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Generalized solutions of dynamical equations in nonlinear theory of thin elastic shells

A. CHRZĘSZCZYK (KIELCE)

THE PROBLEM of existence of generalized solutions for a nonlinear system of partial differential equations describing the vibrations of thin inhomogeneous elastic shells is considered. The influence of initial deflections, initial stresses and temperature is taken into account.

W pracy sformułowano i udowodniono twierdzenia o istnieniu, jednoznaczności i regularności uogólnionych rozwiązań nieliniowego układu równań różniczkowych cząstkowych, opisującego drgania cienkiej niejednorodnej powłoki sprężystej poddanej działaniu pól temperatury oraz obciążeń zewnętrznych. Istnienia i regularności rozwiązań dowodzi się konstruując rozwiązania przybliżone metodą Faedo-Galerkina, dokonując odpowiednich oszacowań *a priori* tych rozwiązań oraz wykazując ich zbieżność do rozwiązania uogólnionego. Jednoznaczność wyprowadza się z odpowiedniej równości energetycznej.

В работе сформулированы и доказаны теоремы существования, единственности и регулярности обобщенных решений нелинейной системы дифференциальных уравнений в частных производных, описывающей колебания тонкой неоднородной упругой оболочки, подвергнутой действию полей температур, а также внешних нагрузок. Существование и регулярность решений доказывается, строя приближенные решения методом Фаэдо-Галеркина, производя соответствующие оценки априори этих решений, а также доказывая их сходимость к обобщенному решению. Единственность выводится из соответствующего энергетического равенства.

1. Introduction

THIS PAPER is concerned with the existence, uniqueness and regularity of generalized solutions of a pair of nonlinear fourth-order, partial differential equations with homogeneous boundary conditions and nonhomogeneous initial conditions, describing the vibrations of this inhomogeneous, prestressed, elastic shells with initial deflection and subjected to an arbitrary distribution of temperature.

The problem of vibrations of a homogeneous plate, but without accounting for temperature, initial stresses and initial deflections was dealt with in [1–3]. The procedure used in the present paper is similar to that in [1–3]. Through the paper we shall consider functions of a point $x = (x_1, x_2)$ of two-dimensional space R^2 and time t. The symbol Ω will denote a bounded domain of the space R^2, $\partial\Omega$ its boundary and $]0, T[$ — an open interval of R^1. The problem to be discussed deals with the following system of equations, see [4] Eqs. (4.17) and (4.18) — in the static case.(1)

$$\varrho h\ddot{w} - \Delta_{D_0}\ddot{w} + \Delta^2_{1D_1} w + \mu\Delta^2_{D_1} w = L(w+w^{(0)}, F+F^{(0)}) + KF + p + \Delta M_T, \tag{1.1}$$

$$\Delta^2_{1D_2} F - \mu\Delta^2_{D_2} F = -\frac{1}{2} L(w+2w^{(0)}, w) - Kw - \Delta(N_T D_2), \tag{1.2}$$

(1) The inclusion of terms $\varrho h\ddot{w} - \Delta_{D_0}\ddot{w}$ and $F^{(0)}$ in Eqs. (1.1), (1.2) is explained in [9] and [10].

where

$$\left.\begin{aligned} &(1.3) \quad \Delta^2_{1G} = \frac{\partial^2}{\partial x_1^2}\left(G\frac{\partial^2}{\partial x_1^2}\right) + 2\frac{\partial^2}{\partial x_1 \partial x_2}\left(G\frac{\partial^2}{\partial x_1 \partial x_2}\right) + \frac{\partial^2}{\partial x_2^2}\left(G\frac{\partial_2}{\partial x_2^2}\right) \\ &(1.4) \quad \Delta^2_G = \frac{\partial^2}{\partial x_1^2}\left(G\frac{\partial^2}{\partial x_2^2}\right) - 2\frac{\partial^2}{\partial x_1 \partial x_2}\left(G\frac{\partial^2}{\partial x_1 \partial x_2}\right) + \frac{\partial^2}{\partial x_2^2}\left(G\frac{\partial^2}{\partial x_1^2}\right) \end{aligned}\right\} \quad G = D_1, D_2,$$

$$\Delta_{D_0} = \frac{\partial}{\partial x_1}\left(D_0\frac{\partial}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(D_0\frac{\partial}{\partial x_2}\right), \quad D_0 = \frac{\varrho h^3}{12}, \tag{1.5}$$

$$K = k_2\frac{\partial^2}{\partial x_1^2} + k_1\frac{\partial^2}{\partial x_2^2}; \tag{1.6}$$

$$\begin{aligned} L(f, g) &= \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 g}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_2^2}\frac{\partial^2 g}{\partial x_1^2} - 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\frac{\partial^2 g}{\partial x_1 \partial x_2}, \\ \Delta &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \\ \ddot{w} &= \frac{\partial^2 w}{\partial t^2}, \end{aligned} \tag{1.7}$$

$$D_1 = \frac{1}{1-\mu^2}\int_{-h/2}^{h/2}(x_3 - x_3^0)^2 E\,dx_3, \tag{1.8}$$

$$D_2 = \left(\int_{-h/2}^{h/2} E\,dx_3\right)^{-1}, \tag{1.9}$$

$$M_T = \frac{1}{1-\mu}\int_{-h/2}^{h/2}(x_3 - x_3^0)E\alpha\,dx_3, \tag{1.10}$$

$$N_T = \frac{1}{1-\mu}\int_{-h/2}^{h/2} E\alpha T\,dx_3 \tag{1.11}$$

and

$\varrho = \varrho(x_1, x_2)$ the mass density,
$h = \text{const} > 0$ the thickness of the shell,
$\mu = \text{const}$ Poisson's ratio,
$\alpha = \alpha(x_1, x_2, x_3)$ the coefficient of thermal expansion,
$T = T(x_1, x_2, x_3)$ the temperature distribution,
$E = E(x_1, x_2, x_3)$ Young's modulus,
x_0^3 the solution of the equation

$$\int_{-h/2}^{h/2}(x_3 - x_3^0)E\,dx_3 = 0,$$

$k_i = k_i(x_1, x_2, t)$, $i = 1, 2$ the curvatures of the shell,

$\left.\begin{aligned} w^{(0)} &= w^{(0)}(x_1, x_2) \\ F^{(0)} &= F^{(0)}(x_1, x_2) \end{aligned}\right\}$ the initial deflection and the initial stress function,

$p = p(x_1, x_2, t)$ the normal load,

$\left.\begin{array}{l} w = w(x_1, x_2, t) \\ F = F(x_1, x_2, t) \end{array}\right\}$ the normal deflection and Airy stress function.

We shall consider the boundary conditions

$$w = \frac{\partial w}{\partial n} = F = \frac{\partial F}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \times]0, T[\tag{1.12}$$

and the initial conditions ($w_0 = w_0(x_1, x_2)$, $w_1 = w_1(x_1, x_2)$, $(x_1, x_2) \in \Omega$)

$$w(0) = w_0, \quad \dot{w}(0) = \frac{\partial w(0)}{\partial t} = w_1 \quad \text{in} \quad \Omega, \tag{1.13}$$

(in general $w_0 \neq w^{(0)}$).

2. Some function spaces

Let Ω be a bounded domain in R^2. Let $W_p^l(\Omega)$ ($l = 0, 1, 2, \ldots; 1 < p < \infty$) be the collection of all functions w on Ω which have generalized derivatives D^α with respect to x_1, x_2 of all orders $|\alpha| \leqslant l$ and lie in $L_p(\Omega)$ where

$$\alpha = (\alpha_1, \alpha_2), \quad \alpha_1, \alpha_2 = 0, 1, 2, \ldots, \quad |\alpha| = \alpha_1 + \alpha_2, \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}.$$

For $w \in W_p^l(\Omega)$ we define the norm

$$||w||_{l,p} = \Big(\int_\Omega \sum_{|\alpha| \leqslant l} |D^\alpha w|^p dx\Big)^{1/p}.$$

By $\mathring{W}_p^l(\Omega)$ we mean the closure in $W_p^l(\Omega)$ of the set of all infinitely differentiable functions with compact support in Ω.

By $W_p^{-l}(\Omega)$ ($l = 0, 1, 2, \ldots; 1 < p < \infty$), we denote the space of all continuous linear functionals on $\mathring{W}_g^l(\Omega)$, $\dfrac{1}{p} + \dfrac{1}{g} = 1$.

If $f \in W_p^{-l}(\Omega)$, $g \in \mathring{W}_q^l(\Omega)$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then (f, g) denotes the value of the linear functional f at g. If $f, g \in L_2(\Omega)$, then $(f, g) = \int_\Omega fg dx$.

Let X be a Banach space with norm $||\cdot||_X$. Let $L^p(0, T; X)$ ($1 \leqslant p \leqslant \infty$) be the space of functions f defined on $[0, T]$ with values in X, strongly measurable over $[0, T]$ and such, that

$$||f||_{L^p(0,T;X)} < \infty,$$

where

$$||f||_{L^p(0,T;X)} = \begin{cases} \Big(\int_0^T ||f(t)||_X^p dt\Big)^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname*{supess}_{t \in [0,T]} ||f(t)||_X, & p = \infty. \end{cases}$$

If $f \in L^p(0, T; X)$, then the derivatives $\dot{f}, \ddot{f}, \dddot{f}, \ldots$ are to be understood in the vector—valued distributional sense (see [5]).

3. Definition of a generalized solution

Let us define the following bilinear forms:

$$B_0[\phi, \psi] = \int_{\Omega} \frac{\varrho h^3}{12}\left(\frac{\partial \phi}{\partial x_1}\frac{\partial \psi}{\partial x_1} + \frac{\partial \phi}{\partial x_2}\frac{\partial \psi}{\partial x_2}\right) dx, \tag{3.1}$$

$$B_1[\phi, \psi] = B_{1D_1}[\phi, \psi] + \mu B_{D_1}[\phi, \psi], \tag{3.2}$$

$$B_2[\phi, \psi] = B_{1D_2}[\phi, \psi] - \mu B_{D_2}[\phi, \psi], \tag{3.3}$$

where

$$B_{1G}[\phi, \psi] = \int_{\Omega} G\left(\frac{\partial^2 \phi}{\partial x_1^2}\frac{\partial^2 \psi}{\partial x_1^2} + 2\frac{\partial^2 \phi}{\partial x_1 \partial x_2}\frac{\partial^2 \psi}{\partial x_1 \partial x_2} + \frac{\partial^2 \phi}{\partial x_2^2}\frac{\partial^2 \psi}{\partial x_2^2}\right) dx, \tag{3.4}$$

$$B_G[\phi, \psi] = \int_{\Omega} G\left(\frac{\partial^2 \phi}{\partial x_1^2}\frac{\partial^2 \psi}{\partial x_2^2} - 2\frac{\partial^2 \phi}{\partial x_1 \partial x_2}\frac{\partial^2 \psi}{\partial x_1 \partial x_2} + \frac{\partial^2 \phi}{\partial x_2^2}\frac{\partial^2 \psi}{\partial x_1^2}\right) dx, \tag{3.5}$$

$$G \in \{D_1, D_2\}.$$

Let us suppose that $\varrho, D_1, D_2 \in L_\infty(\Omega)$(2), $k_1, k_2 \in L_\infty(\Omega \times]0, T[)$; $w^{(0)}, F^{(0)} \in \mathring{W}_2^2(\Omega)$; $p \in L^2(0, T; W_2^{-2}(\Omega))$; $\Delta M_T, \Delta(N_T D_2) \in W_2^{-2}(\Omega)$.

DEFINITION 1. *A pair of functions* $w, F \in L^\infty(0, T; W_2^2(\Omega))$ *such that* $\dot{w} \in L^\infty(0, T; \mathring{W}_2^1(\Omega))$ *is said to be a generalized solution of the problem* (1.1), (1.2), (1.12), (1.13) *if it satisfies the integral identities*

$$-\int_0^T (\varrho h \dot{w}(t), \dot{\eta}(t))dt - \int_0^T B_0[\dot{w}(t), \dot{\eta}(t)]dt + \int_0^T B_1[w(t), \eta(t)]dt$$

$$= \int_0^T \left(L\left(w(t)+w^{(0)}, F(t)+F^{(0)}\right), \eta(t)\right)dt + \int_0^T (KF(t), \eta(t))dt$$

$$+\int_0^T (p(t), \eta(t))dt + \int_0^T (\Delta M_T, \eta(t))dt + (\varrho h w_1, \eta(0)) + B_0[w_1, \eta(0)], \tag{3.6}$$

$$\forall \eta \in V = \{\eta \in L^2(0, T; \mathring{W}_2^2(\Omega)) : \dot{\eta} \in L^2(0, T; \mathring{W}_2^1(\Omega)), \eta(T) = 0\},$$

$$\int_0^T B_2[F(t), \psi(t)]dt = -\frac{1}{2}\int_0^T \left(L\left(w(t)+2w^{(0)}, w(t)\right), \psi(t)\right)dt \tag{3.7}$$

$$-\int_0^T (Kw(t), \psi(t))dt - \int_0^T (\Delta(N_T D_2), \psi(t))dt, \quad \forall \psi \in L^2(0, T; \mathring{W}_2^2(\Omega)),$$

and the condition

$$w(0) = w_0, \quad \textit{in the sense explained in } [3]. \tag{3.8}$$

A correctness of the condition (3.8) can be justified in a manner similar to that for an isothermal homogeneous plate (see [3]).

(2) $f \in L_\infty(Q)$, ($Q = \Omega$ or $Q = \Omega \times]0, T[$ if $|f(z)| \leqslant$ const for a.e. $z \in Q$.

4. Existence of generalized solutions

Let us suppose that

(A.1) $$\Omega \subset R^2$$

is a bounded domain having the cone property ([6] p. 66),

(A.2) $$w^{(0)}, F^{(0)} \in \mathring{W}_2^2(\Omega), \quad D^\alpha F^{(0)} \in L_\infty(\Omega), \quad |\alpha| = 1,$$

(A.3) $$w_0 \in \mathring{W}_2^2(\Omega), \quad w_1 \in \mathring{W}_2^1(\Omega),$$

(A.4) $$p \in L^2(0, T; W_2^{-1}(\Omega)), \quad \Delta M_T \in W_2^{-1}(\Omega),$$
$$\Delta(N_T D_2) \in W_2^{-2}(\Omega),$$

(A.5) $$D^\beta k_i, \ \dot{k}_i \in L_\infty(\Omega \times]0, T[), \quad |\beta| \leqslant 1, \quad i = 1, 2.$$

(A.6) There exist positive constants c_i, $i = 1, 2, 3, 4$ such that
$$c_1 \leqslant \varrho \leqslant C_2,$$
$$c_3 \leqslant E \leqslant c_4.$$

We have the following result:

THEOREM 1. *If the conditions* (A.1)–(A.6) *are satisfied, then there exists at least one generalized solution of the problem* (1.1), (1.2), (1.12), (1.13).

Proof. The proof is realized in three steps (see [2]).

I.

Using the Faedo–Galerkin method we construct an approximate solution (w_m, F_m) in such a way that w_m is postulated in the form

(4.1) $$w_m = \sum_{i=1}^{m} g_{im}\phi_i,$$

where the ϕ_i, $i = 1, 2, \ldots$ form a basis in the space $\mathring{W}_2^2(\Omega)$ (i.e. the set of all linear combinations of functions ϕ_i is dense in $\mathring{W}_2^2(\Omega)$). The g_{im} are real-valued funtions of $t \in [0, T]$ that satisfy the Faedo–Galerkin system of ordinary differential equations

(4.2) $$(\varrho h\ddot{w}_m(t), \phi_j) + B_0[\ddot{w}_m(t), \phi_j] + B_1[w_m(t), \phi_j] = \big(L(w_m(t) + w^{(0)}, F_m(t) + F^{(0)}), \phi_j\big) + \big(KF_m(t), \phi_j\big) + \big(p(t) + \Delta M_T, \phi_j\big), \quad 1 \leqslant j \leqslant m,$$

together with the initial conditions

(4.3) $$w_m(0) = w_{0m},$$

(4.4) $$\dot{w}_m(0) = w_{1m},$$

where w_{im}, $i = 0, 1$ is the orthogonal projection of w_i upon the space spanned by $\{\phi_1, \ldots, \phi_m\}$.

The function F_m on the right-hand side of Eq. (4.2) is the unique solution of the following generalized Dirichlet problem:

(4.5) $$B_2[F_m(t), \phi] = -\frac{1}{2}\big(L(w_m(t) + 2w^{(0)}, w_m(t)), \phi\big) - \big(Kw_m(t) + \Delta(N_T D_2), \phi\big), \quad \forall \phi \in \mathring{W}_2^2(\Omega), \quad \text{(see [7] p. 99).}$$

The existence of local solutions of the problem (4.2), (4.3), (4.4) follows from the general theory of ordinary differential equations.

II.

We obtain *a priori* estimates for the approximate solution. By multiplying the j-th equation of Eqs. (4.2) by $\dot{g}_{jm}$ and taking the sum with respect to j we obtain

$$(4.6)\quad \big(\varrho h\ddot{w}_m(t), \dot{w}_m(t)\big)+B_0[\ddot{w}_m(t), \dot{w}_m(t)]+B_1[w_m(t), \dot{w}_m(t)] = \big(L(w_m(t), F_m(t)), \dot{w}_m(t)\big)+\big(L(w^{(0)}, F_m(t)), \dot{w}_m(t)\big) +\big(L(w_m(t)+w^{(0)}, F^{(0)}), \dot{w}_m(t)\big)+\big(KF_m(t), \dot{w}_m(t)\big)+\big(p(t)+\Delta M_T, \dot{w}_m(t)\big).$$

By differentiating Eq. (4.5) with respect to t and putting $\phi = F_m(t)$ we obtain

$$(4.7)\quad B_2[\dot{F}_m(t), F_m(t)] = -\big(L(w_m(t), \dot{w}_m(t)), F_m(t)\big) -\big(L(w^{(0)}, \dot{w}_m(t)), F_m(t)\big)-\big((Kw_m(t))^{\cdot}, F_m(t)\big).$$

By adding Eqs. (4.6) and (4.7) together side by side and integrating over $[0, t]$ we arrive at the following equality:

$$(4.8)\quad \frac{1}{2}\Big(\big(\varrho h\dot{w}_m(t), \dot{w}_m(t)\big)+B_0[\dot{w}_m(t), \dot{w}_m(t)]+B_1[w_m(t), w_m(t)] +B_2[F_m(t), F_m(t)]\Big) = \frac{1}{2}\Big(\big(\varrho h\dot{w}_m(0), \dot{w}_m(0)\big)+B_0[\dot{w}_m(0), \dot{w}_m(0)] +B_1[w_m(0), w_m(0)]+B_2[F_m(0), F_m(0)]\Big)+\int_0^t \big(L(w_m(s)+w^{(0)}, F^{(0)}), \dot{w}_m(s)\big)ds +\int_0^t \big(KF_m(s), \dot{w}_m(s)\big)ds-\int_0^t \big((Kw_m(s))^{\cdot}, F_m(s)\big)ds+\int_0^t \big(p(s)+\Delta M_T, \dot{w}_m(s)\big)ds.$$

From Eq. (A.3) we have the estimate

$$(4.9)\quad \big(\varrho h\dot{w}_m(0), \dot{w}_m(0)\big)+B_0[\dot{w}_m(0), \dot{w}_m(0)]+B_1[w_m(0), w_m(0)]+B_2[F_m(0), F_m(0)] \leqslant \text{const}.$$

As in [1] we can prove that the following simple inequalities hold: (c_i, $i = 5, 6, \ldots$ the positive constants)

$$(4.10)\quad \big(L(w_m(s)+w^{(0)}, F^{(0)}), \dot{w}_m(s)\big) \leqslant c_5\|w_m(s)+w^{(0)}\|_{2,2} \times \|\dot{w}_m(s)\|_{1,2}\Big(\sum_{|\alpha|=1}\|D^\alpha F^{(0)}\|_{L_\infty(\Omega)}\Big) \leqslant c_6\|w_m(s)\|_{2,2}\|\dot{w}_m(s)\|_{1,2} +c_7\|\dot{w}_m(s)\|_{1,2} \leqslant c_8+c_9\big(\|w_m(s)\|^2_{2,2}+\|\dot{w}_m(s)\|^2_{1,2}\big),$$

$$(4.11)\quad \big((Kw_m(s))^{\cdot}, F_m(s)\big) \leqslant c_{10}\big(\|w_m(s)\|^2_{2,2}+\|\dot{w}_m(s)\|^2_{1,2}+\|F_m(s)\|^2_{2,2}\big),$$

$$(4.12)\quad \big(KF_m(s), \dot{w}_m(s)\big) \leqslant c_{11}\big(\|w_m(s)\|^2_{2,2}+\|\dot{w}_m(s)\|^2_{1,2}+\|F(s)\|^2_{2,2}\big),$$

$$(4.13)\quad \big(p(s)+\Delta M_T, \dot{w}_m(s)\big) \leqslant c_{12}+c_{13}\|\dot{w}_m(s)\|^2_{1,2}.$$

From Eq. (A.6) we have

$$(4.14)\quad \|\dot{w}_m(s)\|^2_{0,2}+\|\dot{w}_m(s)\|^2_{1,2}+\|w_m(s)\|^2_{2,2}+\|F_m(s)\|^2_{2,2} \leqslant c_{14}\Big(\big(\varrho h\dot{w}_m(s), \dot{w}_m(s)\big)+B_0[\dot{w}_m(s), \dot{w}_m(s)]+B_1[w_m(s), w_m(s)]+B_2[F_m(s), F_m(s)]\Big).$$

The relations (4.8)–(4.14) imply

$$(4.15)\qquad \|\dot{w}_m(t)\|_{0,2}^2+\|\dot{w}_m(t)\|_{1,2}^2+\|w_m(t)\|_{2,2}^2+\|F_m(t)\|_{2,2}^2 \leqslant C_{15} + c_{16}\int\limits_0^t \big(\|\dot{w}_m(s)\|_{0,2}^2+\|\dot{w}_m(s)\|_{1,2}^2+\|w_m(s)\|_{2,2}^2+\|F_m(s)\|_{2,2}^2\big)ds.$$

Due to Gronwall's inequality (see [5] p. 298) we have

$$(4.16)\qquad \|\dot{w}_m(t)\|_{1,2} \leqslant \text{const},\qquad \|w_m(t)\|_{2,2} \leqslant \text{const},\qquad \|F_m(t)\|_{2,2} \leqslant \text{const}.$$

From the estimate (4.16) we conclude that the local solutions of Eqs. (4.2), (4.3), and (4.4) can be extended over the entire interval $[0, T]$.

III. Passage to the limit

Another conclusion from the estimates (4.16) is that it is possible to choose subsequences w_k and F_k with the following properties:

$$(4.17)\qquad \begin{aligned} &\dot{w}_k \xrightarrow[k\to\infty]{} \dot{w} \quad \text{weakly star in} \quad L^\infty\big(0, T; \mathring{W}_2^1(\Omega)\big),^{(3)} \\ &w_k \xrightarrow[k\to\infty]{} w \quad \text{weakly star in} \quad L^\infty\big(0, T; \mathring{W}_2^2(\Omega)\big), \\ &F_k \xrightarrow[k\to\infty]{} F \quad \text{weakly star in} \quad L^\infty\big(0, T; \mathring{W}_2^2(\Omega)\big), \end{aligned}$$

for some $w, F \in L^\infty\big(0, T; \mathring{W}_2^2(\Omega)\big)$, $\dot{w} \in L^\infty\big(0, T; \mathring{W}_2^1(\Omega)\big)$.

Let the functions ψ_j, $1 \leqslant j \leqslant j_0$ belong to the space $C^1([0, T])$ and satisfy the condition $\psi_j(T) = 0$. Let us assume that

$$\eta(x, t) = \sum_{j=1}^{j_0} \psi_j(t)\phi_j(x).$$

From the equality (4.2) it follows that for $m = k > j_0$ we have

$$(4.18)\qquad -\int\limits_0^T \big(\varrho h\dot{w}_k(t), \dot{\eta}(t)\big)dt - \int\limits_0^T B_0[\dot{w}_k(t), \dot{\eta}(t)]dt + \int\limits_0^T B_1[w_k(t), \eta(t)]dt = \int\limits_0^T \Big(L\big(w_k(t)+w^{(0)}, F_k(t)+F^{(0)}\big), \eta(t)\Big)dt + \int\limits_0^T \big(KF_k(t), \eta(t)\big)dt + \int\limits_0^T \big(p(t)+\Delta M_T, \eta(t)\big)dt + \big(\varrho h w_{1k}, \eta(0)\big) + B_0[w_{1k}, \eta(0)].$$

By taking the limit of both sides of Eq. (4.18) and using the arguments similar to that in [2] p. 62 we conclude that the functions w, F satisfy the conditions (3.6) and (3.7) of the definition 1. A verification of the condition (3.8) is very similar to that in [2]. Thus we see that the pair w, F is the generalized solution of the problem (1.1), (1.2), (1.12), (1.13), q.e.d.

(3) If X is the reflexive Banach space and X^* its dual space, then $f_n \to f$ weakly star in $L^\infty(0, T; X)$ if

$$\int\limits_0^T (f_n(t), \varphi(t))\,dt \to \int\limits_0^T (f(t), \varphi(t))\,dt, \quad \forall \varphi \in L^1(0, T; X^*).$$

5. Uniqueness of the generalized solution

By imposing somewhat stronger requirments on the "data" we can prove that the generalized solution of the problem (1.1), (1.2), (1.12), (1.13) is unique. To be more precise, let us suppose that in addition to Eqs. (A.1)–(A.6) the following assumptions are satisfied:

(A.7) $$\Omega \subset R^2 \quad \text{is of class} \quad C^3,$$

(A.8) $$F^{(0)} \in W_p^3(\Omega)$$

(A.9) $$\Delta(N_T D_2) \in W_p^{-1}(\Omega) \Bigg\} \quad 1 < p < 2,$$

(A.10) $$E \in C^1(\overline{\Omega}).$$

THEOREM 2. *If the conditions* (A.1)–(A.10) *are satisfied, then there exists one and only one generalized solution of the problem* (1.1), (1.2), (1.12), (1.13).

Proof. Let $\{\hat{w}_1, \hat{F}_1\}$ and $\{\hat{w}_2, \hat{F}_2\}$ be two possible generalized solutions. If we set $\hat{w} = \hat{w}_1 - \hat{w}_2$, $\hat{F} = \hat{F}_1 - \hat{F}_2$, then from Eqs. (3.6), (3.7) it follows that

(5.1) $$-\int_0^T (\varrho h\dot{\hat{w}}, \dot{\eta})\,dt - \int_0^T B_0[\dot{\hat{w}}, \dot{\eta}]\,dt + \int_0^T B_1[\hat{w}, \eta]\,dt = \int_0^T \big(L(\hat{w}_1 + w^{(0)}, \hat{F}), \eta\big)dt + \int_0^T \big(L(\hat{w}, \hat{F}_2 + F^{(0)}), \eta\big)dt + \int_0^T (K\hat{F}, \eta)\,dt, \quad \forall \eta \in V \quad \big(\text{see (3.6)}\big),$$

(5.2) $$\int_0^T B_2[\hat{F}, \psi]\,dt = \frac{1}{2}\int_0^T \big(L(\hat{w} - 2\hat{w}_1 - 2w^{(0)}, \hat{w}), \psi\big)dt - \int_0^T (K\hat{w}, \psi)\,dt, \quad \forall \psi \in L^2\big(0, T; \mathring{W}_2^2(\Omega)\big),$$

(5.3) $$\hat{w}(0) = 0.$$

A simple generalization of considerations of the paper [2] gives

(5.4) $$\hat{f} \in L^\infty\big(0, T; W_2^{-1}(\Omega)\big), \qquad \|\hat{f}(t)\|_{-1,2} \leqslant \hat{c}_1 \|\hat{w}(t)\|_{2,2} \quad \text{a.e.on } [0, T],$$

where

$$\hat{f} = L(\hat{w}_1 + w^{(0)}, \hat{F}) + L(\hat{w}, \hat{F}_2 + F^{(0)}) - K\hat{F}, \qquad \hat{c}_1 > 0 \text{ — a constant}.$$

By using the procedure similar to that used in [2] we can prove that the following equality holds true:

(5.5) $$\frac{1}{2}\big((\varrho h\dot{\hat{w}}(t), \dot{\hat{w}}(t)) + B_0[\dot{\hat{w}}(t), \dot{\hat{w}}(t)] + B_1[\hat{w}(t), \hat{w}(t)]\big) = \int_0^t \big(\hat{f}(\sigma), \dot{\hat{w}}(\sigma)\big)d\sigma.$$

According to the inequality (5.4) we have

(5.6) $$\int_0^t \big(\hat{f}(\sigma), \dot{\hat{w}}(\sigma)\big)d\sigma \leqslant \hat{c}_2 \int_0^t \big(\|\hat{w}(\sigma)\|_{2,2}^2 + \|\dot{\hat{w}}(\sigma)\|_{1,2}^2\big)d\sigma.$$

The assumption (A.6) implies

$$(5.7) \quad \|\dot{\hat{w}}(t)\|_{0,2}^2+\|\dot{\hat{w}}(t)\|_{1,2}^2+\|\hat{w}(t)\|_{2,2}^2 \leqslant \hat{c}_3\big((\varrho h\dot{\hat{w}}(t), \dot{\hat{w}}(t))+B_0[\dot{\hat{w}}(t), \dot{\hat{w}}(t)]+B_1[\hat{w}(t), \hat{w}(t)]\big).$$

The relations (5.5)–(5.7) give the estimate

$$(5.8) \quad \|\dot{\hat{w}}(t)\|_{0,2}^2+\|\dot{\hat{w}}(t)\|_{1,2}^2+\|\dot{\hat{w}}(t)\|_{2,2}^2 \leqslant \hat{c}_4\int_0^t (\|\dot{\hat{w}}(s)\|_{0,2}^2+\|\dot{\hat{w}}(s)\|_{1,2}^2+\|\dot{\hat{w}}(s)\|_{2,2}^2)ds.$$

Thus $\hat{w} = 0$, $\hat{F} = 0$ and the theorem is proved.

6. Regularity of the generalized solution

Concerning the regularity of the generalized solution of the problem (1.1), (1.2), (1.12), (1.13), we have the following result. Let us suppose that the conditions (A.1)–(A.10) are satisfied and consider the following new asumptions.

(A.11) Ω is the bounded domain of class C^4,

(A.12) $w^{(0)}, F^{(0)}, w_0 \in W_2^3(\Omega), \quad w_1 \in \mathring{W}_2^2(\Omega)$,

(A.13) $\dot{p} \in L^2(0, T; W_2^{-1}(\Omega)), \quad \Delta(N_T D_2) \in L_r(\Omega), \quad 2 < r < \infty$,

(A.14) $D^\beta \dot{k}_i \in L_\infty(\Omega\times]0, T[), \quad |\beta| \leqslant 1, \quad i = 1, 2.$

THEOREM 3. *If the conditions* (A.1)–(A.14) *are satisfied, then there exists one and only one generalized solution of the problem* (1.1), (1.2), (1.12), (1.13) *with the property*

$$(6.1) \quad \begin{aligned} & w \in L^\infty(0, T; \mathring{W}_2^2(\Omega)\cap W_2^3(\Omega)), \\ & \dot{w} \in L^\infty(0, T; \mathring{W}_2^2(\Omega)), \\ & \ddot{w} \in L^\infty(0, T; \mathring{W}_2^1(\Omega)), \\ & F \in L^\infty(0, T; \mathring{W}_2^2(\Omega)\cap W_r^4(\Omega)), \quad 2 < r < \infty. \end{aligned}$$

Proof

I. Approximate solutions

Let us suppose that the set $\{\phi_i\}_{i=1,2,\ldots}$ is a basis in the space $\mathring{W}_2^2(\Omega)\cap W_2^3(\Omega)$. As in the proof of Theorem 1 we may construct the Faedo–Galerkin approximations w_m, F_m satisfying the *a priori* estimates (4.16).

II. A priori estimates

We shall show that due to the assumptions (A.1)–(A.13), the following estimates hold:

$$(6.2) \quad \begin{aligned} & \|\ddot{w}_m(t)\|_{1,2} \leqslant \bar{c}, \\ & \|\dot{w}_m(t)\|_{2,2} \leqslant \bar{c}, \end{aligned}$$

where $\bar{c}$ is a positive constant.

By putting in Eq. (4.2) $t = 0$, multiplying the j-th equation of Eqs. (4.2) by $\ddot{g}_{jm}(0)$ and taking the sum wiht respect to j, we obtain

$$(6.3)\quad \big(\varrho h\ddot{w}_m(0), \ddot{w}_m(0)\big)+B_0[\ddot{w}_m(0), \ddot{w}_m(0)]+B_1[w_m(0), \ddot{w}_m(0)] = \big(L(w_m(0)+w^{(0)}, F_m(0)+F^{(0)}), \ddot{w}_m(0)\big)+\big(KF_m(0), \ddot{w}_m(0)\big) + \big(p(0)+\Delta M_T, \ddot{w}_m(0)\big).$$

It follows that ($\bar{c}_i$, $i = 1, 2, \ldots$ — the positive constants)

$$(6.4)\quad \bar{c}_1\|\ddot{w}_m(0)\|^2_{1,2} \leqslant \bar{c}_2(\|w_m(0)\|_{3,2}+\|w_m(0)+w^{(0)})\|_{3,2}\|F_m(0)+F^{(0)}\|_{2,2} +\|F_m(0)\|_{1,2}+\|p(0)+\Delta M_T\|_{-1,2})\|\ddot{w}_m(0)\|_{1,2}$$

and due to the assumptions of Theorem 3,

$$(6.5)\quad \|\ddot{w}_m(0)\|_{1,2} \leqslant \bar{c}_3.$$

Let us now differentiate Eq. (4.2) with respect to t. We obtain

$$(6.6)\quad \big(\varrho h\dddot{w}_m(t), \phi_j\big)+B_0[\dot{\ddot{w}}_m(t), \phi_j]+B_1[\dot{w}_m(t), \phi_j] = L\big(w_m(t)+w^{(0)}, F_m(t)+F^{(0)}\big)^{\cdot}, \phi_j\big)+\big((KF_m(t))^{\cdot}, \phi_j\big)+\big(\dot{p}(t), \phi_j\big).$$

By multiplying the j-th equation of Eq. (6.6) by $g_{jm}(t)$, taking the sum over j, $1 \leqslant j \leqslant m$ and integrating over $[0, t]$ we arrive at the relation

$$(6.7)\quad \frac{1}{2}\Big(\big(\varrho h\ddot{w}_m(t), \ddot{w}_m(t)\big)+B_0[\ddot{w}_m(t), \ddot{w}_m(t)]+B_1[\dot{w}_m(t), \dot{w}_m(t)]\Big) = \frac{1}{2}\Big(\big(\varrho h\ddot{w}_m(0), \ddot{w}_m(0)\big)+B_0[\ddot{w}_m(0), \ddot{w}_m(0)]+B_1[\dot{w}_m(0), \dot{w}_m(0)]\Big) + \int_0^t \big(L(\dot{w}_m(s), F_m(s)+F^{(0)}), \ddot{w}_m(s)\big)ds+\int_0^t \big((KF_m(s))^{\cdot}, \ddot{w}_m(s)\big)ds + \int_0^t \big(L(w_m(s)+w^{(0)}, \dot{F}_m(s)), \ddot{w}_m(s)\big)ds+\int_0^t \big(\dot{p}(s), \ddot{w}_m(s)\big)ds.$$

The assumptions (A.6) imply the inequality

$$(6.8)\quad \bar{c}_4\big(\|\ddot{w}_m(t)\|^2_{0,2}+\|\ddot{w}_m(t)\|^2_{1,2}+\|\dot{w}_m(t)\|^2_{2,2}\big) \leqslant \big(\varrho h\ddot{w}_m(t), \ddot{w}_m(t)\big)+B_0[\ddot{w}_m(t), \ddot{w}_m(t)]+B_1[\dot{w}_m(t), \dot{w}_m(t)].$$

By using the Sobolev Imbedding Theorem we can deduce the estimates ($1 < p < 2$)

$$(6.9)\quad L\big(\dot{w}_m(s), F_m(s)+F^{(0)}\big), \ddot{w}_m(s)\big) \leqslant \bar{c}_5\|\dot{w}_m(s)\|_{2,2}\Big(\sum_{|\alpha|=1}\max|D^{\alpha}\big(F_m(s)+F^{(0)}\big)| \times\|\ddot{w}_m(s)\|_{1,2} \leqslant \bar{c}_6\big(\|F^{(0)}\|_{3,p}+\|F_m(s)\|_{3,p}\big)\|\dot{w}_m(s)\|_{2,2}\|\ddot{w}_m(s)\|_{1,2},$$

$$(6.10)\quad \big(L\big(w_m(s)+w^{(0)}, \dot{F}_m(s)\big), \ddot{w}_m(s)\big)\leqslant \bar{c}_7\big(\|w_m(s)\|_{2,2}+\|w^{(0)}\|_{2,2}\big)\|\dot{F}_m(s)\|_{3,p}\|\ddot{w}_m(s)\|_{1,2}.$$

Due to Agmon's Regularity Theorem for Dirichlet's boundary-value problem (see [8]) and due to the estimates (4.16) we have

$$(6.11)\qquad \|F_m(s)\|_{3,p} \leqslant \bar{c}_8\left(\frac{1}{2}\|w_m(s)+2w^{(0)}\|_{2,2}\|w_m(s)\|_{2,2} + \|Kw_m(s)\|_{-1,p} + \|\Delta(N_T D_2)\|_{-1,p}\right) \leqslant \text{const},$$

$$(6.12)\qquad \|\dot{F}_m(s)\|_{3,p} \leqslant \bar{c}_9\big(\|w_m(s)+w^{(0)}\|_{2,2}\|\dot{w}_m(s)\|_{2,2} + \|(Kw_m(s))^{\boldsymbol{\cdot}}\|_{-1,p} \leqslant \bar{c}_{10}+\bar{c}_{11}\|\dot{w}_m(s)\|_{2,2}.$$

As in Eqs. (4.11)–(4.13) we deduce also

$$(6.13)\qquad \big((KF_m(s))^{\boldsymbol{\cdot}},\ \ddot{w}_m(s)\big) \leqslant \bar{c}_{12}+\bar{c}_{13}\big(\|\dot{w}_m(s)\|^2_{2,2}+\|\ddot{w}_m(s)\|^2_{1,2}\big),$$

$$(6.14)\qquad \big(\dot{p}(s),\ \ddot{w}_m(s)\big) \leqslant \bar{c}_{14}+\bar{c}_{15}\|\ddot{w}_m(s)\|^2_{1,2}.$$

By using the inequalities (6.9)–(6.14), it is not hard to prove that the right-hand side of Eq. (6.7) is no greater than

$$\bar{c}_{16}+\bar{c}_{17}\int_0^t \big(\|\dddot{w}_m(s)\|^2_{0,2}+\|\ddot{w}_m(s)\|^2_{1,2}+\|\dot{w}_m(s)\|^2_{2,2}\big)ds$$

and, consequently, the following relation holds:

$$(6.15)\qquad \|\dddot{w}_m(t)\|^2_{0,2}+\|\ddot{w}_m(t)\|^2_{1,2}+\|\dot{w}_m(t)\|^2_{2,2} \leqslant \bar{c}_{18}+\bar{c}_{19}\int_0^t \big(\|\dddot{w}_m(s)\|^2_{0,2}+\|\ddot{w}_m(s)\|^2_{1,2}+\|\dot{w}_m(s)\|^2_{2,2}\big)ds.$$

Due to Gronwall's Lemma we have the inequalities (6.2).

III. Passage to the limit

Arguing as in the proof of Theorem 1 we can select subsequences w_ν, F_ν such that

$$(6.16)\qquad \begin{aligned} &\ddot{w}_\nu \xrightarrow[\nu\to\infty]{} \ddot{w} \quad \text{weakly star in} \quad L^\infty\big(0, T;\ \mathring{W}^1_2(\Omega)\big),\\ &\dot{w}_\nu \xrightarrow[\nu\to\infty]{} \dot{w} \quad \text{weakly star in} \quad L^\infty\big(0, T;\ \mathring{W}^2_2(\Omega)\big),\\ &w_\nu \xrightarrow[\nu\to\infty]{} w \quad \text{weakly star in} \quad L^\infty\big(0, T;\ \mathring{W}^2_2(\Omega)\big),\\ &F_\nu \xrightarrow[\nu\to\infty]{} F \quad \text{weakly star in} \quad L^\infty\big(0, T;\ \mathring{W}^2_2(\Omega)\big). \end{aligned}$$

Similarly as in the proof of Theorem 1 we can show that the functions w, F form a generalized solution of our problem. Let us write Eq. (1.1) in the form

$$(6.17)\qquad \Delta^2_{1D_1}w+\mu\Delta^2_{D_1}w = L(w+w^{(0)}, F+F^{(0)})+KF+p+\Delta M_T-\varrho h\ddot{w}+\Delta_{D_0}\ddot{w} \equiv h.$$

Due to Eq. (6.16) we have

$$(6.18)\qquad h \in L^\infty\big(0, T;\ W^{-1}_2(\Omega)\big).$$

The regularity theory for generalized solutions of the Dirichlet problem (see [7] for example) implies

$$w \in L^\infty\big(0, T;\ W^3_2(\Omega)\cap\mathring{W}^2_2(\Omega)\big).$$

We have also

$$-\frac{1}{2}L(w+2w^{(0)}, w)-Kw-\Delta(N_T D_2) \in L^\infty\big(0, T; L_r(\Omega)\big), \qquad r > 2.$$

By applying the Regularity Theorem of [8] to Eq. (1.2) we obtain

$$F \in L^\infty\big(0, T)\ W_r^4(\Omega) \cap \mathring{W}_2^2(\Omega)\big).$$

The proof is complete.

From Theorem 3 it follows that the functions w, F satisfy the boundary conditions (1.12) in the classical sense. This is due to the inclusion $W_2^3(\Omega) \subset C^1(\bar{\Omega})$ and the Lemma 9.1 of [7]. Secondly, the relations (6.1) enable us to conclude that in the shell under consideration there exists a classical stress tensor with the components of class $C^1(\bar{\Omega})$. In fact, the assertion follows from the inclusion $W_r^4(\Omega) \subset C^3(\bar{\Omega})$, $r > 2$ and the relations

$$\frac{\partial^2 F}{\partial x_1^2} = \int_{-h/2}^{h/2} \sigma_{22}\, dx_3, \quad \frac{\partial^2 F}{\partial x_2^2} = \int_{-h/2}^{h/2} \sigma_{11}\, dx_3, \quad \frac{\partial^2 F}{\partial x_1 \partial x_2} = -\int_{-h/2}^{h/2} \sigma_{12}\, dx_3.$$

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INSTITUTE OF MATHEMATICS
SCHOOL OF PEDAGOGY, KIELCE

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