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Two existence theorems for a rigid heat conductor

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THEORY of Gårding operators is used to get existence theorems for boundary-value problems of a rigid heat conductor in a weak form.

Zastosowano teorię operatorów Gårdinga do wyprowadzenia w słabej postaci twierdzeń o istnieniu dla zagadnień brzegowych dotyczących sztywnego przewodnika ciepła.

Применена теория операторов Гардинга для вывода в слабом виде теорем существования для краевых задач, касающихся жесткого теплопроводника.

WE APPLY two theorems given by ODEN [3] to a stationary equation of a rigid heat conductor in a weak form.

Generally Ω is a bounded domain in R^3 with smooth boundary $\partial\Omega$, and U, V are reflexive separable Banach spaces such that the injection i: $U \to V$ is dense, continuous and compact. By $u_n \rightharpoonup u$ we denote weak convergence u_n to u and by U' topological dual of U. $\langle \cdot , \cdot \rangle$ is duality pairing on $U' \times U$.

We consider the problem:

For a given $f \in U'$ find $\theta \in U$ that for any $v \in U$

$$\text{(E)} \qquad \int_{\Omega} Q\big(X, \theta(X), \nabla\theta(X)\big) \cdot \nabla v(X) dX + \int_{\Omega} q\big(X, \theta(X)\big) v(X) dX = -\int_{\Omega} f(X) v(X) dX.$$

Here θ denotes the temperature, f is a density of heat sources, Q represents a vector field of flux of heat, q is a scalar field of density of heat sources depending on the temperature. We assume that we know how Q and q depend on X, θ and $\nabla\theta$. The equation (E) was obtained from the local form of the heat equation

$$\mathrm{DIV} Q\big(X, \theta(X), \nabla\theta(X)\big) = q\big(X, \theta(X)\big) + f(X),$$

(see, e.g. MARSDEN–HUGHES [4]).

We recall the following definitions from [3]:

DEFINITION 1. *$A: U \to U'$ is a Gårding operator if A can be expressed in the form $A(u) = \bar{A}(u, u)$, where $\bar{A}: U \times U \to U'$ satisfies:*

1. *$\forall v \in U$ the map $U \ni u \to \bar{A}(u, v) \in U'$ is a radially continuous*(¹) *operator from U into U'.*

2. *There exists a continuous function $H: R^+ \times R^+ \to R^+$, $R^+ = [0, +\infty)$, with the property*

$$\lim_{t \to 0^+} \frac{1}{t} H(x, ty) = 0 \quad \textit{for any} \quad x, y \in R^+$$

(¹) A is radially continuous if the function $R \ni t \to \langle A(u+tv), v \rangle$ is continuous for all $u, v \in U$.

such that for every u and v in the ball $B_m(0) = \{w \in U: ||w||_U < m\}$ *the following inequality holds*:

$$\langle \bar{A}(u, u) - \bar{A}(v, u), u-v \rangle \geqslant -H(m, ||u-v||_V).$$

3. *If* $u_n \to u$ *weakly in* U, *then for any* $v, w \in U$

$$\liminf_{n\to+\infty} \langle \bar{A}(v, u_n) - \bar{A}(v, u), u_n - u \rangle \geqslant 0,$$

$$\liminf_{n\to+\infty} \langle \bar{A}(v, u_n) - \bar{A}(v, u), w \rangle = 0.$$

DEFINITION 2. $A: U \to U'$ *is a variational Gårding operator if* A *can be expressed in the form* $A(u) = \bar{A}(u, u)$ *where* $\bar{A}: U \times U \to U'$ *satisfies*:

1. *The condition* 1. *from Def.* 1 *holds.*
2. *The condition* 2. *from Def.* 1. *holds.*
3. $\forall u \in U$ *the map* $U \ni v \to \bar{A}(u, v) \in U'$ *is a bounded operator.*
4. *If* $u_n \to u$ *weakly in* U *and* $\lim_{n\to\infty} \langle \bar{A}(u_n, u_n) - \bar{A}(u, u_n), u_n - u \rangle = 0$, *then for any* $v \in U$

$$\bar{A}(v, u_n) \rightharpoonup \bar{A}(v, u) \quad \text{weakly in} \quad U'.$$

5. *If* $u_n \to u$ *weakly in* U *and for any* $v \in U \bar{A}(v, u_n)$ *converges weakly to some element* $f \in U'$, then

$$\lim_{n\to\infty} \langle \bar{A}(v, u_n), u_n \rangle = \langle f, u \rangle.$$

We recall the two theorems which were given by ODEN [3].

THEOREM 1. *A coercive* [2] *bounded Gårding operator is surjective.*

THEOREM 2. *A coercive bounded variational Gårding operator is surjective.*

These two theorems will be used to prove the existence of a solution to two problems of the equation (E).

We shall denote by C a positive constant which is not necessarily the same at each occurrence. For a positive integer p we denote by p' such a number that $1/p + 1/p' = 1$. We shall supress the dependence on X in notations.

We denote by $||\cdot||_{W_0} m, p$ the norm in $W_0^{m,p}(\Omega)$ and by $||\cdot||_p$ the norm in $L^p(\Omega)$.

APPLICATION 1. Let $p > 2$, $f \in (W^{1,p}(\Omega))'$. We consider a stationary equation

$$A(\theta) = f,$$

where

$$A: W^{1,p}(\Omega) \to (W^{1,p}(\Omega))', \quad A(\theta) = \operatorname{DIV} Q(X, \theta(X), \nabla\theta(X)),$$

with the boundary condition

$$\theta_{|\partial\Omega} = \theta_0 (\theta_{|\partial\Omega} - \text{trace of } \theta), \quad \theta_0 \in W^{1-\frac{1}{p}, p}(\partial\Omega).$$

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[2] A is a coercive operator if $\lim_{||u||_U \to +\infty} \frac{\langle A(u), u \rangle}{||u||_U} = \infty.$

There exists $\bar{\theta} \in W^{1,p}(\Omega)$ such that $\bar{\theta}|_{\partial\Omega} = \theta_0$ (see e.g. [5, s. VII], [6 s. VI]). If a solution θ exists, then $\bar{\theta}-\theta \in W_0^{1,p}(\Omega)$, so we can formulate our problem in the following form:

For a given $f \in (W^{1,p}(\Omega))'$ find $\theta \in W_0^{1,p}(\Omega)$ such that for any $v \in W_0^{1,p}(\Omega)$

$$\langle A(\bar{\theta}+\theta), v\rangle = \langle f, v\rangle. \tag{1.1}$$

We show that if Q is suitably restricted, there exists at least one solution of the problem (1.1). We assume

I. $Q(X,\theta,\nabla\theta) = Q_0(\nabla\theta, X)+Q_1(\nabla\theta,\theta,X)$, where $Q_0(w,X)$ is Gateaux differentiable in $w \in R^3$ and continuous in $X \in \Omega$ and $Q_1(w,d,X)$ is Gateaux differentiable in $w \in R^3$ and $d \in R$ and continuous in $X \in \Omega$.

II. For any u, $v \in W^{1,p}(\Omega)$ such that $u-\bar{\theta}$, $v-\bar{\theta} \in W_0^{1,p}(\Omega)$ the following inequalities hold

a. $$\int_0^1 \frac{dQ_0(\nabla u+t\nabla v, X)}{dt}\,dt\cdot\nabla v \geqslant a_1|\nabla v|^p \quad \text{for} \quad X \in \Omega,$$

b. $$|Q_0(w,X)| \leqslant b_1(|w|^{p-1}+1) \quad \text{for} \quad w \in R^3, \quad X \in \Omega.$$

III. For any $w \in R^3, \theta \in W^{1,p}(\Omega), X \in \Omega$

a. $$|Q_1(w,\theta,X)| \leqslant a_2(1+|\theta|^q+|w|^q),$$

b. $$\left|\frac{\partial Q_1}{\partial\theta}(w,\theta,X)\right| \leqslant b_2(1+|\theta|^{q-1}+|w|^{q-1}),$$

c. $$\forall a \in R^3, \quad \frac{\partial Q_1}{\partial w}(w,\theta,X)\cdot a\cdot a \leqslant 0.$$ (³)

In these conditions a_1, b_1, a_2, b_2 are positive constants and $0 < q < p-1$.

THEOREM 1.1. *Suppose the assumptions* I, II, III *are fulfilled. Then there exists at least one solution of the problem* (1.1).

Proof. We shall show that $A_1(v) = A(\bar{\theta}+v)$ is a bounded coercitive Gårding operator from $W_0^{1,p}(\Omega)$ into $(W_0^{1,p}(\Omega))'$. We have $W_0^{1,p} \subset L^p(\Omega)$ and the imbedding is dense continuous and compact, so the assumptions about spaces U, V, are fulfilled.

Step 1. A_1 is bounded.

For $u, v \in W_0^{1,p}(\Omega)$ we have

$$\langle A(\bar{\theta}+u), v\rangle = \int_\Omega Q_0(\nabla\bar{\theta}+\nabla u, X)\cdot\nabla v\,dX + \int_\Omega Q_1(\nabla\bar{\theta}+U, \bar{\theta}+u, X)\cdot\nabla v\,dX.$$

From the Hölder's inequality and the assumption IIb we have

$$\left|\int_\Omega Q_0(\nabla\bar{\theta}+\nabla u, X)\cdot\nabla v\,dX\right| \leqslant \|Q_0(\nabla\bar{\theta}+\nabla u, X)\|_{p'}\|\nabla v\|_p \leqslant C(\||\nabla\bar{\theta}+\nabla u|^{p-1}\|_{p'}$$
$$+\|1\|_{p'})\|v\|_{W_0^{1,p}} \leqslant C(\|\bar{\theta}+u\|^{p/p'}_{W^{1,p}}+C)\|v\|_{W_0^{1,p}} \leqslant C(\|u\|^{p/p'}_{W_0^{1,p}}+C)\|v\|_{W_0^{1,p}}.$$

(³) i.e. $\sum_{i,j=1}^{3} \frac{\partial Q_{1j}}{\partial w_i}(w,\theta,X)\cdot a_i\cdot a_j \leqslant 0.$

By similar calculations we get

$$\left|\int_{\Omega} Q_1(\nabla\bar{\theta}+\nabla u, \bar{\theta}+u, X)\cdot\nabla v\,dX\right| \leqslant C(\||\bar{\theta}+u|^q\|_{p'}+\||\nabla\bar{\theta}+\nabla u|^q\|_{p'}+C)\cdot\|\nabla v\|_p.$$

Since $q < p-1$, then $p'\cdot q < p$ and we have

$$\left|\int_{\Omega} Q_1(\nabla\bar{\theta}+\nabla u, \bar{\theta}+u, X)\cdot\nabla v\,dX\right| \leqslant C(\|\bar{\theta}+u\|^q_{W^{1,p}}+C)\|v\|_{W_0^{1,p}}$$

$$\leqslant C(\|u\|^q_{W_0^{1,p}}+C)\|v\|_{W_0^{1,p}}.$$

Taking together these two estimates we have

$$\|A(\bar{\theta}+u)\|_{(W_0^{1,p}(\Omega))'} = \sup_{\|v\|_{W_0^{1,p}}=1} |\langle A(\bar{\theta}+u), v\rangle| \leqslant C(\|u\|^{p/p'}_{W_0^{1,p}}+\|u\|^q_{W_0^{1,p}}+C)$$

so A is bounded.

Step 2. A is a Gårding operator.

Let $\bar{A}(u, v) = A(u)+0\cdot v$. Then $\bar{A}(u, u) = A(u)$ and the conditions 1 and 3 in Def. 1 hold because of the assumption I. We must prove the condition 2. For $u, v \in W_0^{1,p}(\Omega)$ we have

$$\langle \bar{A}(\bar{\theta}+u, \bar{\theta}+u)-\bar{A}(\bar{\theta}+v, \bar{\theta}+u), u-v\rangle = J_0(\bar{\theta}+u, \bar{\theta}+v)+J_1(\bar{\theta}+u, \bar{\theta}+v),$$

where J_0 and J_1 correspond to Q_0 and Q_1, respectively. Let us denote $z = u-v \in W_0^{1,p}(\Omega)$. Using the assumption IIa we get

$$J_0(\bar{\theta}+u, \bar{\theta}+v) = \int_{\Omega} [Q_0(\nabla\bar{\theta}+\nabla v+\nabla z)-Q_0(\nabla\bar{\theta}+\nabla v)]\cdot\nabla z\,dX$$

$$\geqslant a_1 \int_{\Omega} |\nabla z|^p dX \geqslant C\|z\|^p_{W_0^{1,p}}$$

and similarly using the assumption IIIc we obtain ($w = \bar{\theta}+v+tz$)

$$J_1(\bar{\theta}+u, \bar{\theta}+v) = \int_{\Omega} [Q_1(\nabla\bar{\theta}+\nabla u, \bar{\theta}+u, X)-Q_1(\nabla\bar{\theta}+\nabla v, \bar{\theta}+v, X)]\nabla z\,dX$$

$$= \int_{\Omega}\int_0^1 \left[\frac{\partial Q_1(\nabla w, w, X)}{\partial(\nabla w)}\cdot\nabla z\cdot\nabla z+\frac{\partial Q_1(\nabla w_0, w, X)}{\partial w}\, z\cdot\nabla z\right] dt\,dX$$

$$\leqslant \int_{\Omega}\int_0^1 -\frac{Q_1 w, w, X}{w}\, z\cdot\nabla z\,dt\,dX.$$

Therefore by the assumption IIIb

$$J_1(\bar{\theta}+u, \bar{\theta}+v) \leqslant \int_{\Omega}\int_0^1 \left|\frac{\partial Q_1}{\partial w}(\nabla w, w, X)\cdot z\cdot\nabla z\right| dt\,dX \leqslant \int_{\Omega}\int_0^1 b_2(1+|w|^{q-1}+|\nabla w|^{q-1})$$

$$\times |z||\nabla z|\,dt\,dX.$$

We shall estimate all parts of this sum. Let s denote $h = |\nabla w|^{q-1}$ and

$$I = \int_\Omega \int_0^1 |\nabla w|^{q-1} |z| |\nabla z| \, dt \, dX.$$

Since for any $\varepsilon > 0$ and any function g

$$\int_\Omega |\nabla z| |g| \, dX \leqslant ||z||^p_{W_0^{1,p}} \cdot \frac{\varepsilon^p}{p} + \frac{1}{p' \varepsilon^{p'}} ||g||^{p'}_{p'}$$

we get, taking $g = hz$,

$$I \leqslant \int_0^1 \frac{\varepsilon^p}{p} ||z||^p_{W_0^{1,p}} dt + \int_0^1 ||hz||^{p'}_{p'} \cdot \frac{1}{p' \varepsilon^{p'}} dt.$$

We have also

$$||hz||^{p'(p-1)}_{p'} \leqslant ||z||^p_p ||h||^p_{p/(p-2)} \quad \text{and} \quad ||h||^{p/(p-1)}_{p/(p-2)} = ||\nabla w||^{\frac{p(q-1)}{p-1}}_{\frac{p(q-1)}{p-2}}.$$

Because $\dfrac{p(q-1)}{p-2} < p$, then $||\nabla w||_{\frac{p(q-1)}{p-2}} \leqslant C ||\nabla w||_p$ and therefore for $u, v \in B_m(0)$ in $W_0^{1,p}(\Omega)$

$$I \leqslant \frac{\varepsilon^p}{p} ||z||^p_{W_0^{1,p}} + \frac{1}{p' \varepsilon^{p'}} ||z||^{p/(p-1)}_p \cdot C m^{(q-1)p'}. \tag{1.2}$$

Observing that for $u, v \in B_m(0) \subset W_0^{1,p}(\Omega)$

$$\int_0^1 (||\bar{\theta} + v + tz||_p)^k dt \leqslant C \int_0^1 m^k (1 + 2t)^k dt < \bar{C}(m),$$

where $\bar{C}(m)$ is a constant depending only on m, and doing similar calculations we get

$$\int_\Omega \int_0^1 |\bar{\theta} + v + tz|^{q-1} |z| |\nabla z| \, dt \, dX \leqslant C \frac{\varepsilon^p}{p} ||z||^p_p + \bar{C}(m) \frac{1}{p' \varepsilon^{p'}} ||z||^{p/(p-1)}_{p'}. \tag{1.3}$$

Next we have

$$\int_0^1 \int_\Omega |z| |\nabla z| \, dX \, dt \leqslant \frac{\varepsilon^p}{p} ||z||^p_{W_0^{1,p}} + \frac{1}{p' \varepsilon^{p'}} ||z||^{p'}_{p'}. \tag{1.4}$$

Since $p > 2$, therefore

$$||z||^{p'}_{p'} = \int_\Omega |z|^{p'} dX \leqslant C || |z|^p ||_{p-1} \leqslant C ||z||^{p/(p-1)}_p.$$

Putting this estimate into Eq. (1.4) and taking into account Eqs. (1.2) and (1.3) we have

$$J_1(\bar{\theta} + u, \bar{\theta} + v) \leqslant \frac{3\varepsilon^p}{p} ||z||^p_{W_0^{1,p}} + \bar{C}(m) ||z||^{p/(p-1)}_p.$$

Combining thus with the estimate of J_0 we get

$$J_0(\bar{\theta} + u, \bar{\theta} + v) + J_1(\bar{\theta} + u, \bar{\theta} + v) \geqslant \left(C - \frac{3\varepsilon^p}{p} \right) ||z||^p_{W_0^{1,p}} - \bar{C}(m) ||z||^{p/(p-1)}_p.$$

If we choose ε such as to have $C-\frac{3\varepsilon^p}{p} > 0$ for $u, v \in B_m(0)$ we have

$$\langle A(\bar{\theta}+u)-A(\bar{\theta}+v), u-v\rangle \geqslant -\bar{C}(m)||u-v||_p^{p/(p-1)}.$$

Let us define

$$H: R^+ \times R^+ \to R^+, H(x, y) = \bar{C}(x) y^{p/(p-1)}.$$

H is continuous and $\lim_{t\to 0^+} \frac{1}{t} H(x, ty) = 0$ because $p/(p-1) > 1$ for $p > 2$.

Furthermore we have for $u, v \in B_m(0)$

$$\langle A(\bar{\theta}+u)-A(\bar{\theta}+v), u-v\rangle \geqslant -H(m, ||u-v||_p),$$

which means that the condition 2 of Def. 1 is satisfied.

Step 3. A is coercive.

We have for $u \in W_0^{1,p}(\Omega)$:

$$\langle A(\bar{\theta}+u), u\rangle = \langle A(\bar{\theta}), u\rangle + J_0(u+\bar{\theta}, \bar{\theta}) + J_1(u+\bar{\theta}, \bar{\theta})$$
$$\geqslant \langle A(\bar{\theta}), u\rangle + C||u||^p_{W_0^{1,p}} + J_1(u+\bar{\theta}, \bar{\theta}).$$

Using

$$|| |\nabla\bar{\theta}+\nabla u|^{q+1}||^{\frac{p}{q+1}}_{\frac{p}{q+1}} \leqslant ||\bar{\theta}+u||^p_{W^{1,p}} \quad \text{and} \quad || |\bar{\theta}+u|^q||^{p'}_{p'} \leqslant C||\bar{\theta}+u||^{qp'}_p,$$

we have from the assumptions IIb and IIIa

$$\langle A(\bar{\theta}), u\rangle + J_1(u+\bar{\theta}, \bar{\theta}) = \int_\Omega [Q_0(\nabla\bar{\theta}, X) + Q_1(\nabla u+\nabla\bar{\theta}, u+\bar{\theta}, X)]\cdot\nabla u dX$$
$$\leqslant \int_\Omega [b_1(|\nabla\bar{\theta}|^{p-1}+1) + a_2(1+|u+\bar{\theta}|^{q-1}+|\nabla u+\nabla\bar{\theta}|^{q-1})]\,|\nabla u|\,dX$$
$$\leqslant \int_\Omega [C+C(1+|u|^q+|\nabla u|^q)]\,|\nabla u|\,dX \leqslant \int_\Omega (C|\nabla u|+C|u|^q|\nabla u|+C|\nabla u|^{q+1})\,dX$$
$$\leqslant C\left[\frac{\varepsilon^p}{p}||u||^p_{W_0^{1,p}} + \frac{1}{p'\varepsilon^{p'}}(\text{means }(\Omega))^{p'}\right] + C\frac{q+1}{p}\varepsilon^{\frac{p}{q+1}}||u||^p_{W_0^{1,p}}$$
$$+\frac{1}{p''\varepsilon^{p''}}(\text{means }(\Omega))^{p''} + C\frac{\varepsilon^p}{p}||u||^p_{W_0^{1,p}}$$
$$+C\frac{1}{p'\varepsilon^{p'}}||u||^{\hat{p}}_{W_0^{1,p}} \quad \text{where} \quad p'' = \frac{p}{p-q-1}, \quad \hat{p} = p'q.$$

Therefore

$$\langle A(\bar{\theta}+u), u\rangle \geqslant C\left[\left(1-\frac{\varepsilon^p}{p}-\varepsilon^{\frac{p}{q+1}}\right)||u||^p_{W_0^{1,p}} - ||u||^{\hat{p}}_{W_0^{1,p}} - 1\right].$$

If we choose ε such as to have $1-\frac{\varepsilon^p}{p}-\varepsilon^{\frac{p}{q+1}} > 0$ we obtain

$$\lim_{||u||_{W_0^{1,p}}\to+\infty} \frac{\langle A(\bar{\theta}+u), u\rangle}{||u||_{W_0^{1,p}}} = \infty$$

because $p-1 > \hat{p}-1$.

So A_1 is a coercitive bounded Gårding operator and we conclude from Theorem 1 that A_1 is surjective, so there exists $u \in W_0^{1,p}(\Omega)$ that for any

$$v \in W_0^{1,p}(\Omega), \quad \langle A(\bar{\theta}+u), v\rangle = \langle f, v\rangle$$

and $\theta = \bar{\theta}+u \in W^{1,p}(\Omega)$ satisfies $\langle A(\theta), v\rangle = \langle f, v\rangle$ for any $v \in W_0^{1,p}(\Omega)$ and $\theta|_{\partial\Omega} = \theta_0$. Theorem 1.1 is proved.

APPLICATION 2. Let us consider the map

$$A: W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega),$$

where $1 < p < \infty$, and formulate the problem:

For a given $f \in W^{-1,p'}(\Omega)$ find $\theta \in W_0^{1,p}(\Omega)$ such as to have for any $v \in W_0^{1,p}(\Omega)$

$$\langle A(\theta), v\rangle = \int_\Omega Q\big(X, \theta(X), \nabla\theta(X)\big)\cdot \nabla v\, dX + \int_\Omega q\big(X, \theta(X)\big)\cdot v\, dX = -\int_\Omega f(X) v\, dX,$$

where

I. $Q(X, u, v): \Omega\times R\times R^3 \to R^3, \quad q(X, u): \Omega\times R \to R$

are continuous in $u \in R$, $v \in R^3$ for almost every $X \in \Omega$ and measurable in $X \in \Omega$ for every $u \in R$, $v \in R^3$.

II. There exists $k \in L^{p'}(\Omega)$ such that for $X \in \Omega$

$$|Q(X, u, v)| \leqslant C\big(|u|^{p-1}+|v|^{p-1}+|k(X)|\big),$$

$$|q(X, u)| \leqslant C\big(|u|^{p-1}+|k(X)|\big).$$

If these assumptions hold, then we can easily verify that

$$A: W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = \big(W_0^{1,p}(\Omega)\big)'.$$

We give now two lemmata which are simple consequences from the theorem 2.1 in KRASNOSIELSKIJ [1].

LEMMA 1. Let $u_n \to u$ strongly in $L^p(\Omega)$ and $v, w \in W^{1,p}(\Omega)$. Then

$$Q(X, u_n, \nabla v) \to Q(X, u, \nabla v) \quad \text{strongly in} \quad (L^{p'}(\Omega))^3,$$

$$q(X, u_n) \to q(X, u) \quad \text{strongly in} \quad L^{p'}(\Omega).$$

LEMMA 2. Let $u_n \to u$ strongly in $W^{1,p}(\Omega)$, $v \in L^p(\Omega)$. Then

$$Q(X, v, \nabla u_n) \to Q(X, v, \nabla u) \quad \text{strongly in} \quad (L^{p'}(\Omega))^3.$$

Now we can formulate an existence theorem.

THEOREM 2.1. *Let the assumptions* I, II *hold and*

3. *A is a coercitive operator.*

4. *There exists $H: R^+\times R^+ \to R^+$ continuous and such that for any $x, y \in R^+$*

$$\lim_{t\to 0^+} \frac{1}{t} H(x, ty) = 0$$

and for

$$u, v \in B_m(0) = \{w \in W_0^{1,p}(\Omega): \|\ \|_{W_0^{1,p}} < m\},$$

$$[Q(X, u, \nabla u) - Q(X, v, \nabla u)] \cdot (\nabla u - \nabla v) \geqslant -H(m, \|u-v\|_p).$$

Then A is surjective from $W_0^{1,p}(\Omega)$ onto $W^{-1,p'}(\Omega)$

Proof.

Step 1. We show that A is a variational Gårding operator.

Let us put $W_0^{1,p}(\Omega)$ as U and $L^p(\Omega)$ as V in Def. 2. We denote for $u, v, w \in W_0^{1,p}(\Omega)$

$$a_1(u, v, w) = \int_\Omega Q(X, u, \nabla v) \cdot \nabla w \, dX,$$

$$a_0(u, w) = \int_\Omega q(X, u) w(X) \, dX.$$

The map $w \to a(u, v, w) = a_1(u, v, w) + a_0(u, w)$ is continuous from $W_0^{1,p}(\Omega)$ in R for any $u, v \in W_0^{1,p}(\Omega)$. We define

$$\bar{A}: W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega) \quad \text{by} \quad \langle \bar{A}(v, u), w \rangle = a(u, v, w).$$

Then $\bar{A}(u, u) = A(u)$. Other conditions from Def. 2 are proved in the following way.

1. Because of Lemma 2 $\lim_{t \to 0} Q(X, u, \nabla v_1 + t\nabla v_2) \to Q(X, u, \nabla v_1)$ strongly in $(L^{p'}(\Omega))^3$, therefore for any $u \in W_0^{1,p}(\Omega)$ the map $v \to \bar{A}(v, u)$ is radially continuous from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$.

2. We can check this condition easily using the assumption 4.

3. We can verify similarly to the next step of the proof that the operator $u \to \bar{A}(v, u)$ is bounded for any $v \in W_0^{1,p}(\Omega)$.

4. Let $u, v, w, u_n \in W_0^{1,p}(\Omega)$, $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$. Then

$$\langle \bar{A}(v, u_n), w \rangle = \int_\Omega Q(X, u_n, \nabla w) \cdot \nabla w \, dX + \int_\Omega q(X, u_n) w(X) \, dX.$$

If we choose a subsequence u_{n_k}, $u_{n_k} \to u$ strongly in $L^p(\Omega)$, then from Lemma 1 we have $Q(X, u_{n_k}, \nabla v) \to Q(X, u, \nabla v)$ strongly in $(L^{p'}(\Omega))^3$ and $q(X, u_{n_k}) \to q(X, u)$ strongly in $L^{p'}(\Omega)$. Thus we conclude that $\bar{A}(v, u_{n_k}) \to \bar{A}(v, u)$ weakly in $W^{-1,p}(\Omega)$ for any $v \in W_0^{1,p}(\Omega)$. Because this convergence holds for any subsequence which converges strongly in $L^p(\Omega)$, then we have it for the sequence u_n.

5. Let $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$, $\bar{A}(v, u_n) \rightharpoonup f$ weakly in $W^{-1,p'}(\Omega)$ for $v \in W_0^{1,p}(\Omega)$. From Lemma 1 we have

$$Q(X, u_n, \nabla v) \to Q(X, u, \nabla v) \quad \text{strongly in} \quad (L^{p'}(\Omega))^3,$$

$$q(X, u_n) \to q(X, u) \quad \text{strongly in} \quad L^{p'}(\Omega).$$

Hence

$$a_1(u, v, u_n) \to a_1(u, v, u).$$

Using the Hölder's inequality we get from the Assumption II

$$|a_2(u_n, u_n - u)| \leqslant C\|u_n - u\|_p.$$

Choosing a subsequence $u_{n_k} \to u$ strongly in $L^p(\Omega)$ we have

$$a_2(u_{n_k}, u) \to \langle f, u\rangle - a_1(u, v, u).$$

As a result we get for the sequence u_n

$$\langle \bar{A}(v, u_n), u_n\rangle = a_1(u_n, v, u_n) + a_2(u_n, u_n) \to \langle f, u\rangle.$$

Step 2. We must show that A is a bounded operator. From the Assumption II we have

$$|\langle A(u), w\rangle| \leqslant \int_\Omega |Q(X, u, \nabla u)| \cdot |\nabla w|\, dX + \int_\Omega |q(X, u)| \cdot |w(X)|\, dX$$
$$\leqslant \int_\Omega C(|\nabla u|^{p-1} + |u|^{p-1} + |k|) \cdot |\nabla w|\, dX + C \int_\Omega (|u|^{p-1} + |k|) \cdot |w|\, dX$$
$$\leqslant C\big(\big\||u|^{p-1} + |\nabla u|^{p-1} + |k|\big\|_{p'} + \big\||u|^{p-1} + |k|\big\|_{p'}\big) \|w\|_{W_0^{1,p}}.$$

Hence

$$\|A(u)\|_{W^{-1,p'}} = \sup_{\|w\|_{W^{1,p}=}} |\langle A(u), w\rangle| \leqslant C(\|u\|_{W_0^{1,p}} + \|k\|_{p'}).$$

So A is bounded from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$. Therefore all assumptions of Theorem 2 hold and we conclude that A is surjective from $W_0^{1,p}(\Omega)$ onto $W^{-1,p'}(\Omega)$.

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