# 595.

## ON A SENATE-HOUSE PROBLEM.

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THE following was given [5 Jan., 1874,] as a problem of elementary algebra:

"Solve the equations

$$u(2a-x)=x(2a-y)=y(2a-z)=z(2a-u)=b^2,$$

and prove that unless $b^2=2a^2$, $x=y=z=u$, but that if $b^2=2a^2$, the equations are not independent."

This is really a very remarkable theorem in regard to the intersections of a certain set of four quadric surfaces in four-dimensional space; viz. slightly altering the notation, we may write the equations in the form

$$x(2\theta-y)=m\theta^2 \ldots (12),$$
$$y(2\theta-z)=m\theta^2 \ldots (23),$$
$$z(2\theta-w)=m\theta^2 \ldots (34),$$
$$w(2\theta-x)=m\theta^2 \ldots (41),$$

where, regarding $(x, y, z, w, \theta)$ as coordinates in four-dimensional space, each equation represents a quadric surface. I remark that in such a space we have the notions, point-system, curve, subsurface, surface, according as the number of equations is 4, 3, 2, or 1.

Four quadric surfaces intersect in general in 16 points. But for the system in question ($m$ being arbitrary), the common intersection consists of two lines and the two points

$$x=y=z=w=\theta\{1\pm\sqrt{(1-m)}\};$$

and in the case where $m=2$, then the intersection consists of two lines and a certain unicursal quartic curve.

To obtain these results, I consider the four points

$$\theta=0,\quad x=0,\quad y=0,\quad z=0,\ \ldots\ 123,$$

$$\theta=0,\quad y=0,\quad z=0,\quad w=0,\ \ldots\ 234,$$

$$\theta=0,\quad z=0,\quad w=0,\quad x=0,\ \ldots\ 341,$$

$$\theta=0,\quad w=0,\quad x=0,\quad y=0,\ \ldots\ 412:$$

the two points

$$x=y=z=w=\theta\{1\pm\surd(1-m)\},\ \ldots\ PQ:$$

and the six lines

$$\theta=0,\quad x=0,\quad y=0,\ \ldots\ 12,$$

$$\theta=0,\quad y=0,\quad z=0,\ \ldots\ 23,$$

$$\theta=0,\quad z=0,\quad w=0,\ \ldots\ 34,$$

$$\theta=0,\quad w=0,\quad x=0,\ \ldots\ 41,$$

$$\theta=0,\quad x=0,\quad z=0,\ \ldots\ 13,$$

$$\theta=0,\quad y=0,\quad w=0,\ \ldots\ 24,$$

being the edges of a tetrahedron, the vertices of which are the four points, viz. the point 123 is the intersection of the lines 12, 13, 23, and so for the other points.

The surfaces contain the several lines, viz.

| | | | |
|---|---|---|---|
| the surface | 12 | contains | $(12)^2$, 13, 14, 23, 24, |
| " | 23 | " | $(23)^2$, 12, 24, 13, 34, |
| " | 34 | " | $(34)^2$, 13, 23, 14, 24, |
| " | 41 | " | $(41)^2$, 24, 34, 12, 13, |

where $(12)^2$ denotes that 12 is a double line on the surface, and so in other cases. And it thus appears that the surfaces pass all four of them through the lines 13, 24, so that these lines are a part of the common intersection. To obtain the residual intersection, observe that the equations give

$$x=2\theta-m\frac{\theta^2}{w}=\frac{m\theta^2}{2\theta-y},$$

$$z=2\theta-m\frac{\theta^2}{y}=\frac{m\theta^2}{2\theta-w},$$

whence

$$(2\theta-y)\left(2\theta-\frac{m\theta^2}{w}\right)=m\theta^2,$$

$$(2\theta-w)\left(2\theta-\frac{m\theta^2}{y}\right)=m\theta^2,$$

or omitting from each equation the factor $\theta$, the equations become

$$(2\theta - y)(2w - m\theta) = m\theta w,$$

$$(2\theta - w)(2y - m\theta) = m\theta y,$$

that is,

$$(4 - 2m)\,\theta w - 2m\theta^2 - 2yw + m\theta\,(y + w) = 0,$$

$$(4 - 2m)\,\theta y - 2m\theta^2 - 2yw + m\theta\,(y + w) = 0.$$

Whence, $m$ not being $= 2$, we have $y = w$, and then

$$w^2 - 2\theta w + m\theta^2 = 0,$$

or, what is the same thing,

$$2\theta - w = \frac{m\theta^2}{w},$$

giving $x = y = z = w = \theta\,\{1 \pm \sqrt{(1 - m)}\}$, viz. the surfaces each pass through the points $P$, $Q$. As regards the omitted factor $\theta$, it is to be observed that, writing in the equations of the four surfaces $\theta = 0$, the equations become $xy = 0$, $yz = 0$, $zw = 0$, $wx = 0$, satisfied by $x = 0$, $z = 0$, or by $y = 0$, $w = 0$, we have thus $(\theta = 0,\ x = 0,\ z = 0)$ and $(\theta = 0,\ y = 0,\ w = 0)$, viz. the before-mentioned lines 13 and 24.

In the case $m = 2$, we have between $y$, $w$ the single equation

$$yw - \theta\,(y + w) + 2\theta^2 = 0,$$

giving

$$y = \frac{\theta\,(w - 2\theta)}{w - \theta},$$

and thence

$$x = \frac{2\theta\,(w - \theta)}{w},$$

$$z = \frac{-\,2\theta^2}{w - \theta};$$

or, writing for convenience $\alpha = \dfrac{w}{\theta}$, then the equations are

$$\frac{w}{\theta} = \alpha,$$

$$\frac{y}{\theta} = \frac{\alpha - 2}{\alpha - 1},$$

$$\frac{z}{\theta} = \frac{-\,2}{\alpha - 2},$$

$$\frac{x}{\theta} = \frac{2\,(\alpha - 1)}{\alpha};$$

or, what is the same thing,

$$\begin{array}{llll} x = & 2(\alpha-1)^2(\alpha-2) & \left(1-\frac{\alpha}{\infty}\right) \\ : y : & \alpha \quad \ldots & (\alpha-2)^2\left(1-\frac{\alpha}{\infty}\right) \\ : z : & -2\alpha(\alpha-1) \quad \ldots & \left(1-\frac{\alpha}{\infty}\right)^2 \\ : w : & \alpha^2(\alpha-1)(\alpha-2) & \ldots \\ : \theta : & \alpha(\alpha-1)(\alpha-2) & \left(1-\frac{\alpha}{\infty}\right), \end{array}$$

where, for the sake of homogeneity, I have introduced the factors $\left(1-\frac{\alpha}{\infty}\right)$ and $\left(1-\frac{\alpha}{\infty}\right)^2$; viz. we have $x$, $y$, $z$, $w$, $\theta$ proportional to quartic functions of the arbitrary parameter $\alpha$, or the curve is a unicursal quartic. Writing in the equations $\alpha = 0, 1, 2, \infty$ successively, we see that this quartic curve passes through the four points 123, 234, 341, 412 (intersecting at these points the lines 13 and 24 respectively); and writing also $\alpha = 1 \pm i$ we see that the curve passes through the points $P$, $Q$, the coordinates of which now are

$$x = y = z = w = (1 \pm i)\,\theta.$$

It should admit of being proved by general considerations that, in 4-dimensional geometry when 4 quadric surfaces partially intersect in two lines, the residual intersection consists of 2 points; and that, when they intersect in the two lines and in a unicursal quartic met twice by each of the lines, there is no residual intersection—but this theory has not yet been developed.