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NOTE ON A THEOREM OF JACOBI'S FOR THE TRANSFORMATION OF A DOUBLE INTEGRAL.

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JACOBI, in the Memoir "De Transformatione Integralis Duplicis..." &c., *Crelle*, t. VIII. (1832) pp. 253—279 and 321—357, [*Ges. Werke*, t. III., pp. 91—158], after establishing a theorem which includes the addition-theorem of elliptic functions, viz. this last is "the differential equation

$$\frac{d\eta}{\sqrt{(G'^2\cos^2\eta+G''^2\sin^2\eta-G^2)}}+\frac{d\theta}{\sqrt{(G'^2\cos^2\theta+G''^2\sin^2\theta-G^2)}},$$

has for its *complete* integral

$$G+G'\cos\eta\cos\theta+G''\sin\eta\sin\theta=0,"$$

{observe, as to the integral being complete, that the differential equation contains only the constant $G^2-G'^2\div(G^2-G''^2)$, whereas the integral equation contains the two constants $G'\div G$ and $G''\div G$}, obtains a corresponding theorem for double integrals; viz. this, in the corresponding special case, is as follows: If the variables (ϕ, ψ) and (η, θ) are connected by the two equations

$$\begin{array}{ll|ll} \alpha & =0, & \beta & =0, \\ +\alpha'\cos\phi & .\cos\eta & +\beta'\cos\phi & .\cos\eta \\ +\alpha''\sin\phi\cos\psi & .\sin\eta\cos\theta & +\beta''\sin\phi\cos\psi & .\sin\eta\cos\theta \\ +\alpha'''\sin\phi\sin\psi & .\sin\eta\sin\theta & +\beta'''\sin\phi\sin\psi & .\sin\eta\sin\theta \end{array}$$

and if putting for shortness

$$\begin{aligned} \alpha''\beta'''-\alpha'''\beta''&=f, & \alpha\beta'-\alpha'\beta&=a,\\ \alpha'''\beta'-\alpha'\beta'''&=g, & \alpha\beta''-\alpha''\beta&=b,\\ \alpha'\beta''-\alpha''\beta'&=h, & \alpha\beta'''-\alpha'''\beta&=c, \end{aligned}$$

$$\text{(whence } af+bg+ch=0);$$

$$\begin{aligned}R^2 = \;& f^2(\sin\phi\cos\psi)^2(\sin\phi\sin\psi)^2\\ &+g^2(\sin\phi\cos\psi)^2(\cos\phi)^2\\ &+h^2(\cos\phi)^2(\sin\phi\cos\psi)^2\\ &-a^2(\cos\phi)^2\\ &-b^2(\sin\phi\cos\psi)^2\\ &-c^2(\sin\phi\sin\psi)^2,\end{aligned}$$

$$\begin{aligned}S^2 = \;& f^2(\sin\eta\cos\theta)^2(\sin\eta\sin\theta)^2\\ &+g^2(\sin\eta\sin\theta)^2(\cos\eta)^2\\ &+h^2(\cos\eta)^2(\sin\eta\cos\theta)^2\\ &-a^2(\cos\eta)^2\\ &-b^2(\sin\eta\cos\theta)^2\\ &-c^2(\sin\eta\sin\theta)^2,\end{aligned}$$

then we have

$$\frac{\sin\phi\, d\phi\, d\psi}{R} = \frac{\sin\eta\, d\eta\, d\theta}{S}.$$

And it may be added that the integral equations are, so to speak, a complete integral of the differential relation; viz. in virtue of the identity $af+bg+ch=0$, the differential relation contains really only four constants; the integral relations contain the six constants $\alpha : \alpha' : \alpha'' : \alpha'''$ and $\beta : \beta' : \beta'' : \beta'''$, or we have *two* constants introduced by the integration.

The best form of statement is, in the first theorem, to write x, y for $\cos\eta, \sin\eta, (x^2+y^2=1)$, ξ, η for $\cos\theta, \sin\theta, (\xi^2+\eta^2=1)$, and similarly in the second theorem to introduce the variables x, y, z connected by $x^2+y^2+z^2=1$, and ξ, η, ζ connected by $\xi^2+\eta^2+\zeta^2=1$; then in the first theorem $d\eta, d\theta$ represent elements of circular arc, and in the second theorem $\sin\phi\, d\phi\, d\psi$ and $\sin\eta\, d\eta\, d\theta$ represent elements of spherical surface, and the theorems are:

I. If (x, y) are coordinates of a point on the circle $x^2+y^2=1$, and (ξ, η) coordinates of a point on the circle $\xi^2+\eta^2=1$, and if $ds, d\sigma$ are the corresponding circular elements, then

$$\frac{ds}{\sqrt{(ax^2+by^2-c)}} = \frac{d\sigma}{\sqrt{(a\xi^2+b\eta^2-c)}},$$

has for its complete integral

$$ax\xi+by\eta-c=0.$$

II. If (x, y, z) are coordinates of a point on the sphere $x^2+y^2+z^2=1$, and (ξ, η, ζ) coordinates of a point on the sphere $\xi^2+\eta^2+\zeta^2=1$; and if ds, $d\sigma$ are the corresponding spherical elements, and

$$\beta\gamma' - \beta'\gamma = f, \quad \alpha\delta' - \alpha'\delta = a,$$
$$\gamma\alpha' - \gamma'\alpha = g, \quad \beta\delta' - \beta'\delta = b,$$
$$\alpha\beta' - \alpha'\beta = h, \quad \gamma\delta' - \gamma'\delta = c,$$
$$(\text{whence } af + bg + ch = 0);$$

and for shortness

$$S^2 = f^2y^2z^2 + g^2z^2x^2 + h^2x^2y^2 - a^2x^2 - b^2y^2 - c^2z^2,$$
$$\Sigma^2 = f^2\eta^2\zeta^2 + g^2\zeta^2\xi^2 + h^2\xi^2\eta^2 - a^2\xi^2 - b^2\eta^2 - c^2\zeta^2,$$

then the differential relation

$$\frac{ds}{\sqrt{(S)}} = \frac{d\sigma}{\sqrt{(\Sigma)}},$$

has for its complete integral the system

$$\alpha x\xi + \beta y\eta + \gamma z\zeta + \delta = 0,$$
$$\alpha' x\xi + \beta' y\eta + \gamma' z\zeta + \delta' = 0,$$

where by complete integral is meant a system of two equations containing two arbitrary constants.