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ON A DIFFERENTIAL EQUATION IN THE THEORY OF ELLIPTIC FUNCTIONS.

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THE following equation presented itself to me in connexion with the cubic transformation:

$$Q^2 - Q\left(k + \frac{1}{k}\right) - 3 = 3(1-k^2)\frac{dQ}{dk}.$$

Writing as usual $k = u^4$, I was aware that a solution was

$$Q = \frac{v^2}{u^2} + 2uv,$$

where u, v are connected by the modular equation

$$u^4 - v^4 + 2uv(1 - u^2v^2) = 0;$$

but it was no easy matter to verify that the differential equation was satisfied. After a different solution, it occurred to me to obtain the relation between (Q, u); or, what is the same thing, (Q, k), viz. eliminating v, we find

$$Q^4 - 6Q^2 - 4\left(u^4 + \frac{1}{u^4}\right)Q - 3 = 0,$$

or say

$$\frac{1}{Q}(Q^4 - 6Q^2 - 3) = 4\left(k + \frac{1}{k}\right),$$

whence also

$$\frac{1}{Q}(Q^4 - 6Q^2 \pm 8Q - 3) = 4\left(k \pm 2 + \frac{1}{k}\right),$$

that is,

$$\frac{1}{Q}(Q-1)^3(Q+3) = 4\left\{\sqrt{k} + \frac{1}{\sqrt{k}}\right\}^2,$$

and

$$\frac{1}{Q}(Q+1)^3(Q-3) = 4\left\{\sqrt{k} - \frac{1}{\sqrt{k}}\right\}^2,$$

and thence

$$\frac{(Q+1)^3(Q-3)}{(Q-1)^3(Q+3)} = \left(\frac{k-1}{k+1}\right)^2;$$

viz. the value of Q thus determined must satisfy the differential equation. This is easily verified, for, in virtue of the assumed integral, we have

$$Q^2 - 3 - \tfrac{1}{4}(Q^4 - 6Q^2 - 3) = 3(1-k^2)\frac{dQ}{dk};$$

that is,

$$Q^4 - 10Q^2 + 9 = -12(1-k^2)\frac{dQ}{dk},$$

or finally

$$(Q^2-1)(Q^2-9) = -12(1-k^2)\frac{dQ}{dk},$$

an equation which is at once obtained by differentiating logarithmically the former result, and we have thus the verification of the solution. This is, however, a particular integral only; and it appears doubtful whether there exists a general integral of an algebraical form.