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ADDITION TO PROF. HALL'S PAPER "ON THE MOTION OF A PARTICLE TOWARD AN ATTRACTING CENTRE AT WHICH THE FORCE IS INFINITE."

[From the *Messenger of Mathematics*, vol. III. (1874), pp. 149—152.]

I DO not in the passage referred to* expressly profess to interpret Newton's idea. After referring to his investigation I say, "The method has the advantage of explaining the paradoxical result which presents itself in the case force $\propto$ (dist.)$^{-2}$, and in some other cases where the force becomes infinite. According to theory the velocity becomes infinite at the centre, but the direction of the motion is there abruptly reversed, so that the body in its motion does not pass through the centre, but on arriving there forthwith returns towards its original position; of course such a motion cannot occur in nature, where neither a force nor a velocity is actually infinite;" viz. while assuming that the analysis gives a motion as just described, or in Prof. Hall's figure, a reciprocating motion between A and C, I expressly state that the motion is not one that can occur in nature; in fact, my view is that the question (which, to render it precise, I state as follows: "What happens in nature when the moving point arrives at C") presupposes what is inconceivable. But I consider that the analysis gives a motion as above, viz. that it gives x, t each as a one-valued function of a parameter ϕ, such that this parameter ϕ increasing continuously, we have for the moving point a continuous series of positions corresponding to the motion in question, gives in fact the equations $x = a(1 - \cos\phi)$ and $\frac{t\sqrt{(\mu)}}{a^{\frac{3}{2}}} = \phi - \sin\phi$.

In explanation and justification of the assumption, it is interesting to show how the solution just referred to can be obtained from the equation of motion $\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}$, without (in the process) the extraction of the square root of the two sides of an

[* By Professor Hall in his paper (p. 144, *l.c.*) quoted in the title. The passage is an extract from the British Association Report (1862) *On the progress of the solution of certain special problems of dynamics*, p. 186; [298], *Coll. Math. Papers*, vol. IV. p. 515.]

equation. Taking x as the independent variable and writing for a moment $\frac{dt}{dx} = t'$, $\frac{d^2t}{dx^2} = t''$, the equation is

$$\frac{1}{t'^3} t'' = \frac{\mu}{x^2},$$

and if we herein assume $x = a(1 - \cos\phi)$ and transform to ϕ as the independent variable, it becomes

$$\frac{a^2 \sin^2\phi}{\left(\frac{dt}{d\phi}\right)^3} \left\{\frac{1}{a\sin\phi} \frac{d^2t}{d\phi^2} - \frac{\cos\phi}{a\sin^2\phi} \frac{dt}{d\phi}\right\} = \frac{\mu}{a^2(1-\cos\phi)^2};$$

or, what is the same thing,

$$\sin\phi \frac{d}{d\phi}\left(\frac{dt}{d\phi}\right) - \cos\phi \left(\frac{dt}{d\phi}\right) = \frac{\mu}{a^3} \frac{1}{(1-\cos\phi)^2} \left(\frac{dt}{d\phi}\right)^3,$$

a differential equation of the first order for the determination of $\frac{dt}{d\phi}$ as a function of ϕ. Since a is a constant of integration of the original equation, a particular integral only is required, but it is as well to obtain the general integral. For this purpose assume

$$\frac{dt}{d\phi} = \frac{a^{\frac{3}{2}}}{\sqrt{(\mu)}} z(1 - \cos\phi);$$

then, omitting from each side of the equation the factor $\frac{a^{\frac{3}{2}}}{\sqrt{(\mu)}}$, the equation becomes

$$\sin\phi \left\{z\sin\phi + \frac{dz}{d\phi}(1-\cos\phi)\right\} - \cos\phi \,.\, z(1-\cos\phi) = (1-\cos\phi) z^3,$$

viz. the left-hand side being $(1-\cos\phi)\left(z + \frac{dz}{d\phi}\sin\phi\right)$, the whole equation contains the factor $(1-\cos\phi)$, and omitting this, the equation becomes

$$z + \frac{dz}{d\phi}\sin\phi = z^3;$$

or, what is the same thing,

$$\frac{dz}{z^3 - z} = \frac{d\phi}{\sin\phi}.$$

The integral of this is

$$\log\frac{z^2-1}{z^2} = 2\log k + 2\log\tan\tfrac{1}{2}\phi;$$

or, what is the same thing,

$$\frac{z^2-1}{z^2} = k^2\tan^2\tfrac{1}{2}\phi,$$

where k is the constant of integration.

[In explanation of this constant k, observe that the equation gives

$$z = \frac{1}{\sqrt{(1 - k^2\tan^2\frac{1}{2}\phi)}},$$

and that we thence have

$$\frac{dt}{d\phi}=\frac{a^{\frac{3}{2}}}{\sqrt{(\mu)}}\,\frac{1-\cos\phi}{\sqrt{(1-k^2\tan^2\frac{1}{2}\phi)}};$$

that is,

$$\frac{dt}{dx}=\frac{\sqrt{(\mu)}}{a^{\frac{1}{2}}}\,\frac{\sin\phi}{1-\cos\phi}\,\frac{1}{\sqrt{(1-k^2\tan^2\frac{1}{2}\phi)}},\quad=\frac{a^{\frac{1}{2}}}{\sqrt{(\mu)}}\,\frac{\tan\frac{1}{2}\phi}{\sqrt{(1-k^2\tan^2\frac{1}{2}\phi)}},$$

or, since

$$\tan^2\tfrac{1}{2}\phi=\frac{x}{2a-x},$$

this is

$$\frac{dt}{dx}=\frac{a^{\frac{1}{2}}}{\sqrt{(\mu)}}\,\frac{\sqrt{(x)}}{\sqrt{(2a-x-k^2x)}};$$

or, what is the same thing,

$$\frac{dt}{dx}=\frac{\sqrt{\left(\dfrac{a}{1+k^2}\right)}}{\sqrt{(\mu)}}\,\frac{\sqrt{(x)}}{\sqrt{\left(2\dfrac{a}{1+k^2}-x\right)}},$$

viz. we in effect have $\dfrac{a}{1+k^2}$ as a constant of integration in place of the original constant a.]

Recurring to the general solution

$$\frac{z^2-1}{z^2}=k^2\tan^2\tfrac{1}{2}\phi,$$

we may take $z=1$, as a particular solution answering to the value $k=0$ of the constant; and we then have

$$\frac{dt}{d\phi}=\frac{a^{\frac{3}{2}}}{\sqrt{(\mu)}}(1-\cos\phi),$$

viz. reckoning t from the epoch for which ϕ is $=0$, we thus have

$$t=\frac{a^{\frac{3}{2}}}{\sqrt{(\mu)}}(\phi-\sin\phi),$$

which, combined with the assumed equation

$$x=a(1-\cos\phi),$$

gives the foregoing solution.

I quite admit that, considering (with Prof. Hall) the attracted particle as split into two equal particles placed at equal distances above and below the centre C, the motion when the distances become infinitesimal is a motion not as above, but backwards and forwards along the entire line AB; but it remains to be seen whether *at* the limit this can be brought out as an analytical solution of the differential equation $\dfrac{d^2x}{dt^2}=-\dfrac{\mu}{x^2}$. Possibly this may be done, and I remark as an objection, not to the foregoing as an admissible solution of the problem but to its generality as the *only* solution, that, in writing $x=a(1-\cos\phi)$ and assuming that ϕ is real, I in effect assume that x is always positive. But the burthen of the proof is with Prof. Hall, to show that there is an analytical solution in which x acquires negative values.