{\sigma} \in \mathscr{T}.$$

Now let us consider the following problem: find $\mathbf{u} \in \mathscr{V}$ and $\mathbf{T}^* \in \mathscr{S}^*$ such that

(2.11) $$\begin{cases} \mathbf{E}^*\mathbf{T}^* - f_0^* \in -\partial\varphi(\mathbf{u}), \\ \mathbf{K}^0\mathbf{T}^* - \mathbf{E}(\mathbf{u}) \in -\partial\psi(\mathbf{T}^*). \end{cases}$$

It is easy to see that if $(\mathbf{u}, \mathbf{T}^*)$ is a solution of Eq. (2.11) then there exists exactly one $\mathbf{T} \in \mathscr{T}$ such that $\mathbf{T}^* = \mathbf{T}^{(*)}$ and $(\mathbf{u}, \mathbf{T})$ is a solution of Eq. (2.10). It is the consequence of the fact that the operator "$(*)$" is a linear isomorphism between $\mathscr{T}$ and $\mathscr{S}^*$. Therefore, to any solution of Eq. (2.11) corresponds exactly one solution of Eq. (2.10). Conversely, if $(\mathbf{u}, \mathbf{T})$ is a solution of Eq. (2.10), then $(\mathbf{u}, \mathbf{T}^{(*)})$ is a solution of Eq. (2.11). This means that the systems (2.10) and (2.11) are equivalent to each other. Due to this fact, Eq. (2.10) can be replaced by the system (2.11) which will be the subject of the investigation in Sect. 3.

2.2. Governing relations in Case II

Now let us define

$$\tilde{\mathscr{U}} \overset{\mathrm{df}}{=} \mathscr{U} - \mathbf{u}_0 \subset \mathscr{V},$$

$$\tilde{f}_i: \mathscr{V} \to \bar{R}, \quad \tilde{f}_i(\mathbf{v}) \overset{\mathrm{df}}{=} f_i(\mathbf{v}+\mathbf{u}_0), \quad \forall\, \mathbf{v} \in \mathscr{V}, \quad i = 1, 2.$$

It is easy to verify that $\tilde{\mathscr{U}}$ is a nonempty closed convex subset of $\mathscr{V}$ and $\tilde{f}_i$, $i = 1, 2$ are proper convex lower semicontinuous functions on $\mathscr{V}$.

The set of all constraint reactions is given by

$$\begin{aligned}\Phi(\mathbf{u}) &= \{\mathbf{h}^* \in \mathscr{V}^* : \langle \mathbf{h}^*, \mathbf{v}\rangle_1 \geqslant 0, \quad \forall \mathbf{v} \in \operatorname{co}\mathscr{U}_u\} \\ &= \{\mathbf{h}^* \in \mathscr{V}^* : \langle \mathbf{h}^*, \mathbf{v}-\mathbf{u}\rangle_1 \geqslant 0, \quad \forall \mathbf{v} \in \mathscr{U}\} \\ &= \{\mathbf{h}^* \in \mathscr{V}^* : \langle \mathbf{h}^*, \mathbf{v}-(\mathbf{u}-\mathbf{u}_0)\rangle_1 \geqslant 0, \quad \forall \mathbf{v} \in \tilde{\mathscr{U}}\} \\ &= -\partial \operatorname{ind}_{\tilde{\mathscr{U}}}(\mathbf{u}-\mathbf{u}_0), \quad \mathbf{u} \in \mathscr{U},\end{aligned}$$

where

$$\partial \operatorname{ind}_{\tilde{\mathscr{U}}} : \mathscr{V} \to 2^{\mathscr{V}^*}$$

is the subdifferential of the indicator function of $\tilde{\mathscr{U}}$, $\operatorname{ind}_{\tilde{\mathscr{U}}} : \mathscr{V} \to \bar{R}$.

And, similarly, the field reactions and loadings are characterized by

$$\begin{aligned}\Phi_1(\mathbf{u}) &= -\partial(\tilde{f}_1 + \operatorname{ind}_{\tilde{\mathscr{U}}})(\mathbf{u}-\mathbf{u}_0), \\ \Phi_2(\mathbf{u}) &= -\tilde{f}'_2(\mathbf{u}-\mathbf{u}_0) + \mathbf{f}^*_0, \quad \mathbf{u} \in \mathscr{U},\end{aligned}$$

respectively, where $\partial(\tilde{f}_1 + \operatorname{ind}_{\tilde{\mathscr{U}}})$ is the subdifferential of $\tilde{f}_1 + \operatorname{ind}_{\tilde{\mathscr{U}}}$. Thus we arrive at the following form of the external force relation:

$$\mathbf{f}^* \in -\partial \operatorname{ind}_{\tilde{\mathscr{U}}}(\mathbf{u}-\mathbf{u}_0) - \partial(\tilde{f}_1 + \operatorname{ind}_{\tilde{\mathscr{U}}})(\mathbf{u}-\mathbf{u}_0) - \tilde{f}'_2(\mathbf{u}-\mathbf{u}_0) + \mathbf{f}^*_0 .$$

Using Eq. (1.8) it is easy to verify that the above relation can be written as follows:

$$\mathbf{f}^* \in -\partial\varphi(\mathbf{u}-\mathbf{u}_0) + \mathbf{f}^*_0, \tag{2.12}$$

where $\tilde{\varphi} : \mathscr{V} \to \bar{R}$ is defined by $\tilde{\varphi} = \operatorname{ind}_{\tilde{\mathscr{U}}} + f_1 + f_2$ and $\partial\tilde{\varphi}$ is the subdifferential of $\tilde{\varphi}$ (under our assumptions $\tilde{\varphi}$ is a proper convex lower semicontinuous function).

From Eqs. (2.12) and (2.8) we obtain the following form of the governing relations for $\mathbf{u}$ and $\mathbf{T}$:

$$\begin{cases}\mathbf{E}^*_1 \mathbf{T}^{(*)} - \mathbf{f}^*_0 \in -\partial\tilde{\varphi}(\mathbf{u}-\mathbf{u}_0), \\ \mathbf{K}^0 \mathbf{T}^{(*)} - \mathbf{E}(\mathbf{u}) \in -\partial\psi(\mathbf{T}^{(*)}).\end{cases} \tag{2.13}$$

Setting $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_0$ and $\tilde{\mathbf{K}}^0(\cdot) \overset{\mathrm{df}}{=} \mathbf{K}^0(\cdot) - \mathbf{E}(\mathbf{u}_0)$ we obtain the following system of variational inequalities with the basic unknowns $\tilde{\mathbf{u}} \in \mathscr{V}$ and $\mathbf{T} \in \mathscr{T}$:

$$\begin{cases}\mathbf{E}^*_1 \mathbf{T}^{(*)} - \mathbf{f}^*_0 \in \partial\tilde{\varphi}(\mathbf{u}), \\ \tilde{\mathbf{K}}^0 \mathbf{T}^{(*)} - \mathbf{E}_1(\mathbf{u}) \in -\partial\psi(\mathbf{T}^{(*)}).\end{cases} \tag{2.14}$$

To Eq. (2.14) will be assigned the equivalent system defined by

$$\begin{cases}\mathbf{E}^*_1 \mathbf{T}^* - \mathbf{f}^*_0 \in -\partial\tilde{\varphi}(\tilde{\mathbf{u}}), \\ \tilde{\mathbf{K}}^0 \mathbf{T}^* - \mathbf{E}_1(\mathbf{u}) \in -\partial\psi(\mathbf{T}^*),\end{cases} \tag{2.15}$$

where the pair $(\tilde{\mathbf{u}}, \mathbf{T}^*) \in \mathscr{V} \times \mathscr{S}^*$ is the basic unknown.

2.3. Governing relations in Case III

In Case III we obtain the same system of governing relations as that of Case I. The further investigation will be based on the additional condition (2.1). This condition is equivalent to the statement that external forces constitute the self-equilibrated system, i.e.

$$\langle \mathbf{f}^*, \boldsymbol{\rho} \rangle_1 = 0, \quad \forall \boldsymbol{\rho} \in \mathscr{R}.$$

Let $\mathscr{V}^{\cdot} = \mathbf{H}^1(B)/\mathscr{R}$ be the quotient with $\mathscr{R}$. We define a linear continuous operator $\mathbf{E}^{\cdot}: \mathscr{V}^{\cdot} \to \mathscr{S}$ by

$$\mathbf{E}^{\cdot}(\mathbf{v}^{\cdot}) \overset{\mathrm{df}}{=} \mathbf{E}(\mathbf{v}), \quad \mathbf{v} \in \mathbf{v}^{\cdot}, \quad \mathbf{v}^{\cdot} \in \mathscr{V}^{\cdot},$$

and a function $\varphi^{\cdot}: \mathscr{V}^{\cdot} \to \bar{\mathscr{R}}$ by

$$\varphi^{\cdot}(v^{\cdot}) \overset{\mathrm{df}}{=} \varphi(\mathbf{v}), \quad \mathbf{v} \in \mathbf{v}^{\cdot}, \quad \mathbf{v}^{\cdot} \in \mathscr{V}^{\cdot}.$$

We equip the space $\mathscr{V}^{\cdot}$ with the norm

$$\|\mathbf{v}^{\cdot}\|^{\cdot} = \inf_{\boldsymbol{\rho} \in \mathscr{R}} \|\mathbf{v} + \boldsymbol{\rho}\|_1, \quad \mathbf{v} \in \mathbf{v}^{\cdot}.$$

It is easy to verify that $\varphi^{\cdot}: \mathscr{V}^{\cdot} \to \bar{R}$ is a proper convex lower semicontinuous function on $\mathscr{V}^{\cdot}$.

Using the above denotations we obtain the following form of the governing relations for $\mathbf{u}^{\cdot} = \mathscr{V}^{\cdot}$ and $\mathbf{T} \in \mathscr{T}$:

$$\text{(2.16)} \qquad \begin{cases} (\mathbf{E}^{\cdot})^* \mathbf{T}^{(*)} - f_0^* \in -\partial\varphi^{\cdot}(\mathbf{u}^{\cdot}), \\ \mathbf{K}^0 \mathbf{T}^{(*)} - \mathbf{E}^{\cdot}(\mathbf{u}^{\cdot}) \in -\partial\psi(\mathbf{T}^{(*)}), \end{cases}$$

where $(\mathbf{E}^{\cdot})^*: \mathscr{S}^* \to (\mathscr{V}^{\cdot})^*$ is the adjoint of $\mathbf{E}^{\cdot}$ and $(\mathscr{V}^{\cdot})^*$ is the dual of $\mathscr{V}^{\cdot}$.

To Eq. (2.16) there will be assigned the equivalent system for the basic unknown $(\mathbf{u}^{\cdot}, \mathbf{T}^*) \in \mathscr{V}^{\cdot} \times \mathscr{S}^*$, namely

$$\text{(2.17)} \qquad \begin{cases} (\mathbf{E}^{\cdot})^* \mathbf{T}^* - \mathbf{f}_0^* \in -\partial\varphi^{\cdot}(\mathbf{u}^{\cdot}), \\ \mathbf{K}^0 \mathbf{T}^* - \mathbf{E}^{\cdot}(\mathbf{u}^{\cdot}) \in -\partial\psi(\mathbf{T}^*). \end{cases}$$

This system is characterized by the important property: there exists a positive constant c, such that the following inequality holds, [5]:

$$(K_2) \qquad \|\mathbf{E}^{\cdot}(\mathbf{v}^{\cdot})\|_2 \geqslant c\|\mathbf{v}^{\cdot}\|^{\cdot}, \quad \forall \mathbf{v}^{\cdot} \in \mathscr{V}^{\cdot}.$$

Note, that the formal structure of Eqs. (2.10), (2.14) and (2.16) is the same. The fundamental difference between the system (2.10) and the systems (2.14) and (2.16) is due to the fact that for Eqs. (2.14) and (2.16) Korn's type inequalities (K_1) and (K_2) hold. As we shall see in Section 4 this property has an important meaning for the investigation of the solution existence problem.

Summing up, the above theory holds true under the following general assumptions:

$$\mathbf{f}^* \in -\partial\varphi(\mathbf{u}), \quad \check{\mathbf{E}} \in \mathbf{E}(\mathbf{u}) - \partial\psi(\mathbf{T}^{(*)}),$$

where φ and ψ are any proper convex lower semicontinuous functions. The Assumptions 1–5, characterizing the external force relation (1.6) and strain relation (1.9), show great possibilities of choosing external forces which can act at the body and give wide possibilities in modifications of constitutive relations.

3. Fundamental theorems related to the system of variational inequalities

In order to describe various problems of the elasticity theory with displacement-stress constraints considered in the previous section by the one formal scheme, let:

(A1) V and Y be reflexive Banach spaces with the dual V^* and Y^*, respectively. The norms on V and Y, the pairings over $V^* \times V$ and $Y^* \times Y$ will be denoted by $\|\cdot\|_V$, $\|\cdot\|_Y$, $\langle \cdot, \cdot \rangle_V$, $\langle \cdot, \cdot \rangle_Y$, respectively;

(A2) $L: V \to Y$ be a linear continuous operator from V into Y with the domain $D(L) = V$. Its transpose will be denoted by L^*, $L^*: Y^* \to V^*$;

(A3) $A: Y^* \to Y$ be an operator from Y^* into Y with the nonempty effective domain $D(A)$;

(A4) $\varphi: V \to \overline{R}$ and $\psi: Y^* \to \overline{R}$ be proper convex lower semicontinuous functions with effective domains $D(\varphi)$ and $D(\psi)$, respectively. By $\varphi^*: V^* \to \overline{R}$ we denote the conjugate function of φ. The multivalued mappings $\partial\varphi: V \to 2^{V^*}$, $\partial\psi: Y^* \to 2^Y$ are the subdifferentials of φ and ψ with effective domains $D(\partial\varphi)$ and $D(\partial\psi)$, respectively;

(A5) $f \in V^*$ be a given element of V^*.

The problems (2.11), (2.15) and (2.17) equivalent to Eqs. (2.10), (2.14) and (2.16) can now be summarized in the form of a single problem of finding $(u, \sigma) \in V \times Y^*$ such that

$$(P) \qquad \begin{cases} L^*\sigma - f \in -\partial\varphi(u), \\ A\sigma - Lu \in -\partial\psi(\sigma). \end{cases}$$

Let us define a function $\alpha: Y^* \to \overline{R}$ by

$$(3.1) \qquad \alpha(\eta) \overset{\mathrm{df}}{=} \varphi^*(-L^*\eta + f), \qquad \forall \eta \in Y^*.$$

To (P) will be assigned the following two problems:

1) find $u \in V$ such that

$$(P_1) \qquad 0 \in L^*(A + \partial\psi)^{-1}Lu + \partial\varphi(u) - f,$$

where $(A + \partial\psi)^{-1}: Y \to 2^{Y^*}$ is the inverse to $(A + \partial\psi): Y^* \to 2^Y$ $((A + \partial\psi)(\eta) \overset{\mathrm{df}}{=} A\eta + \partial\psi(\eta)$ if $\eta \in D(A) \cap D(\partial\psi)$ and $(A + \partial\psi)(\eta) \overset{\mathrm{df}}{=} \emptyset$ if $\eta \notin D(A) \cap D(\partial\psi)$;

2) find $\sigma \in Y^*$ such that

$$(P_2) \qquad 0 \in A\sigma + \partial\alpha(\sigma) + \partial\psi(\sigma),$$

where $\partial\alpha: Y^* \to 2^Y$ stands for the subdifferential of α.

THEOREM 1. *Under Assumptions* (A1)–(A5) *the following conditions are satisfied*:

(i) *If* $(u, \sigma) \in V \times Y^*$ *is a solution of* (P), *then* u *is a solution of* (P_1) *and* σ *is a solution of* (P_2);

(ii) *If* $u \in V$ *is a solution of* (P_1), *then there exists* $\sigma \in Y^*$ *such that* (u, σ) *is a solution of* (P).

Proof (i). Let $(u, \sigma) \in V \times Y^*$ be a solution of (P). Then $-L^*\sigma + f \in \partial\varphi(u)$. Taking into account that $\partial\varphi^*$ is the inverse mapping to $\partial\varphi$ [6], we obtain $u \in \partial\varphi^*(-L^*\sigma + f)$. Hence

$$(3.2) \qquad \varphi^*(u^*) - \varphi^*(-L^*\sigma + f) \geqslant \langle u^* - (-L^*\sigma + f), u \rangle_V, \qquad \forall u^* \in V^*.$$

Setting $u^* = -L^*\eta + f$, where $\eta \in Y^*$, in (3.2) we arrive at the inequality

$$\varphi^*(-L^*\eta + f) - \varphi^*(-L^*\sigma + f) \geqslant \langle -L^*\eta + L^*\sigma, u \rangle_V, \qquad \forall \eta \in Y^*.$$

Using Eq. (3.1) the above condition can be expressed in the form

$$\alpha(\eta)-\alpha(\sigma) \geqslant \langle \eta-\sigma, -Lu\rangle_Y, \quad \forall \eta \in Y^*.$$

It follows that $-Lu \in \partial\alpha(\sigma)$. This condition together with $(P)_2$ implies (P_2).

In order to prove that u is a solution of (P_1), let us observe that the condition $A\sigma - Lu \in -\partial\psi(\sigma)$ is equivalent to $\sigma \in (A+\partial\psi)^{-1} Lu$. From $(P)_1$ it follows that there exists $u_0^* \in \partial\varphi(u)$ such that $L^*\sigma + u_0^* - f = 0$. Hence we obtain (P_1).

(ii). Now let us suppose that u is a solution of (P_1). Then there exists $\sigma \in (A+\partial\psi)^{-1}Lu$ and $u_0^* \in \partial\varphi(u)$ such that $-L^*\sigma + f = u_0^*$. It implies $Lu \in (A+\partial\psi)(\sigma)$ and $-L^*\sigma + f \in \partial\varphi(u)$. This proves the assertion.

Q.U.D.

REMARK 1. The following two conditions are equivalent:

(k) (P) has at least one solution;

(kk) (P_1) has at least one solution.

REMARK 2. The necessary condition for the existence of solutions of (P) can be written as

$$D(A)\cap D(\partial\alpha)\cap D(\partial\psi) \neq \emptyset. \tag{3.3}$$

Now, in addition to the assumptions (A1)–(A5) we suppose that

(A6) There exists a positive constant $c > 0$ with the property

$$\|Lv\|_Y \geqslant c\,\|v\|_V, \quad \forall v \in V. \tag{3.4}$$

THEOREM 2. *Under the assumptions* (A1)–(A6) *the following conditions are satisfied*:

(k) *If $(u, \sigma) \in V\times Y^*$ is a solution of (P), then u is a solution of (P_1) and σ is a solution of (P_2)*;

(kk) *If $u \in V$ is a solution of (P_1), then there exists $\sigma \in Y^*$ such that (u, σ) is a solution of (P)*;

(kkk) *If $\sigma \in Y^*$ is a solution of (P_2), then there exists $u \in V$ such that (u, σ) is a solution of (P).*

(k) and (kk) follow immediately from Theorem 1. For the proof of (kkk) the reader is referred to [9].

REMARK 3. Under the assumptions (A1)–(A6) the following conditions are equivalent:

(i) (P) has at least one solution;

(ii) (P_1) has at least one solution;

(iii) (P_2) has at least one solution.

From Remarks 1 and 3 it follows that the inequality (3.4) has an important meaning in the formulation of existence results for problems under consideration since it assures the equivalence between (P) and (P_2).

Now let us pass to the solution existence problem of (P). This problem due to Remarks 1 and 3 leads to the discussion of conditions under which 0 belongs to the range of multivalued mapping

$$L^*(A+\partial\psi)^{-1}L+\partial\varphi-f$$

under the assumptions (A1)–(A5) and mappings

$$L^*(A+\partial\psi)^{-1}L+\partial\varphi-f \quad \text{or} \quad A+\partial\psi+\partial\alpha$$

under the assumptions (A1)–(A6). For the full treatment of problems of this kind the reader is referred to [1–4, 8].

Here we confine ourselves to the case in which the assumptions (A1)–(A6) hold. Basing on certain known results concerning maximal monotone multivalued mappings, [4], we can formulate the following theorems.

THEOREM 3. *Let the assumptions* (A1)–(A6) *be satisfied. Suppose that*

1) $D(A) = Y^*$ *and* A *is maximal monotone,*

2) *There exists a positive constant* $c > 0$ *with*

$$\langle A\eta - A\sigma, \eta - \sigma\rangle_Y \geqslant c\|\eta - \sigma\|_Y^2, \quad \forall\, \eta \in Y^*, \quad \forall\, \sigma \in Y^*,$$

3) $\partial\psi + \partial\alpha$ *is a maximal monotone mapping. Then the problem* (P_2) *has exactly one solution.*

From Theorem 3 and Remark 3 it follows immediately:

THEOREM 4. *Let the hypotheses of Theorem* 3 *be satisfied. Then the problem* (P) *has at least one solution.*

It must be stressed that the conditions under which the sum of maximal monotone mappings $\partial\psi$ and $\partial\alpha$ is maximal monotone seems to be of fundamental importance to obtain the existence results for the problem under consideration. Some results in this direction have been proved by BROWDER [2–4], ROCKAFELLAR [13].

4. Existence theorems and constraints compatibility conditions

Throughout this section we shall use the denotations of Sects. 1 and 2.

4.1. Statement of the principal results in Case I

In Case I the governing relations for the unknowns $(\mathbf{u}, \mathbf{T})$ is given by Eq. (2.10). In accordance with the results of Sect. 3 with Eq. (2.10) we can associate the following two problems:

1) find $\mathbf{u} \in \mathscr{V}$ such that

$$\mathbf{0} \in \mathbf{E}^*(\mathbf{K}^0 + \partial\psi)^{-1}\mathbf{E}(\mathbf{u}) + \partial\varphi(\mathbf{u}) - \mathbf{f}_0^*, \tag{4.1}$$

2) find $\mathbf{T} \in \mathscr{T}$ such that

$$\mathbf{0} \in \mathbf{K}^0\mathbf{T}^{(*)} + \partial\psi(\mathbf{T}^{(*)}) + \partial\alpha(\mathbf{T}^{(*)}). \tag{4.2}$$

In the above relations $(\mathbf{K}^0 + \partial\psi)^{-1}: \mathscr{S} \to 2^{\mathscr{S}^*}$ is the inverse of $(\mathbf{K}^0 + \partial\psi): \mathscr{S}^* \to 2^{\mathscr{S}}$ and $\alpha: \mathscr{S}^* \to \bar{R}$ is given by

$$\alpha(\boldsymbol{\sigma}^*) \overset{\text{df}}{=} \varphi^*(-\mathbf{E}^*\boldsymbol{\sigma}^* + \mathbf{f}_0^*), \quad \forall\, \boldsymbol{\sigma}^* \in \mathscr{S}^*,$$

where $\varphi^*: \mathscr{V}^* \to \bar{R}$ is the conjugate of φ, $\partial\alpha$ stands for the subdifferential of α.

The problems (4.1) and (4.2) will be termed the displacement formulation and the stress formulation of the problem under consideration, respectively.

Applying Theorem 1 to Eq. (2.11) and taking into account the equivalence between (2.10) and (2.11), we obtain the following result:

THEOREM 4.1. *The following conditions are satisfied*:

(i) *If* $(\mathbf{u}, \mathbf{T})$ *is a solution of Eq.* (2.10), *then* $\mathbf{u}$ *is a solution of Eq.* (4.1) *and* $\mathbf{T}$ *is a solution of Eq.* (4.2).

(ii) *If* $\mathbf{u}$ *is a solution of Eq.* (4.1), *then there exists* $\mathbf{T}$ *such that* $(\mathbf{u}, \mathbf{T})$ *is a solution of Eq.* (2.10).

REMARK 4.2. From Theorem 4.1 it follows that Eq. (2.10) has at least one solution if and only if Theorem (4.1) has at least one solution.

REMARK 4.3. The necessary condition for the existence of solutions of Eq. (2.10) takes the following form:

$$D(\partial\alpha)\cap\Sigma^{(*)} \neq \emptyset, \tag{4.3}$$

where $D(\partial\alpha)$ is the effective domain of $\partial\alpha$.

The condition (4.3) will be called a constraint compatibility condition. We can distinguish two aspects of this condition:

1) for the given loadings and field reactions we have to choose only such sets of all admissible displacements and of all admissible stresses that the condition (4.3) holds;

2) for the given sets of all admissible displacements and of all admissible stresses we have to choose such a system of loadings and field reactions that the condition (4.3) is satisfied.

REMARK 4.4. In Case I the existence of solutions of Eq. (4.2) does not imply in general the existence of solutions of Eq. (2.10).

4.2. Statement of the principal results in Case II

In Case II the governing relations for the basic unknown $(\tilde{\mathbf{u}}, \mathbf{T})$ is given by the system (2.14). With the system (2.14) can be associated the following two problems:

1) find $\tilde{\mathbf{u}}$ such that

$$\mathbf{0} \in \mathbf{E}_1^*(\tilde{\mathbf{K}}^0 + \partial\psi)^{-1}\mathbf{E}_1(\mathbf{u}) + \partial\varphi(\tilde{\mathbf{u}}) - \mathbf{f}_0^*; \tag{4.4}$$

2) find $\mathbf{T}$ such that

$$\mathbf{0} \in \tilde{\mathbf{K}}^0\mathbf{T}^{(*)} + \partial\psi(\mathbf{T}^{(*)}) + \partial\alpha(\mathbf{T}^{(*)}). \tag{4.5}$$

In the above relation $\tilde{\alpha}:\mathcal{S}^* \to \overline{R}$ is given by

$$\tilde{\alpha}(\boldsymbol{\sigma}^*) \overset{\mathrm{df}}{=} \tilde{\varphi}^*(-\mathbf{E}_1^*\boldsymbol{\sigma}^* + \mathbf{f}_0^*), \quad \forall\, \boldsymbol{\sigma}^* \in \mathcal{S}^*,$$

where $\tilde{\varphi}^*:\mathcal{V}^* \to \overline{R}$ is the conjugate of $\tilde{\varphi}$.

In this case, due to Korn's inequality (K_1), we can formulate stronger results than that of Case I. Namely, applying Theorem 1 and Theorem 2 to the system (2.15) and taking into account the equivalence between the system (2.14) and (2.15) we obtain

THEOREM 4.5. *The following conditions are satisfied*:

(*i*) *If* $(\tilde{\mathbf{u}}, \mathbf{T})$ *is a solution of the system* (2.14), *then* $\tilde{\mathbf{u}}$ *is a solution of Eq.* (4.4) *and* $\mathbf{T}$ *is a solution of Eq.* (4.5);

(ii) *If* $\tilde{\mathbf{u}}$ *is a solution of Eq.* (4.4), *then there exists* $\mathbf{T}$ *such that* $(\tilde{\mathbf{u}}, \mathbf{T})$ *is a solution of the system* (2.14);

(iii) *If* $\mathbf{T}$ *is a solution of Eq.* (4.5), *then there exists* $\tilde{\mathbf{u}}$ *such that* $(\mathbf{u}, \mathbf{T})$ *is a solution of the system* (2.14).

REMARK 4.6. In Case II the following conditions are equivalent:

(i) the system (2.14) has solutions;

(ii) Eq. (4.4) has solutions;

(iii) Eq. (4.5) has solutions.

REMARK 4.7. The necessary condition for the existence of solutions of the system (2.14) takes the form

$$\Sigma^{(*)} \cap D(\partial\tilde{\alpha}) \neq \emptyset. \tag{4.6}$$

The above condition will be called the constraint compatibility condition. Its interpretation is the same as that of Eq. (4.3).

THEOREM 4.8. *Let* $\mathbf{K}^0$ *be a linear continuous operator with the properties*: $D(\mathbf{K}^0) = \mathscr{S}^*$ *and*

$$\langle \mathbf{K}^0\mathbf{T}^*, \mathbf{T}^*\rangle_2 \geqslant c\|\mathbf{T}^*\|_2^2, \quad \forall\, \mathbf{T}^* \in \mathscr{S}^*.$$

Suppose that $\partial\psi + \partial\tilde{\alpha}$ *is a maximal monotone mapping. Then the system* (2.14) *has at least one solution.*

Proof. Under the assumptions $\tilde{\mathbf{K}}^0$ is maximal monotone with $D(\tilde{\mathbf{K}}^0) = \mathscr{S}^*$ and the assertion follows immediately from Theorem 4.

The basic results concerning the system (2.16) can be easily obtained in the same way as that of the system (2.14) (Korn's type inequality (K_2) holds) and therefore will be omitted.

EXAMPLE. Let us suppose that the assumptions of Case II hold (for instance, this situation takes a place if admissible displacements satisfy the classical boundary conditions on a part of the boundary of the body) and that the compliance field tensor is such that

$$m\,\mathrm{tr}(\mathbf{TT}) \leqslant \mathrm{tr}\big(\mathbf{K}(\mathbf{T})\mathbf{T}\big) \leqslant M\,\mathrm{tr}(\mathbf{TT})$$

for any regular $\mathbf{T}$, where m, M are positive constants. Then $\mathbf{K}$ can be uniquely extended to the linear continuous operator again denoted by $\mathbf{K}: \mathscr{T} \to \mathscr{S}$ with $D(\mathbf{K}) = \mathscr{T}$. It follows that the corresponding to $\mathbf{K}$ operator $\mathbf{K}^0: \mathscr{S}^* \to \mathscr{S}$, defined by Eq. (1.2), is linear continuous with $D(\mathbf{K}^0) = \mathscr{S}^*$ and $\langle \mathbf{K}^0\mathbf{T}^*, \mathbf{T}^*\rangle_2 \geqslant m\|\mathbf{T}^*\|_2^2$, $\forall \mathbf{T}^* \in \mathscr{S}^*$. Hence $\mathbf{K}^0$ is maximal monotone and due to Theorem 4.8 the existence solution problem of the system (2.14) reduces to the formulation conditions under which the sum $\partial\psi + \partial\tilde{\alpha}$ is a maximal monotone mapping.

Final remarks

The existence and uniqueness theorems for elasticity problems with constraints for displacements and stresses which lead to "two-dimensional" boundary value problems (plate and shell theory) and, in general, to some class of problems with generalized coordinates and generalized forces will be presented in [10, 11]. Minimization and minimax problems in the elasticity theory with constraints for displacements and stresses can be found in [12].

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INSTITUTE OF MECHANICS
UNIVERSITY OF WARSAW.

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