591.

A SMITH'S PRIZE PAPER AND DISSERTATION; SOLUTIONS AND REMARKS.

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1. *Find the triangular numbers which are also square.*

The "*mise en équation*" is immediate; we have to find n, m such that

$$\tfrac{1}{2}n(n+1) = m^2;$$

or, what is the same thing,

$$(2n+1)^2 - 8m^2 = 1.$$

Observing that this is satisfied by $n = m = 1$, that is, $2n+1=3$, $2m=2$, we have the general solution given by

$$2n+1+2m\sqrt{(2)} = \{3+2\sqrt{(2)}\}^p,$$

where p is any positive integer; viz. $2n+1$, $2m$ being rational, this implies

$$2n+1-2m\sqrt{(2)} = \{3-2\sqrt{(2)}\}^p,$$

and thence the equation in question. The successive powers

$$3+2\sqrt{(2)},\quad 17+12\sqrt{(2)},\quad 99+70\sqrt{(2)},\ \&c.,$$

give the solutions

$$n, m = \ 1, 1\ ,\qquad 8, 6\ ,\qquad 48, 35\ ,\ \&c.;$$

viz. the square triangular numbers are

$$1^2, =\tfrac{1}{2}\,1\cdot2;\quad 6^2, =\tfrac{1}{2}\,8\cdot9;\quad 35^2, =\tfrac{1}{2}\,49\cdot50,\ \&c.$$

2. *Show how to express any symmetrical function of the roots of an equation in terms of the coefficients. What objection is there to the method which employs the sums of the powers of the roots?*

The ordinary method is that referred to, employing the sums of the powers of the roots; *but it is a very bad one.* In fact, writing

$$x^n - bx^{n-1} + cx^{n-2} - \&\text{c}., = (x-\alpha)(x-\beta)(x-\gamma)\ldots = 0,$$

leading to

$$\begin{aligned} S_1 &= b, \\ S_2 &= b^2 - 2c, \\ S_3 &= b^3 - 3bc + 3d, \\ &\vdots \end{aligned}$$

then if the method were employed throughout, we should have for instance to find $S\alpha\beta\gamma$, that is, d, from the formula

$$\begin{array}{rll} 6S\alpha\beta\gamma = & S_1^3 = & b^3 \\ & -3S_1S_2 & -3b(b^2-2c) \\ & +2\ \ S_3 & +2(b^3-3bc+3d) \\ & = & 6d, \text{ which is right,} \end{array}$$

but the process introduces terms b^3 and bc each of a higher order than d (reckoning the order of each coefficient as unity), with numerical coefficients which destroy each other. And, so again, $S\alpha^2\beta$ would be calculated from the formula

$$\begin{array}{rll} S\alpha^2\beta = & S_1S_2 = & b(b^2-2c) \\ - & S_3 & -(b^3-3bc+3d) \\ & = & bc - 3d, \text{ which is right,} \end{array}$$

but there is here also a term b^3 of a higher order, with numerical coefficients which destroy each other. And the order in which the several expressions are derived the one from the other is a non-natural one; S_3 is required for the determination of $S\alpha^2\beta$, whereas (as will be seen) it is properly $S\alpha^2\beta$ which leads to the value of S_3.

The true method is as follows: we have

$$S\alpha = b, \quad S\alpha\beta = c, \quad S\alpha\beta\gamma = d, \ \&\text{c}.,$$

and we thence derive the sets of equations

$$\begin{array}{rl} b = & S\alpha; \\ c = & S\alpha\beta, \\ b^2 = & S\alpha^2 + 2S\alpha\beta; \\ d = & S\alpha\beta\gamma, \\ bc = & S\alpha^2\beta + 3S\alpha\beta\gamma, \\ b^3 = & S\alpha^3 + 3S\alpha^2\beta + 6S\alpha\beta\gamma; \\ \vdots & \end{array}$$

viz. we thus have 1 equation to give $S\alpha$; 2 equations to give $S\alpha\beta$ and $S\alpha^2$; 3 equations to give $S\alpha\beta\gamma$, $S\alpha^2\beta$, $S\alpha^3$; and so on. And taking for instance the third set of equations, the first equation gives $S\alpha\beta\gamma$, the second then gives $S\alpha^2\beta$, and the third then gives $S\alpha^3$, viz. we have

$$\begin{aligned} S\alpha\beta\gamma &= d, \\ S\alpha^2\beta &= bc - 3d, \\ S\alpha^3 &= b^3 - 3(bc - 3d) - 6d, \\ &= b^3 - 3bc + 3d. \end{aligned}$$

Of course the process for the formation of the successive sets of equations would require further explanation and development.

3. *Given a point P in the interior of an ellipsoid, show that it is possible to determine an exterior point Q such that for every chord RS through P, the relation $QR : QS = PR : PS$ may hold good.*

There is no difficulty in the analytical solution and in showing thereby that the point Q is determined as the intersection of the polar plane of P by the perpendicular let fall from P on this plane. But a simple and elegant geometrical solution was given in the Examination. Constructing Q as above, let the chord RS meet the polar plane of P in Z; then the polar plane of Z passes through P, that is, the line ZP is harmonically divided in R, S, or we have

$$ZR : ZS = PR : PS.$$

Again ZQP being a right angle, the sphere on ZP as diameter will pass through Q; and R, S being points on the diameter, and Z, Q points on the surface, $ZR : ZS = QR : QS$; whence the required relation $QR : QS = PR : PS$.

4. *Find the number of regions into which infinite space is divided by n planes.*

The number $\frac{1}{6}(n^3 + 5n + 6)$ is a known result, but not a generally known one, and I intended the question as a problem; I do not think it is a difficult one.

Consider the analogous problem for lines in a plane: the first line divides the plane into 2 regions.

The second line is by the first divided into 2 parts, and therefore adds 2 regions.

The third line is by the other two divided into 3 parts, and therefore adds 3 regions; and so on.

That is, the number of regions for

$$\begin{array}{llll} 1 \text{ line is} & = 2 & = 2 & \text{regions,} \\ 2 \text{ lines} & = 2 + 2 & = 4 & ,, \\ 3 \text{ lines} & = 2 + 2 + 3 & = 7 & ,, \\ \vdots & & & \\ n \text{ lines} & = 2 + 2 + 3 + \ldots + n & = \frac{1}{2}(n^2 + n + 2) & ,, \end{array}$$

In exactly the same way for the problem in space:

The first plane divides space into 2 regions.

The second plane is by the first plane divided into 2 regions, and therefore adds 2 regions.

The third plane is by the other two planes divided into 4 regions, and therefore adds 4 regions.

The fourth plane is by the other three planes divided into 7 regions, and therefore adds 7 regions: and so on.

That is the number of regions for

$$\begin{array}{lll}
1 \text{ plane is} & = 2 & = 2 \text{ regions} \\
2 \text{ planes} & = 2 + 2 & = 4 \quad ,, \\
3 \text{ planes} & = 2 + 2 + 4 & = 8 \quad ,, \\
4 \text{ planes} & = 2 + 2 + 4 + 7 & = 15 \quad ,, \\
\vdots & & \\
n \text{ planes} & = 2 + 2 + 4 + 7 + \ldots + \tfrac{1}{2}(n^2 - n + 2) & = \tfrac{1}{6}(n^3 + 5n + 6),
\end{array}$$

where, for effecting the summation, observe that the series is

$$\begin{aligned}
&= 2 + \{1 + 1 + 1 \ldots (n-1) \text{ terms}\} \\
&\quad + \{1 + 3 + 6 \ldots + \tfrac{1}{2}n(n-1)\}, \\
&= 2 + (n-1) + \tfrac{1}{6}(n+1)\,n\,(n-1), \text{ = as above.}
\end{aligned}$$

5. *In the theory of Elliptic Functions, explain and connect together the notations* $F(\theta)$, am u (sinam u, cosam u, Δam u), *illustrating them by reference to the circular functions*.*

What is asked for is an explanation of the fundamental notations of Elliptic Functions. To a student acquainted with the subject, the only difficulty is to say enough to bring the meaning fully out, and not to say more than enough.

Defining $F(x)$ by the equation

$$F(x) = \int_0 \frac{dx}{\sqrt{\{(1 - x^2)(1 - k^2x^2)\}}},$$

(viz. the integral is taken from 0 up to the indefinite value x), then the fundamental property of elliptic functions (derived from consideration of the differential equation

$$\frac{dx}{\sqrt{\{(1 - x^2)(1 - k^2x^2)\}}} + \frac{dy}{\sqrt{\{(1 - y^2)(1 - k^2y^2)\}}} = 0)$$

consists herein, that the functional relation

$$F(x) + F(y) = F(z)$$

* It would have been better in the question to have written $F(x)$ instead of $F(\theta)$.

is equivalent to an algebraic equation between the arguments x, y, z. $F(x)$ as defined by the foregoing equation is properly an inverse function; this at once appears from a particular case, viz. writing $k=0$, $F(x)=\sin^{-1}x$, and the theory of the function $F(x)$ in the general case corresponds to what the theory of circular functions would be, if writing $F(x)$ to denote $\sin^{-1}x$, we were to work with the equation

$$F(x)+F(y)=F(z)$$

as equivalent to the algebraical equations (one a transformation of the other)

$$z=x\sqrt{(1-y^2)}+y\sqrt{(1-x^2)},$$
$$\sqrt{(1-z^2)}=\sqrt{(1-x^2)}\sqrt{(1-y^2)}-xy.$$

But in the actual theory of circular functions, we introduce the direct symbols sin, cos; writing $F(x)=\theta$, that is, $x=\sin\theta$, $\sqrt{(1-x^2)}=\cos\theta$, and similarly $F(y)=\phi$, that is, $y=\sin\phi$ and $\sqrt{(1-y^2)}=\cos\phi$, then the equation

$$F(x)+F(y)=F(z)$$

becomes $F(z)=\theta+\phi$, that is, $z=\sin(\theta+\phi)$, $\sqrt{(1-z^2)}=\cos(\theta+\phi)$, and the other two equations become

$$\sin(\theta+\phi)=\sin\theta\cos\phi+\sin\phi\cos\theta,$$
$$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi,$$

viz. these are the addition-equations for the functions sin and cos.

In passing from the original notation $F(x)$ to the notation am u, we make the like step of passing from an inverse to a set of direct functions; first modifying the meaning of F, so as to denote by $F(\theta)$ what was originally $F(\sin\theta)$, we have as the new definition

$$F(\theta)=\int_0\frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}}=\int_0\frac{d\theta}{\Delta(\theta)},$$

(if as usual $\Delta\theta$ denotes $\sqrt{(1-k^2\sin^2\theta)}$), and this being so, the relation $F(\theta)+F(\phi)=F(\mu)$ is equivalent to a relation between the sine, cosine, and Δ of θ, ϕ, μ. Writing then $F(\theta)=u$, and considering this equation as determining θ as a function of u, $\theta=\text{am}\,u$, we have $\sin\theta=\sin.\text{am}\,u$, $\cos\theta=\cos.\text{am}\,u$, and $\Delta\theta=\Delta.\text{am}\,u$, and similarly $F(\phi)=v$, $\phi=\text{am}\,v$, &c., then the equation $F(\theta)+F(\phi)=F(\mu)$ becomes $F(\mu)=u+v$, that is, $\mu=\text{am}(u+v)$; and the algebraic relation in its various forms gives the values of $\sin.\text{am}(u+v)$, $\cos.\text{am}(u+v)$, $\Delta.\text{am}(u+v)$ in terms of the like functions of u, v respectively, viz. it is the addition-theorem for the function am.

Observe that am u is considered as a certain function of u, $\sin.\text{am}\,u$ is its sine, $\cos.\text{am}\,u$ its cosine, and

$$\Delta.\text{am}\,u=\sqrt{(1-k^2\sin^2.\text{am}\,u)},$$

a function analogous to a cosine. But making only a slight change in the point of view, we have sinam u, a certain function of u, and

$$\text{cosam}\,u\,\{=\sqrt{(1-\text{sinam}^2u)}\},\ \Delta\text{am}\,u\,\{=\sqrt{(1-k^2\,\text{sinam}^2u)}\},$$

two allied functions, viz. sinam u is analogous to a sine, and the other two functions to cosines; the algebraical equations give the sinam, cosam, and Δam of $u+v$ in terms of the like functions of u and v respectively, viz. they constitute the addition-theorem for these functions.

6. *Find the differential equation satisfied by a hypergeometric series, and express by means of such series the coefficients of the expansion of* $(1-2a\cos\theta+a^2)^{-n}$ *according to multiple cosines of* θ.

I understand the expression "hypergeometric series" in the restricted sense in which it signifies the series

$$F(\alpha,\ \beta,\ \gamma,\ x)=1+\frac{\alpha\,.\,\beta}{1\,.\,\gamma}x+\frac{\alpha(\alpha+1)\,\beta(\beta+1)}{1\,.\,2\gamma(\gamma+1)}x^2+\&\text{c.}$$

I find it was understood in the more general sense of a series

$$u=a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots,$$

where the coefficient a_{n+1} is given in terms of the preceding one a_n by an equation of the form $a_{n+1}=\phi(n)\,.\,a_n$. In this latter sense, but supposing for greater simplicity, that $\phi(n)$ is a rational and integral function of n, the solution is as follows: we operate on the series with the symbol $\phi\left(x\dfrac{d}{dx}\right)$; viz. $x\dfrac{d}{dx}$ is regarded as a single symbol of operation; $x\dfrac{d}{dx}\,.\,x^n=nx^n$, $\left(x\dfrac{d}{dx}\right)^2x^n=n^2x^n$, &c.; thus $x\dfrac{d}{dx}$ is, as regards x^n, $=n$, and therefore $\phi\left(x\dfrac{d}{dx}\right)=\phi(n)$. We thence have

$$\begin{aligned}\phi\left(x\frac{d}{dx}\right)u&=\phi(0)\,a_0+\phi(1)\,a_1x+\phi(2)\,a_2x^2\ldots+\phi(n)\,a_nx^n+\ldots\\&=\quad a_1\quad+\quad a_2x\quad+\quad a_3x^2\quad\ldots+\quad a_{n+1}x^n\ +\ldots,\end{aligned}$$

and consequently

$$x\phi\left(x\frac{d}{dx}\right)u=u-a_0,$$

which is the required differential equation. This is equivalent to the process given in Boole, only he writes $x=e^\theta$, in order to reduce $x\dfrac{d}{dx}$ to a mere differentiation $\dfrac{d}{d\theta}$. I regard this introduction of a new variable θ *as most unfortunate;* the effect is entirely to conceal the real nature of the operation; the notion of $x\dfrac{d}{dx}$ as a single symbol of operation is quite as simple as that of $\dfrac{d}{d\theta}$; and by means of it we retain the original variable.

The process is substantially the same when $\phi(n)$ is a rational fraction, but I give the investigation directly for the hypergeometric series in the restricted sense, viz. writing u for the series $F(\alpha,\ \beta,\ \gamma,\ x)$, we find

$$x\left(x\frac{d}{dx}+\alpha\right)\left(x\frac{d}{dx}+\beta\right)u=x\frac{d}{dx}\left(x\frac{d}{dx}+\gamma-1\right)u;$$

or, what is the same thing,

$$\left\{\left(x\frac{d}{dx}+\alpha\right)\left(x\frac{d}{dx}+\beta\right)-\frac{d}{dx}\left(x\frac{d}{dx}+\gamma-1\right)\right\}u=0,$$

as at once appears by writing the general term successively under the two forms

$$\frac{\alpha\,.\,\alpha+1\ldots\alpha+n-1\,.\,\beta\,.\,\beta+1\ldots\beta+n-1}{1\,.\,2\ \ldots\ \ \ n\qquad\quad \gamma\,.\,\gamma+1\ldots\gamma+n-1}x^n,$$

and

$$\frac{\alpha\,.\,\alpha+1\ldots\alpha+n\,.\,\beta\,.\,\beta+1\ldots\beta+n}{1\,.\,2\ \ldots n+1\,.\,\gamma\,.\,\gamma+1\ldots\gamma+n}x^{n+1}.$$

The differential equation may also be written

$$\left[(x^2-x)\frac{d^2}{dx^2}+\{(1+\alpha+\beta)\,x-\gamma\}\frac{d}{dx}+\alpha\beta\right]u=0.$$

Take next the function

$$\begin{aligned}
&(1-2a\cos\theta+a^2)^{-n},\\
&=\left\{1-a\left(x+\frac{1}{x}\right)+a^2\right\}^{-n}\\
&=\left\{(1-ax)\left(1-a\frac{1}{x}\right)\right\}^{-n},\ \text{if } x+\frac{1}{x}=2\cos\theta,\\
&=\left(1+\frac{n}{1}ax+\frac{n\,.\,n+1}{1\,.\,2}a^2x^2+\ldots\right)\left(1+\frac{n}{1}\frac{a}{x}+\frac{n\,.\,n+1}{1\,.\,2}\frac{a^2}{x^2}+\ldots\right)\\
&=\left\{1^2+\left(\frac{n}{1}\right)^2a^2+\left(\frac{n\,.\,n+1}{1\,.\,2}\right)^2a^4\ldots\right\}\\
&+\left\{1\,.\,\frac{n}{1}+\frac{n}{1}\,\frac{n\,.\,n+1}{1\,.\,2}a^2+\frac{n\,.\,n+1}{1\,.\,2}\,\frac{n\,.\,n+1\,.\,n+2}{1\,.\,2\,.\,3}a^4+\ldots\right\}a\left(x+\frac{1}{x}\right)(=2a\cos\theta)\\
&+\left\{1\,.\,\frac{n\,.\,n+1}{1\,.\,2}+\&\text{c.}\qquad\qquad\qquad\qquad\qquad\qquad\right\}a^2\left(x^2+\frac{1}{x^2}\right)(=2a^2\cos 2\theta),\\
&\&\text{c.}\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\&\text{c.}
\end{aligned}$$

where the second term contains the factor $\frac{n}{1}a$, the third the factor $\frac{n\,.\,n+1}{1\,.\,2}a^2$, and so on. Throwing these out, the remaining factors are each of them a hypergeometric series, viz. representing the whole expression by

$$A_0+2A_1\cos\theta+2A_2\cos 2\theta+\&\text{c.},$$

we have

$$A_0=F(n,\ n,\ 1,\ a^2),$$

$$A_1=\frac{n}{1}aF(n,\ n+1,\ 2,\ a^2),$$

$$\vdots$$

and generally

$$A_r=\frac{n\,.\,n+1\ldots n+r-1}{1\,.\,2\ldots r}a^rF(n,\ n+r,\ r+1,\ a^2).$$

7. *The function* $e^{-\frac{1}{(x-a)^2}}$ *has been suggested as an exception to the theorem that if a function and all its differential coefficients vanish for a given value of the variable, then the function is identically* $=0$; *discuss the question as regards the precise meaning of the theorem, and validity of the exception.*

The suggestion was made by Sir W. R. Hamilton; the following remarks arise in regard to it:

The function $e^{-\frac{1}{(x-a)^2}}$ is a function which *in a certain sense* satisfies the condition that for a given value $(=a)$ of the variable, the function and all its differential coefficients vanish; viz. each differential coefficient is of the form $Xe^{-\frac{1}{(x-a)^2}}$, where X is a finite series of negative powers of $x-a$; if then $x=a\pm r$, where r is real and positive, and if r continually diminishes to zero, then $(x-a)^2$, remaining always real and positive, continually diminishes to zero, that is, $-\dfrac{1}{(x-a)^2}$ remaining always real and negative continually increases to $-\infty$, and $e^{-\frac{1}{(x-a)^2}}$ remaining always real and positive continually diminishes to zero. And, moreover, (X containing only a finite series of negative powers of $x-a$) the expression $Xe^{-\frac{1}{(x-a)^2}}$ will in like manner, remaining always real, continually approximate to zero. But assume $x=a+r(\cos\theta+i\sin\theta)$, r real and positive, θ real; then $(x-a)^2=r^2(\cos 2\theta+i\sin 2\theta)$, and if $\cos 2\theta$ be positive, then the real part of $(x-a)^2$, being always positive, continually diminishes to zero, and the like conclusions follow. If however $\cos 2\theta$ be negative, then the real part of $(x-a)^2$ is negative, and the real part of $-\dfrac{1}{(x-a)^2}$ is positive, and as r diminishes continually approximates to $+\infty$; so far from $e^{-\frac{1}{(x-a)^2}}$ continually approximating to zero, it is in general an imaginary quantity continually approximating to infinity; and the like is the case with its successive differential coefficients; the conclusion is, it is not true *simpliciter* that the function $e^{-\frac{1}{(x-a)^2}}$, or any one of its successive differential coefficients, vanishes for the value a of the variable.

Generally, if a real or imaginary quantity $\alpha+\beta i$ is represented by the point whose rectangular coordinates are α, β; say if the value a of the variable x is represented by the point P, and any other value $a+h+ki$, by the point Q (h, k being therefore the coordinates of Q measured from the origin P), then a function $F(x)$ which as Q (no matter in what direction) approaches and ultimately coincides with P, tends to become and becomes ultimately $=0$, may be said to vanish *simpliciter* for the value a of the variable; but if this is only the case when Q approaches P in a certain direction or within certain limits of direction, the function not becoming zero when Q approaches in a different direction, then the function may be said to vanish *sub modo* for the value a of the variable.

Taking the theorem to mean "If for a given value a of the variable, a function and its differential coefficients vanish *sub modo*, the function is identically $=0$," the instance of the function $e^{-\frac{1}{(x-a)^2}}$ shows that the theorem is certainly not true; but taking the theorem to mean "If for a given value a of the variable, the function and its differential coefficients vanish *simpliciter*, then the function is identically $=0$"; the instance does not apply to it, and the truth of the theorem remains an open question.

The above view is consistent with a theorem obtained by Cauchy and others, defining within what limits of h the expansion by Taylor's theorem of the function $F(a+h)$ is applicable, viz. a and h being in general imaginary as above, if the function (or ? the function and its successive differential coefficients) is (or are) finite and continuous so long as the distance PQ does not exceed a certain real and positive value ρ, then the expansion is applicable for any point Q, whose distance PQ does not exceed this value ρ: but it ceases to be applicable for a point Q, the distance of which is equal to or exceeds ρ. In the case of a function such as $e^{-\frac{1}{(x-a)^2}}$, discontinuity arises *at* the point P, that is, for the value $\rho=0$, and according to the theorem in question, the expansion is not applicable for any value of ρ however small.

I wish to remark on a view which appears to me to be founded on a radical misconception of the notion of convergence. Writing $F(x)=e^{-\frac{1}{(x-a)^2}}$, consider the series

$$F(a)+F'(a)\frac{h}{1}+F''(a)\frac{h^2}{1\,.\,2}+\&\text{c.}\,\ldots$$

Then admitting that the exponential $e^{-\frac{1}{(x-a)^2}}$ becomes $=0$ for $x=a$, the successive functions $F(a)$, $F'(a)$, $F''(a)$, ... are each $=0$ as containing this exponential: but inasmuch as the successive differentiations introduce negative powers of $x-a$, each successive function is regarded as an infinitesimal of a lower order than those which precede it; say $F(a)$ being $=0^{\mu}$, the successive terms are multiples 0^{μ}, $0^{\mu-3}$, $0^{\mu-6}$, $0^{\mu-9}$, &c. respectively; where however μ is infinite, so that the several exponents μ, $\mu-3$, $\mu-6$, &c., however far the series is continued, remain all of them positive. This being so, it is said that the series $F(a)+F'(a)\frac{h}{1}+\&\text{c.}$, as being really of the form $0^{\mu}+0^{\mu-3}+0^{\mu-6}+\ldots$ is divergent, and for this reason fails to give a correct value of $F(a+h)$. I apprehend that the notion of divergence is a strictly numerical one; a series of numbers $a+b+c+d+\ldots$ is divergent when the successive sums a, $a+b$, $a+b+c$, $a+b+c+d$, &c., are numbers not continually tending to a determinate limit. In the actual case the series is $0+0+0+0+\ldots$, viz. each term is by hypothesis an absolute zero; the successive sums 0, $0+0$, $0+0+0$, ... are each $=0$, and we cannot, by the process of numerical summation, make the sum of the series to be anything else than 0. If it could, there would be an end of all numerical equality between infinite series; for taking any convergent series $a+b+c+d+\ldots$, if 0 means 0, this is the same thing as the series, also a convergent one,

$$(a+0)+(b+0)+(c+0)+\&\text{c.},$$

and their difference $0+0+0+\ldots$ must be $=0$. I regard the view as a mere failure to reconcile the equation

$$F(a+h)=Fa+\frac{h}{1}F'(a)+\&c.,$$

with the *supposed* fact in regard to the function $e^{-\frac{1}{(x-a)^2}}$.

8. *Find the value of the definite integrals*

$$\int e^{-x^2}\,dx,\ \int \sin x^2\,dx,\ \int \cos x^2\,dx,$$

the limits being in each case ∞, $-\infty$. *Examine whether the last two integrals can be found by a process such as Laplace's (depending on a double integral) for the first integral.*

Laplace's process for the integral $\int e^{-x^2}\,dx$ is as follows: write $u=\int e^{-x^2}\,dx$, then also $u=\int e^{-y^2}\,dy$, and thence

$$u^2=\iint e^{-(x^2+y^2)}\,dx\,dy,$$

which, considering x, y as rectangular coordinates and substituting for them the polar coordinates r, θ, becomes

$$u^2=\iint e^{-r^2}\,r\,dr\,d\theta\,;$$

and then considering the double integral as extending over the infinite plane, and taking the limits to be $r=0$ to $r=\infty$, $\theta=0$ to $\theta=2\pi$, we obtain

$$u^2=(-\tfrac{1}{2}e^{-r^2})_0^\infty\,2\pi,\ =\tfrac{1}{2}\,.\,2\pi,\ =\pi,$$

that is,

$$u=\int e^{-x^2}\,dx=\sqrt{(\pi)}.$$

There is an assumption the validity of which requires examination. We have u the limit of the integral $\int_{-a}^{a} e^{-x^2}\,dx$, as a approaches to ∞; and this being so, we have u^2 the limit of

$$\int_{-a}^{a}\int_{-a}^{a} e^{-(x^2+y^2)}\,dx\,dy,$$

viz. u^2 is the integral of $e^{-(x^2+y^2)}$ taken over a square, the side of which is $2a$, a being ultimately infinite. But making the transformation to polar coordinates, and integrating as above, we in fact take the integral over a circle radius $=\beta$, β being ultimately infinite. And we assume that the two values are equal; or, generally, that taking the integral over an area bounded by a curve which is such that the distance of every point from the origin is ultimately infinite, the value of the integral is independent of the form of the curve.

This is really the case under the following conditions: 1°. For a curve of a given form, the integral tends to a fixed limit, as the size is continually increased. 2°. The quantity under the integral sign is always of the same sign (say always positive); (the last condition is sufficient, but not necessary). For, to fix the ideas, let the curves be as before the square and the circle: take a square; surrounding this, a circle; and surrounding the circle, a square. Imagine the two squares and the circle continually to increase in magnitude; the integral over the smaller square and that over the larger square, each tend to the same fixed limit; consequently the integral over the area enclosed between the two squares tends to the limit zero; and *à fortiori* the integral over the area enclosed between the circle and either of the two squares tends to the limit zero; that is, the integral over the square, and that over the circle, tend to the same limit. In the case under consideration, the function $e^{-(x^2+y^2)}$ is always positive; and the integral $\iint e^{-(x^2+y^2)}\,dxdy$, taken over the circle, tends (as in effect shown above) to the limit π: hence the process is a legitimate one.

But endeavour to apply it to the other two integrals; write

$$\begin{array}{l|l} u = \int \sin x^2 dx & v = \int \cos x^2 dx \\ \quad = \int \sin y^2 dy, & \quad = \int \cos y^2 dy, \end{array}$$

then

$$\iint \sin(x^2+y^2)\,dxdy = 2uv, \quad \iint \cos(x^2+y^2)\,dxdy = v^2 - u^2,$$

where the double integrals on the left-hand side really denote integrals taken over a square and are not equal to the like integrals taken over a circle. This appears *à posteriori* if we only assume that the integrals u, v have determinate values; for taking the integrals over a circle they would be

$$\iint \frac{\sin}{\cos} r^2 . r\,dr d\theta,$$

and would involve the indeterminate functions $\frac{\sin}{\cos}\infty$; that is, if it were allowable to take the integrals over a circle, we should have $2uv$ and $v^2 - u^2$ indeterminate instead of determinate.

A process of finding them is as follows: in the equation $\int e^{-x^2} dx = \sqrt{(\pi)}$, substituting in the first instance $x\sqrt{(a)}$ for x, a real and positive, we have

$$\int e^{-ax^2} dx = \frac{\sqrt{(\pi)}}{\sqrt{(a)}},$$

and if it be assumed that this equation extends to the case where $a = \alpha + \beta i$, *the* *real part a real and positive**; or, what is the same thing, $a=\rho(\cos\theta+i\sin\theta)$, ρ real and positive, θ between the limits 0 and $\frac{1}{2}\pi$, then we have

$$\int e^{-\rho(\cos\theta+i\sin\theta)\,x^2}\,dx=\frac{\sqrt{(\pi)}}{\sqrt{(\rho)}}(\cos\tfrac{1}{2}\theta-i\sin\tfrac{1}{2}\theta),$$

or, separating the real and imaginary parts and taking $\rho=1$, we have

$$\int e^{-x^2\cos\theta}\cos(x^2\sin\theta)\,dx=\sqrt{(\pi)}\cos\tfrac{1}{2}\theta,$$

$$\int e^{-x^2\cos\theta}\sin(x^2\sin\theta)\,dx=\sqrt{(\pi)}\sin\tfrac{1}{2}\theta.$$

Admitting these formulæ to be true in general, there is still considerable difficulty in seeing that they hold good in the limiting case $\theta=\frac{1}{2}\pi$. But assuming that they do, the formulæ then become

$$\int\cos x^2\,dx=\frac{\sqrt{(\pi)}}{\sqrt{(2)}},\quad \int\sin x^2\,dx=\frac{\sqrt{(\pi)}}{\sqrt{(2)}},$$

which are the values of the integrals in question.

9. *Considering in a solid body a system of two, three, four, five, or six lines, determine in each case the relations between the lines in order that it may be possible to find along them forces to hold the body in equilibrium.*

If there are two lines, the condition obviously is that these must be one and the same line.

If three lines, then these must lie in a plane, and meet in a point.

The conditions in the other cases ought to be in the text-books; they in fact are not, and I assumed that they would not be known, and considered the question as a problem; it is, in regard to the cases of four and five lines, *a very easy problem when the solution is seen.*

In the case of four lines; imagine in the solid body an axis meeting any three of the lines, and let this axis be fixed; the condition of equilibrium about this axis is that the fourth line shall meet the axis. The required condition therefore is that every line meeting three of the four lines shall meet the fourth line; or, what is the same thing, the four lines must be generators (of the same kind) of a skew hyperboloid.

In the case of five lines, taking any four of them, we have two lines (tractors) each meeting the four lines; and taking either of the two lines as an axis, then for equilibrium the fifth line must also meet this axis; the required relations therefore are that the fifth line shall meet each of the two lines which meet the other four lines; or, what is the same thing, that there shall be two lines each meeting the five given lines.

* The equation is clearly *not true* unless this is so: for a being negative, then in virtue of the factor e^{-ax^2}, the exponential, instead of decreasing will increase, and ultimately become infinite as x increases to $\pm\infty$

The case of six lines is one the answer to which could not have been discovered in an examination; the relations in fact are that the six lines shall form an involution; viz. this is a system such that taking five of the lines as given, then if the sixth line is taken to pass through a given point it may be any line whatever in a determinate plane through this point; or, what is the same thing, if the sixth line is taken to be in a given plane, it may be any line whatever through a determinate point in this plane. But in a particular case, the answer is easy; suppose five of the six given lines to be met by a single line, then the sixth line may be any line whatever meeting this single line.

10. *If $X, Y, Z, \ldots$ are the roots of the equation*

$$(1,\ P,\ Q, \ldots)(c,\ 1)^n = 0,$$

show that the differential equation obtained by the elimination of c is $\zeta X'Y'Z' = 0$, where ζ denotes the product of the squared differences of the roots $X, Y, Z, \ldots$, and $X', Y', Z', \ldots$ are the derived functions of these roots; and connect this result with the theory of singular solutions.

We have identically

$$(1,\ P,\ Q \ldots)(c,\ 1)^n = (c - X)(c - Y)(c - Z) \ldots;$$

the original equation and its derived equation

$$(0,\ P',\ Q', \ldots)(c,\ 1)^n = 0$$

(the latter of them of degree $n - 1$) may therefore be written

$$(c - X)(c - Y)(c - Z) \ldots = 0,$$

$$X'(c - Y)(c - Z) \ldots + Y'(c - X)(c - Z) \ldots + \&\text{c.} = 0.$$

To eliminate c, we have in the *nilfactum* of the second equation to substitute successively the values $c = X$, $c = Y$, &c., multiply the several functions together and equate the result to zero; the factors are evidently

$$X'(X - Y)(X - Z) \ldots,\quad Y'(Y - X)(Y - Z) \ldots,\ \&\text{c.},$$

where each difference occurs twice, e.g. $X - Y$ under the two forms $X - Y$ and $Y - X$ respectively; the result thus is

$$X'Y'Z' \ldots (X - Y)^2 (X - Z)^2 (Y - Z)^2 \ldots = 0;$$

that is,

$$\zeta . X'Y'Z' \ldots = 0.$$

Thus in particular in the case of a quadric equation

$$(1,\ P,\ Q)(c,\ 1)^2,\ = (c - X)(c - Y),\ = 0,$$

the differential equation is

$$(X - Y)^2 X'Y' = 0;$$

viz. since $X + Y = -P$, and $XY = Q$, this is

$$(P^2 - 4Q) X'Y' = 0,$$

and writing also

$$X = -\tfrac{1}{2}\{P + \sqrt{(P^2 - 4Q)}\}, \quad Y = -\tfrac{1}{2}\{P - \sqrt{(P^2 - 4Q)}\},$$

we find

$$X'Y' = \tfrac{1}{4}\left\{P'^2 - \frac{(PP' - 2Q')^2}{P^2 - 4Q}\right\};$$

the differential equation thus is

$$(P^2 - 4Q)\left\{P'^2 - \frac{(PP' - 2Q')^2}{P^2 - 4Q}\right\} = 0.$$

The application to the theory of singular solutions is that, in the case where the function $(1, P, Q \ldots)(c, 1)^n$ breaks up into rational factors $c - X$, $c - Y, \ldots$, the factor $\zeta = (X - Y)^2 (X - Z)^2 \ldots$ divides out and should be rejected from the differential equation, which in its true form is $X'Y'Z' \ldots = 0$; viz. this is what we obtain immediately, considering the given integral equation as meaning the system of curves $c - X = 0$, $c - Y = 0, \ldots$, and there is not really any singular solution; whereas in the case where the factors are not rational, the factor in question, when the product $X'Y'Z' \ldots$ is expressed in terms of the coefficients $P, Q, \ldots$, and their derived coefficients does not divide out from the equation; and in this case, equated to zero, it gives a proper singular solution of the equation.

11. *In the theory of elliptic motion, v denoting the mean anomaly and e the eccentricity, if m' be an angle such that* $\tan \tfrac{1}{2}v = \dfrac{1+e}{1-e} \tan \tfrac{1}{2}m'$, *find in terms of e, m' the mean anomaly m.*

Taking as usual u for the eccentric anomaly, to commence the solution write down

$$\tan \tfrac{1}{2}v = \sqrt{\left(\frac{1+e}{1-e}\right)} \tan \tfrac{1}{2}u$$

$$= \frac{1+e}{1-e} \tan \tfrac{1}{2}m',$$

that is,

$$\tan \tfrac{1}{2}u = \sqrt{\left(\frac{1+e}{1-e}\right)} \tan \tfrac{1}{2}m',$$

and u being given hereby as a function of m', we have by substitution in the equation $m = u - e \sin u$, to find m as a function of m'.

A creditable approximate solution would be $m = m' + 0 \,.\, e$, viz. this would be to show that neglecting terms in e^2, &c., we have $m = m'$. In fact, taking e small, we have

$$\tan \tfrac{1}{2}u = (1 + e) \tan \tfrac{1}{2}m',$$

and thence if $u = m' + x$, we have

$$\tan \tfrac{1}{2}m' + \tfrac{1}{2}x \sec^2 \tfrac{1}{2}m' = (1 + e) \tan \tfrac{1}{2}m',$$

that is,

$$x = 2e\cos^2\tfrac{1}{2}m' \tan\tfrac{1}{2}m' = e\sin m'\,;\ \ u = m' + e\sin m',$$

and

$$\begin{aligned} m &= m' + e\sin m' \\ &\quad - e\sin(m' + \ldots) \\ &= m' + 0\,.\,e. \end{aligned}$$

The complete solution would be obtained by expanding u in terms of e, m' from the equation $\tan\tfrac{1}{2}u = \sqrt{\left(\dfrac{1+e}{1-e}\right)}\tan\tfrac{1}{2}m'$ (which is of the form $\tan\tfrac{1}{2}u = n\tan\tfrac{1}{2}m'$, giving for u a known series $= m' +$ multiple sines of m'), and then observing that the same equation leads to

$$\sin u = \frac{\sqrt{(1-e^2)}\sin m'}{1 - e\cos m'},$$

we have

$$m = \text{series} - \frac{e\sqrt{(1-e^2)}\sin m'}{1 - e\cos m'},$$

where the second term has also to be expanded in a series of multiple sines of m'; which can be done without difficulty.

12. *If* (u, v) *are given functions of the coordinates* (x, y), *neither of them a maximum or a minimum at a given point* O; *and if through* O *we draw* Ox' *in the direction in which* v *is constant and* u *increases, and* Oy' *in the direction in which* u *is constant and* v *increases; then the rotation* (*through an angle not greater than* π), *from* Ox' *to* Oy' *is in the same direction with that from* Ox *to* Oy, *or in the contrary direction, according as* $\dfrac{du}{dx}\dfrac{dv}{dy} - \dfrac{du}{dy}\dfrac{dv}{dx}$ *is positive or negative.*

The theorem has not, so far as I am aware, been noticed, and it seems to be one of some importance; there is no difficulty in it, but the answer requires some care in writing out; of course where the whole question is one of sign and direction, the omission to state that a subsidiary quantity is positive may render an answer worthless.

It depends on the following lemma: Consider the triangle $OX'Y'$, such that Ox, Oy being any rectangular axes through the origin O, the coordinates of X' are h, k, and those of Y' are h_1, k_1: then considering the area as positive, the double area is $= \pm(hk_1 - h_1k)$, viz. the sign is $+$ or $-$ according as the rotation from OX' to OY' (through an angle less than π) is in the same direction with that from Ox to Oy, or in the contrary direction; or, what is the same thing, $hk_1 - h_1k$ is in the first case positive and in the second case negative.

To show this, suppose for a moment that the lines OX', OY' are each of them in the quadrant xOy, say in the first quadrant, the inclination of OY' to Ox exceeding that of OX' to Ox; then h, k, h_1, k_1 are all positive, and $\dfrac{k_1}{h_1} > \dfrac{k}{h}$, that is, $hk_1 - h_1k$ is $+$, and the rotation from OX' to OY' is in the same direction as that from Ox to Oy; or the lemma holds good. Now OX' remaining fixed, let OY' revolve in the direction Ox to Oy; so long as OY' remains in the first quadrant, $\frac{k_1}{h_1}$ continues to increase, and we have always $\frac{k_1}{h_1} > \frac{k}{h}$, and $hk_1 - h_1k = +$; when OY' comes into the second quadrant (h, k being always positive), h_1 is negative and k_1 positive, consequently $hk_1 - h_1k$ is the sum of two positive terms, and therefore $= +$; as OY' continues to revolve and passes into the third quadrant, we have h_1, k_1 each negative, but $\frac{k_1}{h_1} < \frac{k}{h}$, and therefore $hk_1 - h_1k$ still $= +$; when, however, OY' comes into the position opposite to OX', then $\frac{k_1}{h_1} = \frac{k}{h}$, and $hk_1 - h_1k$ is $= 0$; and when OY', continuing in the third quadrant, has passed the position in question, we have $\frac{k_1}{h_1} > \frac{k}{h}$, and therefore $hk_1 - h_1k = -$, but now the angle $X'OY'$ measured in the original direction has become $> \pi$, and the rotation OX' to OY' through an angle less than π will be in the opposite direction, that is, in the direction opposite to that from Ox to Oy; and, similarly, when OY' passes into the fourth quadrant, and until, passing into the first quadrant, it approaches the position OX', the sign of $hk_1 - h_1k$ will be $-$, and the rotation will be in the direction contrary to that from Ox to Oy. The lemma is thus true for any position of OX' in the first quadrant; and the like reasoning would show that it is true for any position of OX' in the second, the third, or the fourth quadrant; hence the lemma is true generally.

This being so, taking a new origin, let the coordinates of O be x, y; and drawing through O the axes Ox', Oy' as directed, let X' be the point belonging to the values $u + \delta u$, v of (u, v), and Y' the point belonging to the values u, $v + \delta v$ of (u, v); taking δu positive, X' will be on Ox' in the direction O to x', and similarly taking δv positive, Y' will be on Oy' in the direction O to y'. Taking as before (h, k) for the coordinates of X', and (h_1, k_1) for the coordinates of Y', these coordinates being measured from the point O as origin, we have

$$\delta u = \frac{du}{dx} h + \frac{du}{dy} k,$$

$$0 = \frac{dv}{dx} h + \frac{dv}{dy} k,$$

whence, writing for a moment $J = \frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx}$, we have $Jh = +\frac{dv}{dy}\delta u$, $Jk = -\frac{dv}{dx}\delta u$. And in like manner

$$0 = \frac{du}{dx} h_1 + \frac{du}{dy} k_1,$$

$$\delta v = \frac{dv}{dx} h_1 + \frac{dv}{dy} k_1,$$

whence

$$Jh_1 = -\frac{du}{dy}\delta v, \quad Jk_1 = \frac{du}{dx}\delta v;$$

and hence

$$J^2(hk_1 - h_1k) = \left(\frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx}\right)\delta u\,\delta v, \ = J\delta u\delta v,$$

that is,

$$hk_1 - h_1k = \frac{1}{J}\delta u\delta v,$$

and δu, δv being as above each of them positive, J has the same sign as $hk_1 - h_1k$. But the rotation from OX' to OY' is in the same direction as that from Ox to Oy, or in the contrary direction, according as $hk_1 - h_1k$ is + or −, that is, according as $J, = \frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx}$, is + or −; which is the theorem in question.

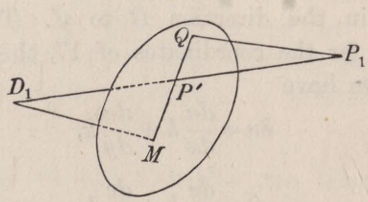
13. Write a dissertation on:

The theory and constructions of Perspective.

In Perspective we represent an object in space by means of its central projection upon a plane: viz. any point P_1* of the object is represented by P', the intersection with the plane of projection of the line D_1P_1 from the centre of projection (or say the eye) D_1 to the point P_1; and considering any line or curve in the object, this is represented by the line or curve which is the locus of the points P', the projections of the corresponding points P_1 of the line or the curve in the object.

The fundamental construction in perspective is derived from the following considerations: viz. considering through P_1 (fig. 1) a line meeting the plane of projection in Q, and drawing parallel thereto through D_1 a line to meet the plane of projection in M and joining the points M, Q, then the lines D_1M, MQ, QP_1 are in a plane; that is, the plane through D_1 and the line P_1Q meets the plane of projection in MQ;

Fig. 1.



and consequently the projection P' of any point P_1 in the line P_1Q lies in the line QM; and not only so, but considering only the points P_1 of this line which lie behind the plane of projection (D_1 being considered as in front of it), the projections of all these points lie on the terminated line MQ; viz. Q is the projection of the point Q, and M the projection of the point at infinity on the line QP_1; or, if we please, the finite line QM is the projection of the line $QP_1\infty$.

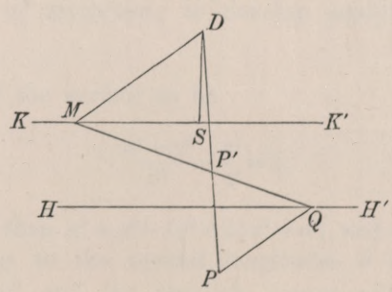
* The subscript unity is used to denote a point not in the plane of projection, considered as a point out of this plane; a point in the plane of projection, used in the constructions of perspective as a conventional representation of a point P_1, will be denoted by the same letter P without the subscript unity. And the like as regards D_1 and D.

If we consider a set of lines parallel to P_1Q, these all give rise to the same point M, and thus their projections MQ all pass through this point M, which is said to be the "vanishing point" of the system of parallel lines. Again, if we consider any two or more lines through P_1, to each of these there correspond different points M and Q, and, therefore, a different line MQ, but these all intersect in a common point P' which is the projection of P_1. If the lines are all in one and the same plane through P_1, then the locus of the points Q is a line, the intersection of this plane with the plane of projection, say the "trace" line; and the locus of the points M is a parallel line, the intersection of the parallel plane through D_1 with the plane of projection; say this is the "vanishing line" for the plane in question.

A construction in perspective presupposes a conventional representation on the plane of projection (or say on the paper) as well of the position of the eye as of the object to be projected. If for simplicity we suppose the object to be a figure in one plane, then this plane intersects the paper in a trace line, and we may imagine the plane made to rotate about the trace line until it comes to coincide with the paper, and we have thus the plane object conventionally represented on the paper. Similarly considering the parallel plane through the eye D_1, and regarding D_1 as a point of this plane, the plane meets the paper in the vanishing line, and we may imagine the plane made to rotate (in the direction opposite to that of the first rotation) until it comes to coincide with the paper, bringing the point D_1 to coincide with a point D of the paper. We have thus the "point of distance" D, being a conventional representation on the paper of the position of the eye D_1; but which point D has, observe, a different position for different directions of the plane of the object.

To fix the ideas, suppose the plane of projection to be vertical, and the plane of the object to be a horizontal plane situate below the eye. The trace line will be represented by a horizontal line HH' (fig. 2), and the object by a figure in the plane

Fig. 2.



of the paper below the line HH' such that, bending this portion of the paper backwards through a right angle round HH', the figure would be brought to coincide with the object*. The vanishing line will be a horizontal line KK' above HH', and the

* It is assumed in the text, that the figure on the paper is equal in magnitude to the object; but practically the figure is drawn on a reduced scale, the distance between the lines KK', HH', and the distance DS (representing respectively the distance between the parallel planes, and the distance of the eye from the plane of projection) being drawn on the same reduced scale.

eye will be represented by a point D above KK', in suchwise that, bending the upper part of the paper round KK' forwards through a right angle, the point D would come to coincide with the position D_1 of the eye. This being so, taking any line PQ in the representation of the object, we draw through D the parallel line DM, and then joining the points M and Q, we have MQ as the perspective representation of the line $QP\infty$, which represents a line $QP_1\infty$ of the object. And drawing through P any number of lines, each of these gives a point Q and a point M, but the lines MQ all meet in a common point P', which is the perspective representation of the point P; which point P' may, it is clear, be obtained as the intersection with any one line MQ of the line DP drawn to join P with the point of distance D. The plane of the object has for convenience been taken to be horizontal; but its position may be any whatever, and in particular the construction is equally applicable in the case where the plane is vertical.

In the case of an object not in one plane, any point Q_1 of the object may be determined by means of its projection by a vertical line upon a given horizontal plane, say this is P_1, and of its altitude Q_1P_1 above this plane. We in fact determine the object by means of its groundplan, and of the altitudes of the several points thereof. It is easy, from the foregoing principles, to see that, drawing through P the vertical line PQ equal to the altitude, and joining the points Q, D, then the vertical line through P' meets this line QD in a point Q', which will be the perspective representation of Q_1. We have thus a construction applicable to any solid figure whatever.