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Method of fundamental solutions
A novel theory of lifting surface in a subsonic flow

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IN THIS PAPER the method offunda mental solutions known from [4] in the case of the three-dimensional motion of fluids in the presence of bodies is developed by replacing the bodies surfaces by continuous distributions of momentum sources. To this purpose the perturbation of the uniform motion of a fluid caused by the presence of a stationary point momentum (the steady fundamental solution) is determined. Using the method of superposition, the perturbation of the uniform motion of a fluid in the presence of a three-dimensional body which is assimilable with the domain of intersection of this body by the plane xOy (the lifting surface theory) is obtained. This perturbation is given by the formula (3.2). From the boundary conditions follows the integral equation of the lifting surface in the form (4.6) or (4.9). As a particular case the solution of the plane problem as well as the Prandtl's representation and his integro-differential equation from the lifting line theory are found.

W pracy rozwinięto metodę rozwiązań podstawowych znaną z [4] dla przypadku trójwymiarowego ruchu płynów w obecności ciał drogą zastąpienia powierzchni tych ciał przez ciągłe rozkłady źródeł pędu. Określono w tym celu perturbację równomiernego ruchu płynu wywołaną obecnością stacjonarnego pędu punktowego (ustalone rozwiązanie podstawowe). Posługując się metodą superpozycji otrzymano perturbację równomiernego ruchu płynu w obecności ciała trójwymiarowego zgodnie z obszarem przecięcia tego ciała płaszczyzną xOy (teoria powierzchni nośnej). Perturbację tę podaje wzór (3.2). Z warunków brzegowych wynika równanie całkowe dla powierzchni nośnych w postaci (4.6) lub (4.9). Jako przypadek szczególny rozpatrzono rozwiązanie zagadnienia płaskiego, jak również reprezentację Pandtla i jego równanie różniczkowo-całkowe z teorii linii nośnych.

В работе развернут метод фундаментальных решений, известный из [4], для случая трехмерного движения жидкостей в присутствии тел путем замены поверхности этих тел непрерывными распределениями источников импульса. С этой целью определена пертурбация равномерного движения жидкости, вызванная присутствием стационарного точечного импульса (установившееся фундаментальное решение). Послуживаясь методом суперпозиции, получена пертурбация равномерного движения жидкости в присутствии трехмерного тела согласно с областью пересечения этого тела плоскостью xOy (теория несущей поверхности). Эту пертурбацию дает формула (3.2). Из граничных условий следует интегральное уравнение для несущей поверхности в виде (4.6) или (4.9). Как частный случай рассмотрено решение плоской задачи, как тоже представление Прандтля и его дифференциально-интегральное уравнение из теории несущих линий.

1. Introduction

THE CLASSICAL METHOD of replacing the presence of a body in an incompressible fluid by a continuous distribution of sources or vortices situated on the body surface is now well-known. The solution of the equations of motion has then the form of an integral on the body surface (continuous superposition) with *a priori* unknown density of this kind of sources of vortices. The boundary conditions lead to an integral equation for the determination of the density. From the mathematical point of view this method is rigorous. It consists in representing the solution of the Laplace equation by potentials of a simple and double layer (in the case of the incompressible fluid the continuity equation in the hypothesis of irrotational flow leads to the Laplace equation). Nevertheless from the point of view of mechanics it has no justification since the body is not a distribution of mass or vortex sources. It is true that the experiments show that some vortices are separating from a body moving in a fluid, but the appearance of these vortices is an effect due to the presence of the body. In fact, the body transmits some momentum to the fluid and this leads to the formation of these vortices. Therefore it seems natural to replace the presence of the body by a continuous distribution of momentum sources. On the other hand, this idea stems from the Cauchy principle [15] by virtue of which we admit that there exists a continuous distribution of forces of the body surface of a priori unknown density which replaces the action of this body upon the fluid. This distribution must be introduced in the equations of motion of the fluid. Another fact which pleads against the idea of assimilation of a body by a distribution of mass or vortex sources is: in order to obtain a solution satisfactory from the point of view of mechanics, we have to take also into account the layer of free vortices behind the body (see for instance the Prandtl theory of the lifting line). This idea is still artificial since the vortices behind the body are a consequence of the body. From the mathematical point of view the representation of the solution as continuous distributions of sources or vortices on the body is too poor to be able to take into account all the necessary boundary condition.

In [3] we suggested the replacement of the body surface by a continuous distribution of momentum. We applied this idea in [4] to the case of the plane steady motion of a compressible fluid using the linearized equations; the theory of thin airfoils is given as illustration. In the present paper we apply this method to the three-dimensional problem of compressible (or incompressible) fluids in subsonic flow (the supersonic case will be treated elsewhere) and, as an example, the theory of the lifting surface is considered. In this way we regain as naturally and simply as possible the more particular representations of the incompressible flow obtained in the hypothesis that the body is replaced by distributions of connected and free vortices [16, 17, 2] as well as the general representation of the compressible motion given by HOMENTCOVSCHI in [8] by means of distributional equations. We regain also the integral equations given by the quoted authors. The solution given here is valid for compressible fluids in the frame of the linearized theory and for thick airfoils.

2. Stationary fundamental solution

The perturbation $\varrho_0 V_0^2 p$, $V_0 \mathbf{v}$ produced by a point momentum source of intensity $\mathbf{f}_0$, which acts stationary at the origin of the system of coordinates, in a uniform motion whose velocity is V_0 along the Ox-axis, of a compressible fluid which obeys the law of perfect gas, is determined, in the linear approximation by the following system [3]:

$$
\begin{gathered}
M^2 \partial p/\partial x + \operatorname{div}\mathbf{v} = 0,\\
\partial \mathbf{v}/\partial x + \operatorname{grad} p = \mathbf{f}\delta(\mathbf{x}),\\
\lim_{x\to -\infty} (p, \mathbf{v}) = 0,
\end{gathered}
\tag{2.1}
$$

where ϱ_0 is the density of the unperturbed motion, $M = V_0/c_0$ ($c_0^2 \equiv \gamma p_0/\varrho_0$) is the Mach number ($c_0$ stands for the velocity of the sound) of the unperturbed motion and $\mathbf{f}$ differs from $\mathbf{f}_0$ by a scalar factor which has no importance for the subsequent considerations. The solution of the system (2.1) has been determined in [5]. In the subsonic case ($M < 1$) we obtain

$$p = -\frac{1}{4\pi}\left(f_1\frac{\partial}{\partial x}+f_2\frac{\partial}{\partial y}+f_3\frac{\partial}{\partial z}\right)\frac{1}{R},$$

$$v_1 = f_1 H(x)\,\delta(y,z)-p,$$

(2.2)
$$v_2 = f_2 H(x)\,\delta(y,z)+\frac{f_1}{4\pi}\frac{\partial}{\partial y}\frac{1}{R}+\frac{f_2}{4\pi}\frac{\partial^2}{\partial y^2}\int\limits_{-\infty}^{x}\frac{dx}{R}+\frac{f_3}{4\pi}\frac{\partial^2}{\partial y\,\partial z}\int\limits_{-\infty}^{x}\frac{dx}{R},$$

$$v_3 = f_3 H(z)\,\delta(y,z)+\frac{f_1}{4\pi}\frac{\partial}{\partial z}\frac{1}{R}+\frac{f_2}{4\pi}\frac{\partial^2}{\partial y\partial z}\int\limits_{-\infty}^{x}\frac{dx}{R}+\frac{f_3}{4\pi}\frac{\partial^2}{\partial z^2}\int\limits_{-\infty}^{x}\frac{dx}{R},$$

where

(2.2′)
$$R = \sqrt{x^2+\beta^2(y^2+z^2)}, \qquad \beta^2 = 1-M^2$$

or, after some elementary calculations,

$$v_2 = f_2 H(x)\,\delta(y,z)+\frac{f_1}{4\pi}\frac{\partial}{\partial y}\frac{1}{R}-\frac{f_2}{4\pi}\frac{\partial}{\partial y}\frac{y}{y^2+z^2}\left(1+\frac{x}{R}\right)-\frac{f_3}{4\pi}\frac{\partial}{\partial y}\frac{z}{y^2+z^2}\left(1+\frac{x}{R}\right),$$

(2.3)
$$v_3 = f_3 H(x)\,\delta(y,z)+\frac{f_1}{4\pi}\frac{\partial}{\partial z}\frac{1}{R}-\frac{f_2}{4\pi}\frac{\partial}{\partial z}\frac{y}{y^2+z^2}\left(1+\frac{x}{R}\right)-\frac{f_3}{4\pi}\frac{\partial}{\partial z}\frac{z}{y^2+z^2}\left(1+\frac{x}{R}\right).$$

This is the form of the solution given in [5]. From its form it can be seen that the perturbation is irrotational except for the Ox-axis downstream the perturbation source ($x > 0$). This result explains the presence of the vortices downstream the body which is the momentum source if we determine the perturbation of the fluid as discontinuous superposition of perturbations of the form (2.3). Nevertheless in the following we shall not use the form of the solution which follows from Eq. (2.3).

Since

$$\mathscr{F}^{-1}\left(\frac{1}{\beta^2\alpha_1^2+\alpha_2^2+\alpha_3^2}\right)=\frac{1}{4\pi R}$$

it follows

$$\left(\beta^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\frac{1}{4\pi R} = -\delta(\mathbf{x})$$

and therefore

$$\frac{1}{4\pi}\frac{\partial^2}{\partial z^2}\int\limits_{-\infty}^{x}\frac{dx}{R} = -\frac{1}{4\pi}\int\limits_{-\infty}^{x}\left(\beta^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{dx}{R}-\int\limits_{-\infty}^{x}\delta(\mathbf{x})\,dx$$
$$= -\frac{\beta^2}{4\pi}\frac{\partial}{\partial x}\frac{1}{R}-\frac{1}{4\pi}\frac{\partial^2}{\partial y^2}\int\limits_{-\infty}^{x}\frac{dx}{R}-H(x)\,\delta(y,z).$$

Substituting this into Eq. (2.2) and taking into account

$$\frac{\partial}{\partial y}\int\limits_{-\infty}^{x}\frac{dx}{R} = -\frac{y}{y^2+z^2}\left(1+\frac{x}{R}\right),$$

it follows

$$(2.4)\qquad v_3(x, y, z) = \frac{f_1}{4\pi}\frac{\partial}{\partial z}\frac{1}{R} - \frac{1}{4\pi}\left(f_2\frac{\partial}{\partial z} - f_3\frac{\partial}{\partial y}\right)\frac{y}{y^2+z^2}\left(1+\frac{x}{R}\right) - \frac{\beta^2}{4\pi}f_3\frac{\partial}{\partial x}\frac{1}{R}$$

and, similarly,

$$(2.5)\qquad v_2(x, y, z) = \frac{f_1}{4\pi}\frac{\partial}{\partial y}\frac{1}{R} + \frac{1}{4\pi}\left(f_2\frac{\partial}{\partial z} - f_3\frac{\partial}{\partial y}\right)\frac{z}{y^2+z^2}\left(1+\frac{x}{R}\right) - \frac{\beta^2}{4\pi}f_2\frac{\partial}{\partial x}\frac{1}{R}.$$

This is the form which shall be used in the following.

Obviously the fundamental solution is important not only for the theory presented in this paper but also for some other problems.

3. Theory of lifting surface. Representation of the solution

Let us now determine the perturbation produced in a subsonic uniform motion of velocity V_0 along the Ox-axis, of a compressible fluid, by the presence of a three-dimensional body whose projection on the xOy plane forms the domain D (Fig. 1). We denote by

$$(3.1)\qquad z = h_{\pm}(x, y), \quad (x, y) \in D$$

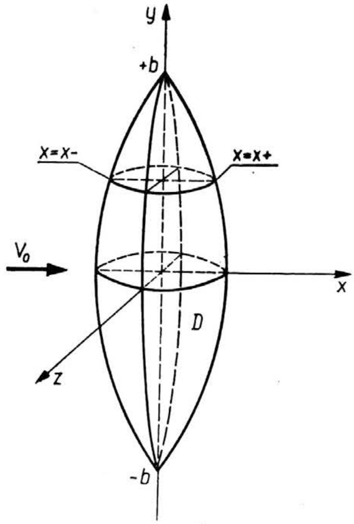

FIG. 1.

the equations of the upper and lower edge of this airfoil, the functions $h_\pm(x, y)$ are defined on D and they are smooth and uniformly bounded by a small number such that we may assume that the perturbation produced by the airfoil in the fluid satisfies the linearized equations of motion. By virtue of the discussion in the introduction we shall replace the presence of this body by continuous distributions of momentum sources of unknown intensity **f**. For the sake of simplicity we define this distribution on D and neglect the component f_2 along the span since this component is not essential (i.e. it does not influence the lifting). Taking into account Eqs. $(2.2)_1$, (2.4) and (2.5), in the exterior of D, we obtain the following solution:

$$p(x,y,z) = -\frac{1}{4\pi}\iint\limits_D g(\xi,\eta)\frac{\partial}{\partial x}\left(\frac{1}{r}\right)d\xi\,d\eta - \frac{1}{4\pi}\iint\limits_D f(\xi,\eta)\frac{\partial}{\partial z}\left(\frac{1}{r}\right)d\xi d\eta,$$

(3.2)
$$v_2(x,y,z) = \frac{1}{4\pi}\iint\limits_D \left[g(\xi,\eta)\frac{\partial}{\partial y}\left(\frac{1}{r}\right) - f(\xi,\eta)\frac{\partial}{\partial y}\frac{z}{(y-\eta)^2+z^2}\left(1+\frac{x-\xi}{r}\right)\right]d\xi\,d\eta,$$

$$v_3(x,y,z) = \frac{1}{4\pi}\iint\limits_D g(\xi,\eta)\frac{\partial}{\partial z}\left(\frac{1}{r}\right)d\xi d\eta + \frac{1}{4\pi}\iint\limits_D f(\xi,\eta)\frac{\partial}{\partial y}\left[\frac{y-\eta}{(y-\eta)^2+z^2}\left(1+\frac{x-\xi}{r}\right)\right]d\xi d\eta - \frac{\beta^2}{4\pi}\iint\limits_D f(\xi,\eta)\frac{\partial}{\partial x}\left(\frac{1}{r}\right)d\xi d\eta,$$

where, throughout the paper, we denote

(3.3)
$$r = \sqrt{(x-\xi)^2+\beta^2(y-\eta)^2+\beta^2z^2}, \quad r_0 = \sqrt{(x-\xi)^2+\beta^2(y-\eta)^2},$$
$$r_1 = \sqrt{x^2+\beta^2(y-\eta)^2+\beta^2z^2}, \quad f = f_3, \quad g = f_1.$$

The representation (3.2) reduces to that known in the literature for the incompressible fluid in the presence of the airfoil without thickness [2, 16, 17], if we take $\beta = 1$ and $g = 0$ and to that given by Homentcovschi [8] for the general case of the compressible fluid in the presence of the airfoil with thickness.

Applying the Green formula we obtain

$$\iint\limits_D f(\xi,\eta)\frac{\partial}{\partial y}\left[\frac{y-\eta}{(y-\eta)^2+z^2}\left(1+\frac{x-\xi}{r}\right)\right]d\xi d\eta = -\iint\limits_D f\frac{\partial}{\partial\eta}\left[\frac{y-\eta}{(y-\eta)^2+z^2}\left(1+\frac{x-\xi}{r}\right)\right]d\xi d\eta$$
$$= -\iint\limits_D \frac{\partial}{\partial\eta}\left[f\frac{y-\eta}{(y-\eta)^2+z^2}\left(1+\frac{x-\xi}{r}\right)\right]d\xi\,d\eta + \iint\limits_D \frac{\partial f}{\partial\eta}\frac{y-\eta}{(y-\eta)^2+z^2}\left(1+\frac{x-\xi}{r}\right)d\xi d\eta$$
$$= \oint\limits_{\partial D} f(\xi,\eta)\frac{y-\eta}{(y-\eta)^2+z^2}\left(1+\frac{x-\xi}{r}\right)d\xi + \iint\limits_D \frac{\partial f}{\partial\eta}\frac{y-\eta}{(y-\eta)^2+z^2}\left(1+\frac{x-\xi}{r}\right)d\xi d\eta$$

and from the identity

$$f\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\partial}{\partial\eta}\left(\frac{f}{r}\frac{y-\eta}{x-\xi}\right) - \frac{\partial f}{\partial\eta}\frac{y-\eta}{x-\xi}\frac{1}{r}$$

it follows that

$$-\iint_D f\frac{\partial}{\partial x}\left(\frac{1}{r}\right)d\xi\, d\eta = \oint_{\partial D}\frac{f}{r}\,\frac{y-\eta}{x-\xi}\,d\xi+\iint_D \frac{\partial f}{\partial\eta}\,\frac{y-\eta}{x-\xi}\,\frac{d\xi\, d\eta}{r}$$

therefore v_3 may be written in the form

$$v_3(x, y. z) = \frac{1}{4\pi}\iint_D g\frac{\partial}{\partial z}\left(\frac{1}{r}\right)d\xi d\eta+\frac{1}{4\pi}\oint_{\partial D} f\,\frac{y-\eta}{(y-\eta)^2+z^2}\left(1+\frac{x-\xi}{r}\right)d\xi$$
$$+\frac{1}{4\pi}\iint_D\frac{\partial f}{\partial\eta}\,\frac{y-\eta}{(y-\eta)^2+z^2}\left(1+\frac{x-\xi}{r}\right)d\xi\, d\eta+\frac{\beta^2}{4\pi}\oint_{\partial D}\frac{f}{r}\,\frac{y-\eta}{x-\xi}\,d\xi$$
$$+\frac{\beta^2}{4\pi}\iint_D\frac{\partial f}{\partial\eta}\,\frac{y-\eta}{x-\xi}\,\frac{d\xi\, d\eta}{r}. \tag{3.4}$$

This form of v_3 will lead us to the integral equation given by HOMENTCOVSCHI in [8]. We preferred the transformations prior to the passage to the limit because in the limit the ˽rals become singular.

4. Integral equation of the lifting surface

Let us now impose the boundary conditions

$$v_3(x, y, \pm 0) = h'_{\pm}(x, y), \qquad (x, y)\in D, \tag{4.1}$$

where $h' = \partial h/\partial x$, which follows from the slip condition of the fluid on the surface of the airfoil. Therefore it is necessary to calculate $p(x, y, \pm 0)$, $v_3(x, y, \pm 0)$ for $(x, y)\in D$. The first term from the expression of the perturbation p is the tangential derivative of a single layer potential, hence it is a continuous function. The second term is the normal derivative of the same potential; it then follows:

$$p(x, y, \pm 0) = -\frac{1}{4\pi}\iint_D g(\xi, \eta)\,\frac{\partial}{\partial x}\left(\frac{1}{r_0}\right)d\xi\, d\eta\pm\frac{1}{2}f(x, y) \tag{4.2}$$

whence we obtain the significance of the density $f(x, y)$

$$f(x, y) = p(x, y, +0)-p(x, y, -0) \tag{4.3}$$

which is of fundamental importance for the calculation of the lifting.

In the expression of v_3 given by Eq. (3.2) we shall use the following formula proved in the Appendix:

$$\lim_{\substack{z\to\pm 0\\ (x, y)\in D}}\iint_D f(\xi, \eta)\,\frac{\partial}{\partial y}\left[\frac{y-\eta}{(y-\eta)^2+z^2}\left(1+\frac{x-\xi}{r}\right)\right]d\xi d\eta$$
$$= PF\iint_D\frac{f(\xi, \eta)}{(y-\eta)^2}\left(1+\frac{x-\xi}{r_0}\right)d\xi\, d\eta+\beta^2\iint_D f(\xi, \eta)\,\frac{\partial}{\partial x}\left(\frac{1}{r_0}\right)d\xi\, d\eta, \tag{4.4}$$

where PF is the symbol for the "Finite Part" in the Hadamard sense. By passing to the limit it follows that

$$v_3(x, y, \pm 0) = \mp\frac{1}{2}g(x, y)+\frac{1}{4\pi}PF\iint_D\frac{f(\xi, \eta)}{(y-\eta)^2}\left(1+\frac{x-\xi}{r_0}\right)d\xi d\eta. \tag{4.5}$$

Introducing this into Eq. (4.1), adding and subtracting the two obtained equations, we find for $(x, y) \in D$

$$\frac{1}{2\pi} PF \iint_D \frac{f(\xi, \eta)}{(y-\eta)^2}\left(1+\frac{x-\xi}{r_0}\right) d\xi\, d\eta = H(x, y), \tag{4.6}$$

$$g(x, y) = h'_-(x, y) - h'_+(x, y), \tag{4.7}$$

where

$$H(x, y) = h'_+(x, y) + h'_-(x, y). \tag{4.6$_1$}$$

Equation (4.6) is the equation of the lifting surface from which f has to be determined. Under various forms for the incompressible fluid and for the airfoil without thickness it can be found in [17, 2, 1] and may be in the works of some other authors. For fluids, with chemical reactions it has a more general form given in [11]. Obviously, for airfoils without thickness we have in addition $g = 0$.

Passing to the limit in Eq. (3.4) and taking into account Eqs. (A.3) and (A.6) it follows for $(x, y) \in D$

$$\begin{aligned} v_3(x, y, \pm 0) = \mp \frac{1}{2} g(x, y) + \frac{1}{4\pi} \oint_{\partial D} \frac{f(\xi, \eta)}{y-\eta}\left(1+\frac{x-\xi}{r_0}\right) d\xi \\ + \frac{1}{4\pi} \iint_{D_=} \frac{\partial f}{\partial \eta}\left(1+\frac{x-\xi}{r_0}\right) \frac{d\xi\, d\eta}{y-\eta} + \frac{\beta^2}{4\pi} \oint_{\partial D} \frac{f(\xi, \eta)}{r_0} \frac{y-\eta}{x-\xi} d\xi \\ + \frac{\beta^2}{4\pi} \iint_{D_{\parallel}} \frac{\partial f}{\partial \eta} \frac{y-\eta}{x-\xi} \frac{d\xi\, d\eta}{r_0}, \end{aligned} \tag{4.8}$$

where we denote [8]

$$\iint_{D_=} k(x, y, \xi, \eta)\, d\xi\, d\eta = \lim_{\varepsilon \to 0} \iint_{D - D^{\varepsilon}_=} k(x, y, \xi, \eta)\, d\xi\, d\eta, \tag{4.8$_1$}$$

$D^{\varepsilon}_=$ being a strip parallel to Ox, centered in (x, y) and having the thickness 2ε. Introducing Eq. (4.8) into Eq. (4.1) and proceeding as in the above, the Homentcovschi equation [8] follows:

$$\begin{aligned} \frac{1}{2\pi} \oint_{\partial D} f(\xi, \eta) \frac{x-\xi+r_0}{(x-\xi)(y-\eta)} d\xi + \frac{1}{2\pi} \iint_{D_=} \frac{\partial f}{\partial \eta} \frac{d\xi\, d\eta}{y-\eta} \\ + \frac{1}{2\pi} \iint_{D_{\#}} \frac{\partial f}{\partial \eta} \frac{r_0\, d\xi\, d\eta}{(x-\xi)(y-\eta)} = H(x, y), \quad (x, y) \in D. \end{aligned} \tag{4.9}$$

This derivation avoids the calculation with singular integrals.

The form of the domain D imposes restrictions on the unknown $f(x, y)$. We shall denote by $x = x_-(y)$ the equation of the leading edge and by $x = x_+(y)$ the equation of the trailing edge in the xOy plane (Fig. 1). Obviously $x_-(y)$ and $x_+(y)$ are assigned functions. If the leading edge and the trailing edge are meeting at the points $(0, -b)$ and $(0, +b)$, then

$$x_-(\pm b) = x_+(\pm b). \tag{4.10}$$

If at the ends $y = \pm b$ the domain D is bounded by segments parallel to Ox, as it is for instance in the case of the rectangular wing, then we shall put the condition

$$f(x, \pm b) = 0, \quad x_- < x < x_+ \tag{4.11}$$

which, as it follows from Eq. (4.3), expresses the condition of continuity of the pressure at the side edges $y = \pm b$ (condition of the Kutta–Jucovschi type).

In the following we shall use the identity

$$\frac{1}{(y-\eta)^2}\left(1+\frac{x-\xi}{r_0}\right) = \frac{\partial}{\partial\eta}\frac{x-\xi+r_0}{(x-\xi)(y-\eta)}, \tag{4.12}$$

too. It may be easily checked.

5. Wing of infinite span (Plane problem)

In order to obtain the theory of the wing of infinite span (the plane solution), we have to assume that in any plane parallel to xOz the airfoil has the same form, that is to suppose that Eqs. (3.1) are of the form $z = h_\pm(x)$ and D is a rectangular domain $(-a < x < a, -b < y < b)$ for which $b \to \infty$. This implies to put $g = g(\xi)$ and $f = f(\xi)$ in Eq. (3.2). By denoting $s^2 = (x-\xi)^2+\beta^2 z^2$, we have

$$\lim_{b\to\infty}\int_{-b}^{b}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)d\eta = (x-\xi)\lim_{b\to\infty}\frac{y-\eta}{s^2\sqrt{s^2+\beta^2(y-\eta)^2}}\Bigg|_{\eta=-b}^{\eta=b} = -\frac{2(x-\xi)}{s^2\beta},$$

then, from Eq. (3.2) the representation given in [4] follows:

$$\begin{aligned} p(x, z) &= \frac{1}{2\pi\beta}\int_{-a}^{a}\frac{(x-\xi)g(\xi)+\beta^2 z f(\xi)}{(x-\xi)^2+\beta^2 z^2}\,d\xi, \\ v_3(x, z) &= \frac{\beta}{2\pi}\int_{-a}^{a}\frac{(x-\xi)f(\xi)-zg(\xi)}{(x-\xi)^2+\beta^2 z^2}\,d\xi. \end{aligned} \tag{5.1}$$

In the integral equation (4.6) we shall use Eq. (4.12) and we shall remark that

$$\lim_{b\to\infty}\frac{x-\xi+r_0}{(x-\xi)(y-\eta)}\Bigg|_{\eta=-b}^{\eta=b} = -\frac{2\beta}{x-\xi}.$$

The known integral equation then follows:

$$\frac{\beta}{\pi}{\int_{-a}^{a}}'\frac{f(\xi)}{\xi-x}\,d\xi = h'_+(x)+h'_-(x), \quad |x| < a. \tag{5.2}$$

6. Theory of lifting line

In the theory of the lifting line (PRANDTL's theory [14]) the unknown function is the circulation $C(y)$ on the curve which delimitates the airfoil in the section of the body with the plane $y = $ const and the thickness of the airfoil is neglected ($g = 0$). This last condition implies $v_1 = -p$, as it follows from Eq. $(2.2)_2$. In the approximation of the linear theory it follows that

$$C(y) = \oint \mathbf{v}\cdot d\mathbf{x} = \int_{x_+(y)}^{x_-(y)} v_1(x, y, -0)dx + \int_{x_-}^{x_+} v_1(x, y, +0)dx$$

$$= \int_{x_-}^{x_+} [|v_1|]dx = -\int_{x_-}^{x_+} [|p|]dx = -\int_{x_-}^{x_+} f(x, y)dx.$$

Therefore

$$C(\eta) = -\int_{\xi_-(\eta)}^{\xi_+(\eta)} f(\xi, \eta)d\xi. \tag{6.1}$$

From Eqs. (4.10) or (4.11) it follows

$$C(\pm b) = 0. \tag{6.2}$$

In the Prandtl's theory the airfoil in the plane xOy, therefore the domain D, is assimilated with the segment $(-b, +b)$. This is the hypothesis of the lifting line. In order to see what happens to the integrals from Eq. (3.2) as $[\xi_-(\eta), \eta] \to [0, \eta] \leftarrow [\xi_+(\eta), \eta]$, we shall apply the formula of the average. For a continuous function k we have

$$\lim \iint_D f(\xi, \eta)k(x, y, z, \xi, \eta)d\xi d\eta = \lim \int_{-b}^{b} d\eta \int_{\xi_-}^{\xi_+} f(\xi, \eta)k(x, y, z, \xi, \eta)d\xi$$

$$= \lim \int_{-b}^{b} k(x, y, z, \xi^*, \eta)C(\eta)d\eta = \int_{-b}^{b} k(x, y, z, 0, \eta)C(\eta)d\eta,$$

where $\xi^* \in (\xi_-, \xi_+)$. Applying this reasoning and taking into account Eq. (6.2), it follows that

$$v_1 = -p = \frac{\beta^2}{4\pi}\int_{-b}^{b} C(\eta)\frac{z}{r_1^3}\,d\eta,$$

$$v_2 = \frac{1}{4\pi}\frac{\partial}{\partial y}\int_{-b}^{b} C(\eta)\frac{z}{(y-\eta)^2+z^2}\left(1+\frac{x}{r_1}\right)d\eta$$

$$= -\frac{1}{4\pi}\int_{-b}^{b} C(\eta)\frac{\partial}{\partial\eta}\frac{z}{(y-\eta)^2+z^2}\left(1+\frac{x}{r_1}\right)d\eta = \frac{1}{4\pi}\int_{-b}^{b}\frac{dC}{d\eta}\frac{z}{(y-\eta)^2+z^2}\left(1+\frac{x}{r_1}\right)d\eta, \tag{6.3}$$

$$v_3 = -\frac{1}{4\pi}\frac{\partial}{\partial y}\int_{-b}^{b} C(\eta)\frac{y-\eta}{(y-\eta)^2+z^2}\left(1+\frac{x}{r_1}\right)d\eta - \frac{\beta^2}{4\pi}C(\eta)\frac{x}{r_1^3}\,d\eta$$

$$= -\frac{1}{4\pi}\int_{-b}^{b}\frac{dC}{d\eta}\frac{y-\eta}{(y-\eta)^2+z^2}\left(1+\frac{x}{r_1}\right)d\eta - \frac{\beta^2}{4\pi}\int_{-b}^{b} C(\eta)\frac{x}{r_1^3}\,d\eta.$$

For $\beta = 1$ the famous Prandtl's representation [12] is obtained.

In order to see what becomes of Eq. (4.6), we shall make use of a reasoning which exists in essence in the papers of WEISINGER [17] and HOMENTCOVSCHI [9, 10, 11]. On the basis of the identity (4.12) and assuming that PF commutes with the differentiation (for the simple integral this is proved in [13]), it follows that

$$PF\iint_D \frac{f(\xi,\eta)}{(y-\eta)^2}\left(1+\frac{x-\xi}{r_0}\right)d\xi\,d\eta = -\frac{\partial}{\partial y}\iint_{D_\#} f(\xi,\eta)\frac{x-\xi+r_0}{(x-\xi)(y-\eta)}\,d\xi\,d\eta$$

$$= \frac{\partial}{\partial y}\,{}'\!\!\int_{-b}^{b}\frac{C(\eta)}{y-\eta}\,d\eta - \frac{\partial}{\partial y}\iint_{D_\#} f(\xi,\eta)\frac{r_0}{(x-\xi)(y-\eta)}\,d\xi\,d\eta$$

$$= PF\int_{-b}^{b} C(\eta)\frac{\partial}{\partial y}\left(\frac{1}{y-\eta}\right)d\eta - \frac{\partial}{\partial y}\iint_{D_=} \beta f(\xi,\eta)\frac{\operatorname{sgn}(y-\eta)}{x-\xi}\,d\xi\,d\eta + O(\varepsilon^2\ln\varepsilon)$$

$$= {}'\!\!\int_{-b}^{b}\frac{dC}{d\eta}\frac{d\eta}{y-\eta} - 2\beta\iint_{D_=} f(\xi,\eta)\frac{\delta(y-\eta)}{x-\xi}\,d\xi\,d\eta + O(\varepsilon^2\ln\varepsilon)$$

$$= {}'\!\!\int_{-b}^{b}\frac{dC}{d\eta}\frac{d\eta}{y-\eta} - 2\beta\,{}'\!\!\int_{x_-(y)}^{x_+(y)}\frac{f(\xi,y)}{x-\xi}\,d\xi + O(\varepsilon^2\ln\varepsilon)$$

the significance of the term $O(\varepsilon^2\ln\varepsilon)$ being rigorously shown by HOMENTCOVSCHI in [9].

Up to this term, the integral equation (4.6) reduces to

$$\frac{\beta}{\pi}\,{}'\!\!\int_{x_-}^{x_+}\frac{f(\xi,y)}{x-\xi}\,d\xi = \frac{1}{2\pi}\,{}'\!\!\int_{-b}^{b}\frac{dC}{d\eta}\frac{d\eta}{y-\eta} - H(x,y) \equiv L(x,y). \tag{6.4}$$

Considering the right-hand side of (6.4) as known, in (6.4) we have the equation of the thin profiles from the plane flow. Imposing at the trailing edge Kutta-Jucovschi condition (i.e. f must be bounded for $x = x_+$) it follows

$$\beta f(x,y) = -\frac{1}{\pi}\sqrt{\frac{x_+-x}{x-x_+}}\,{}'\!\!\int_{x_-}^{x_+}\sqrt{\frac{t-x_-}{x_+-t}}\frac{L(t,y)}{t-x}\,dt \tag{6.5}$$

$$= -\frac{1}{2\pi}\sqrt{\frac{x_+-x}{x-x_-}}\,{}'\!\!\int_{-b}^{b}\frac{dC}{d\eta}\frac{d\eta}{y-\eta} + \frac{1}{\pi}\sqrt{\frac{x_+-x}{x-x_-}}\,{}'\!\!\int_{x_-}^{x_+}\sqrt{\frac{t-x_-}{x_+-t}}\frac{H(t,y)}{t-x}\,dt.$$

Integrating the equation on the interval (x_-, x_+) and taking into account the relation

$$\int_{x_-}^{x_+}\sqrt{\frac{x_+-x}{x-x_-}}\,dx = \frac{\pi}{2}(x_+-x_-), \tag{6.6}$$

we obtain the Prandtl's equation [12]

$$\beta C(y) = a(y)\,{}'\!\!\int_{-b}^{b}\frac{dC}{y-\eta} + 4\pi a(y)j(y), \tag{6.7}$$

where $4a(y) = x_+(y)-x_-(y)$ and

$$-4\pi a(y)j(y) = \int_{x_-}^{x_+}\sqrt{\frac{t-x_-}{x_+-t}}\,H(t,y)\,dt.$$

Appendix

Using the notation

$$K(x, y, z, \eta) = \int_{\xi_-(\eta)}^{\xi_+(\eta)} f(\xi, \eta)\left(1 + \frac{x-\xi}{r}\right) d\xi, \tag{A.1}$$

where, as in Eq. (6.1), $\xi = \xi_-(\eta)$ is the equation of the leading edge in D and $\xi = \xi_+(\eta)$ is the equation of the trailing edge, from Eqs. (4.10) or (4.11) it follows:

$$K(x, y, z, \pm b) = 0. \tag{A.2}$$

We have also

$$F(x, y, z) \equiv \iint_D f(\xi, \eta)\frac{y-\eta}{(y-\eta)^2 + z^2}\left(1 + \frac{x-\xi}{r}\right) d\xi\, d\eta = \int_{-b}^{b} \frac{(y-\eta)K(x, y, z, \eta)}{(y-\eta)^2 + z^2} d\eta. \tag{A.3}$$

As in [2] we shall introduce in addition the function

$$G(x, y, z) = \int_{-b}^{+b} \frac{zK(x, y, z, \eta)}{(y-\eta)^2 + z^2} d\eta \tag{A.4}$$

such that we have

$$G + iF = \int_{-b}^{+b} \frac{[z + i(y-\eta)]K(x, y, z, \eta)}{(y-\eta+iz)(y-\eta-iz)} d\eta = \frac{1}{i}\int_{-b}^{+b} \frac{K(x, y, z, \eta)}{\eta - \zeta} d\eta, \tag{A.5}$$

where $\zeta = y + iz$. Equation (A.5) is an integral of the Cauchy type for which we may use the Plemelj formulae of passage to the limit for $z \to 0[\zeta \to y \in (-b, +b)]$. Also, the density K depends on the variable z which tends to the limit, but as it is shown in [7] p. 77, the Plemelj formulae are valid in this case, too. Applying these formulae and separating the imaginary part, it follows that

$$F(x, y, \pm 0) = {}' \int_{-b}^{b} \frac{K(x, y, 0, \eta)}{y - \eta} d\eta = \iint_{D_=} \frac{f(\xi, \eta)}{y-\eta}\left(1 + \frac{x-\xi}{r_0}\right) d\xi\, d\eta \tag{A.6}$$

the integral in $D_=$ being defined in Eq. $(4.8)_1$. It exists because the principal value in the Cauchy sense, from which it is derived, exists.

Let us now differentiate the integral (A.5) with respect to y. It follows that

$$\frac{\partial G}{\partial y} + i\frac{\partial F}{\partial y} = \frac{1}{i}\int_{-b}^{+b} \frac{\partial K}{\partial y}\frac{d\eta}{\eta - \zeta} - \frac{1}{i}\int_{-b}^{+b} \frac{K(x, y, z, \eta)}{(\eta-\zeta)^2} d\eta.$$

The first integral is of the Cauchy type, the second is of the type of those studied by Fox [6]. Using the formulae of passage to the limit for this kind of integrals we obtain

$$\left(\frac{\partial G}{\partial y}+i\frac{\partial F}{\partial y}\right)_{\substack{z=\pm 0\\ y\in(-b,+b)}} = \pm\pi\frac{\partial K}{\partial y}(x,y,0,y)+\frac{1}{i}\,{}'\!\!\int_{-b}^{b}\frac{\partial K}{\partial y}(x,y,0,\eta)\frac{d\eta}{\eta-y}$$

$$\mp\pi\left(\frac{\partial K}{\partial\eta}\right)_{\substack{z=0\\ \eta=y}}-\frac{1}{i}\,PF\int_{-b}^{b}\frac{K(x,y,0,\eta)}{(y-\eta)^2}\,d\eta,$$

whence

$$\left(\frac{\partial F}{\partial y}\right)_{\substack{z=\pm 0\\ y\in(-b,+b)}} = {}'\!\!\int_{-b}^{b}\frac{\partial K}{\partial y}(x,y,0,\eta)\frac{d\eta}{y-\eta}+PF\int_{-b}^{b}\frac{K(x,y,0,\eta)}{(y-\eta)^2}\,d\eta.$$

This result is proved by the formula (4.4).

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