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Existence and uniqueness of solutions of the initial boundary value problem for the flow of a barotropic viscous fluid, local in time

W. FISZDON and W. M. ZAJĄCZKOWSKI (WARSZAWA)

IN THIS PAPER the existence and uniqueness of local in time solutions of the initial boundary value problem for the flow of barotropic viscous fluid in a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is proved. Only the case where the fluid enters the domain is considered. Density and the velocity vector are assumed as initial and boundary conditions. The paper is divided into three main parts: 1) for a given velocity vector, the proof of existence and estimate for density; 2) for a given density, the proof of existence and estimate for the velocity vector; 3) the existence and uniqueness of solutions of the considered problem using the method of successive approximations. The existence of generalized solutions such that density and the velocity vector belong to $L_\infty(0, T; H^2(\Omega))$ and $L_\infty(0, T; H^2(\Omega))\cap L_2(0, T; H^3(\Omega))$, respectively, is shown.

W pracy pokazano istnienie i jednoznaczność rozwiązań lokalnych w czasie problemu początkowo-brzegowego dla przepływu ściśliwego barotropowego w ograniczonym obszarze $\Omega \subset \mathbb{R}^n$, $n = 2, 3$. Rozpatrzony został przypadek, gdy ciecz tylko wpływa do rozważanego obszaru. Jako warunki początkowe i brzegowe przyjęto gęstość i prędkość. Praca dzieli się na trzy główne części: 1) dowód istnienia i oszacowanie na gęstość przy danej prędkości, 2) dowód istnienia i oszacowanie na prędkość przy danej gęstości, 3) dowód istnienia rozwiązań rozpatrywanego problemu przy zastosowaniu metody kolejnych przybliżeń. Pokazano istnienie rozwiązań uogólnionych takich, że gęstość należy do $L_\infty(0, T; H^2(\Omega))$, a prędkość do $L_\infty(0, T; H^2(\Omega))\cap$ $\cap L_2(0, T; H^3(\Omega))$.

В работе показано существование и единственность решений, локальных во времени, начально-краевой задачи для сжимаемого баротропного течения в ограниченной области $\Omega \subset \mathbb{R}^n$, $n = 2, 3$. Рассмотрен случай, когда жидкость только втекает в рассматриваемую область. Как начальные и граничные условия задаются плотность и скорость. Работа разделяется на три главные части: 1) доказательство существования и оценка плотности при заданной скорости, 2) доказательство существования и оценка скорости при заданной плотности, 3) доказательство существования решений рассматриваемой задачи, применяя метод последовательных приближений. Показано существование обобщенных решений таких, что плотность принадлежит к $L_\infty(0, T; H^2(\Omega))$, а скорость к $L_\infty(0, T; H^2(\Omega)) \cap L\ (0, T; H^3(\Omega))$.

1. Introduction

IN THIS PAPER we consider the initial boundary value problem for a compressible viscous barotropic fluid flow in a bounded domain. We assume that the fluid enters the domain only. This paper is an introduction to show the existence of global solutions of the problem, what will be the matter of the next paper.

The existence and uniqueness of local in time solutions were obtained by NASH [15] and ITAYA [4, 5] for the initial value problem. Theorems for the first initial boundary value problem, local in time, were obtained by TANI [18], SOLONNIKOW [16] and LUKASZEWICZ [13].

The existence of solutions global in time for the initial boundary value problem in the one-dimensional case was considered by KANEL [6] and KAZHIHOW [7, 8, 9]. In the three-dimensional case the existence of global in time solutions for the Cauchy problem was considered by MATSUMURA and NISHIDA [14] for small initial values only.

In this paper we consider the two- and three-dimensional initial boundary value problem in a bounded simply connected domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$. The conservation equations of a compressible viscous fluid flow are [12]

(1.1) $$\varrho'_t + (\varrho' v^i)_{,x^i} = 0,$$

(1.2) $$\varrho' v^i_t + \varrho' v^j v^i_{x^j} - \mu' v^i_{x^j x^j} - \nu' v^j_{x^i x^j} + p'_{x^i} = \varrho' f^i,$$

where ϱ' — density, $\varrho' > 0$, $v = (v^1, \ldots, v^n)$ — velocity, p' — pressure, $f = (f^1, \ldots, f^n)$— external force, μ', ν' — shear and bulk viscosities, $\mu' > 0$, $\nu' > 0$.

The summation convention is used and the subscripts x^i, t denote partial derivation with respect to the corresponding variable, $i = 1, \ldots, n$.

We assume the barotropy condition

(1.3) $$p' = R' \varrho'^{\gamma},$$

where R' is a constant. Moreover, the following initial and boundary conditions are assumed:

(1.4) $$\varrho'|_{t=0} = \sigma'(x), \quad v|_{t=0} = a(x),$$

(1.5) $$v|_{\partial\Omega} = \eta(x', t), \quad \varrho'|_{\partial\Omega} = b'(x', t), \quad x' \in \partial\Omega.$$

Equations (1.4) and (1.5) imply the following compatibility conditions:

(1.6) $$\eta|_{t=0} = a|_{\partial\Omega}, \quad \sigma'|_{\partial\Omega} = b'|_{t=0}.$$

As will be seen in the next paper about the existence of global solutions, to have a global estimate and hence a global existence theorem, we must assume that $\varrho'|_{\partial\Omega}$ is given and the condition

(1.7) $$-\eta \cdot \bar{n} = d(x', t) \geqslant d_0 > 0, \quad x' \in \partial\Omega, \quad d_0 = \text{const},$$

has to be imposed. Here $\bar{n}$ is the outward unit normal vector to the boundary and the above relation says the fluid enters the considered domain Ω.

This paper is an extended version of [3].

2. Notations and basic lemmas

We introduce the function β such that

(2.1) $$\beta|_{\partial\Omega} = \eta(x', t), \quad x' \in \partial\Omega,$$

and a new dependent variable

(2.2) $$u = v - \beta.$$

Moreover, we assume that density and its initial and boundary values have small deviation from the equilibrium condition, hence we can denote

(2.3) $$\varrho' = \varrho_0(1+\varrho), \quad b' = \varrho_0(1+b), \quad \sigma' = \varrho_0(1+\sigma),$$

where ϱ_0 is a constant and denotes the equilibrium magnitude. Using the above new variables, Eqs. (1.1) and (1.2) can be written in the form

(2.4) $$\varrho_t = -[(1+\varrho)v^i]_{,x^i},$$

(2.5) $$u^i_t = \frac{\mu}{1+\varrho} u^i_{x^k x^k} + \frac{\nu}{1+\varrho} u^j_{x^i x^j} - u^j u^i_{x^j} - \frac{R[(1+\varrho^\gamma)]_{,x^i}}{1+\varrho} - u^j \beta^i_{x^j} - \beta^j u^i_{x^j} + f'^i,$$

where

$$\mu = \frac{\mu'}{\varrho_0}, \quad \nu = \frac{\nu'}{\varrho_0}, \quad R = R'\varrho_0^{\gamma-1},$$

(2.6) $$f'^i = f^i + \frac{\mu}{1+\varrho} \beta^i_{x^j x^j} + \frac{\nu}{1+\varrho} \beta^j_{x^i x^j} - \beta^i_t - \beta^j \beta^i_{x^j},$$

with the following initial and boundary conditions:

(2.7) $$u|_{t=0} = a - \beta|_{t=0}, \quad u|_{\partial\Omega} = 0,$$

and

(2.8) $$\varrho|_{t=0} = \sigma, \quad \varrho|_{\partial\Omega} = b.$$

To prove the existence and uniqueness of solutions of the problem (2.4) ÷ (2.8), we shall use the following method of successive approximations:

(2.9) $$\overset{m}{\varrho}_t = -[(1+\overset{m}{\varrho})\overset{m-1}{v^i}]_{,x^i},$$

(2.10) $$\overset{m}{\varrho}|_{t=0} = \sigma, \quad \overset{m}{\varrho}|_{\partial\Omega} = b,$$

where $\overset{m-1}{v} = \overset{m-1}{u} + \beta$ is a given function, and

(2.11) $$\overset{m}{u^i_t} = \frac{\mu}{1+\overset{m}{\varrho}} u^i_{x^j x^j} + \frac{\nu}{1+\overset{m}{\varrho}} \overset{m}{u^j_{x^i x^j}} - \overset{m-1}{u^j}\overset{m}{u^i_{x^j}} - \overset{m-1}{u^j}\beta^i_{x^j} - \beta^j \overset{m}{u^i_{x^j}} - \frac{R[(1+\overset{m}{\varrho})^\gamma]_{,x^i}}{1+\overset{m}{\varrho}} + \overset{m}{f'^i},$$

(2.12) $$\overset{m}{u}|_{t=0} = a - \beta|_{t=0}, \quad \overset{m}{u}|_{\partial\Omega} = 0,$$

where

(2.13) $$\overset{m}{f'^i} = f^i + \frac{\mu}{1+\overset{m}{\varrho}} \beta^i_{x^j x^j} + \frac{\nu}{1+\overset{m}{\varrho}} \beta^j_{x^i x^j} - \beta^i_t - \beta^j \beta^i_{x^j},$$

and $\overset{m}{\varrho}$, $\overset{m-1}{u}$ are given functions. Moreover, we assume that

(2.14) $$\overset{\circ}{u} = 0,$$

and m is a natural number.

In the neighbourhood $U(q)$, $q \in \partial\Omega$ of the boundary let us introduce the curvilinear coordinates $\tau_1(x), \ldots, \tau_{n-1}(x)$, $n(x)$, corresponding to the orthonormal vectors $\bar{\tau}_1(x), \ldots, \bar{\tau}_{n-1}(x)$, $\bar{n}(x)$, such that $\bar{\tau}_1(x), \ldots, \bar{\tau}_{n-1}(x)$, $x \in U(q)$ are tangent to $\partial\Omega$ and $\bar{n}(x)$ is the outward vector normal to $\partial\Omega$.

Now we introduce some spaces and inequalities. The norms of spaces $W_p^l(\Omega)$ and $L_p(\Omega)$ will be denoted by $\|\ \|_{l,p,\Omega}$ and $\|\ \|_{p,\Omega}$, respectively. We denote $H^l(\Omega) = W_2^l(\Omega)$. We shall use the following Poincaré inequality [10, Chapt. 2]:

$$\|u\|_{p,\Omega} \leqslant C(p)|\Omega|^{1/n-(1/2-1/p)}\|u_x\|_{2,\Omega} \equiv \alpha_p\|u_x\|_{2,\Omega}, \tag{2.15}$$

which is valid for $u \in H_0^1(\Omega) = \{w \in H^1(\Omega): w|_{\partial\Omega} = 0\}$, $|\Omega| = \mathrm{mes}\,\Omega < \infty$. From [2, Chapt. 3] we have the following interpolation inequality:

$$a\|u\|_{p,\Omega}^2 \leqslant \varepsilon\|D_x^l u\|_{2,\Omega}^2 + Ca^{\frac{1}{1-\varkappa}}\varepsilon^{-\frac{\varkappa}{1-\varkappa}}\|u\|_{2,\Omega}^2, \tag{2.16}$$

where $l = 1, 2$, $\varkappa = \dfrac{n}{l}(1/2-1/p) < 1$, $0 < a \in \mathrm{R}$, $n = 2, 3$ and $\varepsilon > 0$, is in general a small number. Moreover, from [2, Chapt. 3] we have

$$a\|u\|_{p,\partial\Omega}^2 \leqslant \varepsilon\|D_x^l u\|_{2,\Omega} + Ca^{\frac{1}{1-\sigma}}\varepsilon^{\frac{\sigma}{1-\sigma}}\|u\|_{2,\Omega}^2, \tag{2.17}$$

where $l = 1, 2$, $\sigma = \dfrac{1}{l}\left(\dfrac{n}{2} - \dfrac{n-1}{p}\right) < 1$, $0 < a \in \mathrm{R}$, $n=2, 3$, and $\varepsilon > 0$. Let B be a Banach space, k a nonnegative integer and T some positive constant. $L_p^k(0, T; B)$ is a Banach space of functions $f(t)$ on $[0, T]$ which have values in B for every fixed $t \in [0, T]$ and its k-time derivative has the norm $\left(\int_0^T \|D_t^k f\|_B^p\,dt\right)^{1/p}$ bounded. Now we introduce the space $\Pi_{k,p}^l(\Omega^T) = \bigcap_{i=k}^{l} L_p^{l-i}(0, T; H^i(\Omega))$, with the norm

$$|u|_{l,k,p,\Omega^T} = \sum_{i=k}^{l}\left(\int_0^T \|D_t^{l-i}u\|_{i,2,\Omega}^p\,dt\right)^{1/p}, \tag{2.18}$$

where $\Omega^T = \Omega \times [0, T]$, and the space $\Gamma_k^l(\Omega)$ with the norm

$$|u|_{l,k,\Omega} = \sum_{i=k}^{l}\|D_t^{l-i}u\|_{i,2,\Omega}. \tag{2.19}$$

For functions defined on the boundary $\partial\Omega$ we introduce the spaces $\Pi_{k,p}^{l+1/2}(\partial\Omega^T) = \bigcap_{i=k}^{l} L_p^{l-i}(0, T; H^{i+1/2}(\partial\Omega))$ and $\Gamma_k^{l+1/2}(\partial\Omega)$ with the norm

$$|u|_{l+1/2,k,\partial\Omega} = \sum_{i=k}^{l}\|D_t^{l-i}u\|_{i+1/2,2,\partial\Omega}, \tag{2.20}$$

where l, k, i are natural numbers.

3. The existence of local solutions of the problem (2.9) and (2.10)

At first we consider the lemma.

LEMMA 3.1.

Let $\partial\Omega$ be of class C^2, $d \geqslant d_0 = \mathrm{const}$, $b \in \Gamma_0^2(\partial\Omega)$, $\eta \in \Gamma_1^2(\partial\Omega)$, $\overset{m-1}{v} \in \Gamma_1^3(\Omega)$, then for an arbitrary smooth solution of the problem (2.9) and (2.10) the following differential inequality holds:

$$(3.1)\qquad \begin{aligned}\frac{d}{dt}|\overset{m}{\varrho}(t)|^2_{2,0,\Omega} \leqslant C_1[(1+|\eta|^6_{2,1,\Omega})|b|^2_{2,0,\partial\Omega} + (1+|b|^2_{2,0,\partial\Omega})|\overset{m-1}{v}|^2_{3,2,\Omega}\\ \cdot(1+|\overset{m-1}{v}|^2_{2,1,\Omega})] + C_2|\overset{m}{\varrho}|^2_{2,0,\Omega}(1+|\overset{m-1}{v}|_{3,1\Omega}) + C_2|\overset{m-1}{v}|^2_{3,1,\Omega},\end{aligned}$$

where $C_1 = C_1(d_0, \max\limits_{\partial\Omega} d, \max\limits_{\partial\Omega}(|\bar{h}_x|, |\tau_x|))$, $C_2 = C_2(r, \Omega)$.

P r o o f. To prove the estimate (3.1) we assume that b, η, $\overset{m-1}{v}$, $\overset{m}{\varrho}$ are smooth functions in $\bar{\Omega}$. Then from Eq. (2.9) we have

$$(3.2)\qquad \sum_{|\nu|=j\leqslant 2}\int_\Omega D^j_{x^\nu}(\overset{m}{\varrho}_t + \overset{m-1}{v}\cdot\nabla\overset{m}{\varrho} + (1+\overset{m}{\varrho})\operatorname{div}\overset{m-1}{v})D^j_{x^\nu}\overset{m}{\varrho}\,dx = 0,$$

where

$$\nu = (\nu_0, \ldots, \nu_n), \qquad D_{x^\nu} = \left(\frac{\partial}{\partial t}\right)^{\nu_0}\ldots\left(\frac{\partial}{\partial x^n}\right)^{\nu_n}, \qquad \nu_0 + \ldots + \nu_n = j.$$

From Eqs. (3.2) and (1.7) we obtain

$$(3.3)\qquad \frac{d}{dt}|\overset{m}{\varrho}|^2_{2,0,\Omega} \leqslant \sum_{|\nu|=j\leqslant 2}\int_{\partial\Omega} d|D^j_{x^\nu}\overset{m}{\varrho}|^2 ds + C(|\overset{m}{\varrho}|^2_{2,0,\Omega} + |\overset{m}{\varrho}|_{2,0,\Omega})|\overset{m-1}{v}|_{3,1,\Omega}.$$

To estimate the surface integral appearing in Eq. (3.3) we assume that Eq. (2.9) is satisfied on $\partial\Omega$. Then using the curvilinear coordinates introduced in Section 2, we have

$$(3.4)\qquad \overset{m}{\varrho}_{,n} = -\frac{1}{v_n}[\overset{m}{\varrho}_t + \overset{m-1}{v}_\mu\overset{m}{\varrho}_{,\tau_\mu} + (1+\overset{m}{\varrho})\operatorname{div}\overset{m-1}{v}],$$

where $v_\mu = v\cdot\bar{\tau}_\mu$, $v_n = v\cdot\bar{n}$. Using Eqs. (1.5) and (2.10) the expression (3.4) yields

$$(3.5)\qquad \overset{m}{\varrho}_{,n}|_{\partial\Omega} = \frac{1}{d}[b_t + \eta_\mu b_{,\tau_\mu} + (1+b)(\operatorname{div}\overset{m-1}{v})|_{\partial\Omega}],$$

$$(3.6)\qquad \begin{aligned}\overset{m}{\varrho}_{,nn}|_{\partial\Omega} = \frac{1}{d^2}\overset{m-1}{v}_{n,n}|_{\partial\Omega}[b_t + \eta_\mu b_{,\tau_\mu} + (1+b)(\operatorname{div}\overset{m-1}{v})|_{\partial\Omega}] + \frac{1}{d}\overset{m}{\varrho}_{,nt}\\ + \overset{m-1}{v}_{\mu,n}|_{\partial\Omega}b_{,\tau_\mu} + \eta_\mu\overset{m}{\varrho}_{,\tau_\mu,n} + \overset{m}{\varrho}_{,n}(\operatorname{div}\overset{m-1}{v})|_{\partial\Omega} + (1+b)(\operatorname{div}\overset{m-1}{v})_{,n}|_{\partial\Omega}],\end{aligned}$$

$$(3.7)\qquad \begin{aligned}\overset{m}{\varrho}_{,nt}|_{\partial\Omega} = -\frac{1}{d^2}d_t[b_t + \eta_\mu b_{\tau_\mu} + (1+b)(\operatorname{div}\overset{m-1}{v})|_{\partial\Omega}] + \frac{1}{d}[b_{tt} + \eta_\mu b_{,t\tau_\mu}\\ + \eta_{\mu,t}b_{,\tau_\mu} + b_t(\operatorname{div}\overset{m-1}{v})|_{\partial\Omega} + (1+b)(\operatorname{div}\overset{m-1}{v}_t)|_{\partial\Omega}],\end{aligned}$$

$$(3.8)\qquad \begin{aligned}\overset{m}{\varrho}_{,n\tau_\nu}|_{\partial\Omega} = -\frac{1}{d^2}d_{,\tau_\nu}[b_t + \eta_\mu b_{,\tau_\mu} + (1+b)(\operatorname{div}\overset{m-1}{v})|_{\partial\Omega}] + \frac{1}{d}[b_{,t\tau_\nu}\\ + \eta_{\mu,\tau_\nu}b_{,\tau_\mu} + \eta_\mu b_{,\tau_\mu\tau_\nu} + b_{,\tau_\nu}(\operatorname{div}\overset{m-1}{v})|_{\partial\Omega} + (1+b)(\operatorname{div}\overset{m-1}{v})_{,\tau_\nu}|_{\partial\Omega}].\end{aligned}$$

Moreover, we have

$$\varrho_{,n\tau_\nu} = \varrho_{,\tau_\nu n} + (\tau^k_\nu n^i_{x^k} - \tau^i_{\nu,x^k} n^k)\varrho_{,x^i}. \tag{3.9}$$

From Eqs. (3.5)÷(3.9) we have

$$\int_{\partial\Omega} d|\overset{m}{\varrho}_{,n}|^2 ds \leqslant \frac{1}{d_0}\left[(1+|\eta|^2_{2,1,\partial\Omega})|b|^2_{2,0,\partial\Omega} + (1+|b|^2_{2,0,\partial\Omega})\|\overset{m-1}{v}_x|_{\partial\Omega}\|^2_{2,\partial\Omega}\right],$$

$$\int_{\partial\Omega} d|\overset{m}{\varrho}_{,n\tau}|^2 ds \leqslant C(d_0)(1+\|d\|^2_{2,2,\partial\Omega})\left[(1+|\eta|^2_{2,1,\partial\Omega})|b|^2_{2,0,\partial\Omega} + (1+|b|^2_{2,0,\partial\Omega}) \cdot \|\overset{m-1}{v}_x|_{\partial\Omega}\|^2_{1,2,\partial\Omega}\right],$$

$$\int_{\partial\Omega} d|\overset{m}{\varrho}_{,nt}|^2 ds \leqslant C(d_0)(1+|d|^2_{2,1,\partial\Omega})\left[(1+|\eta|^2_{2,1,\partial\Omega})|b|^2_{2,0,\partial\Omega} + (1+|b|^2_{2,0,\partial\Omega} \cdot (\|\overset{m-1}{v}_x|_{\partial\Omega}\|^2_{1,2,\partial\Omega} + \|\overset{m-1}{v}_{,tx}|_{\partial\Omega}\|^2_{2,\partial\Omega}\right], \tag{3.10}$$

$$\int_{\partial\Omega} d|\overset{m}{\varrho}_{,nn}|^2 ds \leqslant C(d_0, |\tau_x|, |n_x|)\left[(1+|\eta|^2_{2,1,\partial\Omega} + |\eta|^4_{2,1,\partial\Omega} + |\eta|^6_{2,1,\partial\Omega}) \cdot |b|^2_{2,0,\partial\Omega} + (1+|b|^2_{2,0,\partial\Omega} + |\eta|^2_{2,1,\partial\Omega} + |\eta|^2_{2,1,\partial\Omega}|b|^2_{2,0,\partial\Omega} + |\eta|^4_{2,1,\partial\Omega}|b|^2_{2,0,\partial\Omega}) \cdot (\|\overset{m-1}{v}_x|_{\partial\Omega}\|^2_{1,2,\partial\Omega} + \|\overset{m-1}{v}_x|_{\partial\Omega}\|^2_{2,\partial\Omega}(\sup_{\partial\Omega}|\overset{m-1}{v}_x|_{\partial\Omega}|)^2 + \|\overset{m-1}{v}_{xx}|_{\partial\Omega}\|^2_{2,\partial\Omega} + \|\overset{m-1}{v}_{xx}|_{\partial\Omega}\|^2_{2,\partial\Omega} + \|\overset{m-1}{v}_{xt}|_{\partial\Omega}\|^2_{2,\partial\Omega})\right].$$

Using Eqs. (3.10) and (2.17) the surface integral in Eq. (3.3) is estimated by

$$\sum_{|\nu|=j\leqslant 2}\int_{\partial\Omega} d|D^j_{x^\nu}\overset{m}{\varrho}|^2 ds \leqslant C\big(d_0 \max_{\partial\Omega} d, \max_{\partial\Omega}(|n_x|, |\tau_x|)\big)\left[(1+|\eta|^2_{2,1,\partial\Omega} + |\eta|^4_{2,1,\partial\Omega} + |\eta|^6_{2,1,\partial\Omega})|b|^2_{2,0,\partial\Omega} + (1+|\eta|^2_{2,1,\partial\Omega} + (1+|\eta|^2_{2,1,\partial\Omega} + |\eta|^4_{2,1,\partial\Omega}) \cdot |b|^2_{2,0,\partial\Omega})|\overset{m-1}{v}|^2_{3,2,\Omega}(1+|\overset{m-1}{v}|^2_{2,1,\Omega})\right]. \tag{3.11}$$

Therefore Eqs. (3.3) and (3.11) give Eq. (3.1). This completes the proof.

Now we prove the existence of solutions of the problem (2.9) and (2.10).

LEMMA 3.2.

Let $\partial\Omega$ be of class C^2, $d \geqslant d_0 = \text{const}$, $b \in \Pi^2_{0,2}(\partial\Omega^T)\cap\Pi^2_{0,\infty}(\partial\Omega^T)$, $\eta \in \Pi^2_{1,\infty}(\partial\Omega^T)$, $\overset{m-1}{v} \in \Pi^3_{1,2}(\Omega^T)\cap\Pi^2_{1,\infty}(\Omega^T)$, $\sigma \in H^2(\Omega)$, $a \in H^3(\Omega)$, $f(0) \in H^1(\Omega)$ and $|\sigma| \leqslant 1/2$, then there exists a unique solution of the problem (2.9) and (2.10) such that $\overset{m}{\varrho} \in \Pi^2_{0,\infty}(\Omega^T)$ and the following estimate is valid:

$$|\overset{m}{\varrho}(t)|^2_{2,0,\Omega} \leqslant \exp[C_2(T+T^{1/2}|\overset{m-1}{v}|_{3,1,2,\Omega^T})]\left[C_1(1+|\eta|^6_{2,1,\infty,\partial\Omega^T})|b|^2_{2,0,2,\partial\Omega^T} + (1+|\eta|^4_{2,1,\infty,\partial\Omega^T})(1+|b|^2_{2,0,\infty,\partial\Omega^T})|\overset{m-1}{v}|_{3,2,2,\Omega^T}(1+|\overset{m-1}{v}|^2_{2,1,\infty,\Omega^T}) + C_2|\overset{m-1}{v}|^2_{3,1,2,\Omega^T} + |\overset{m}{\varrho}(0)|^2_{2,0,\Omega}\right], \tag{3.12}$$

for $t \in [0, T]$, where

$$(3.13)\qquad |\overset{m}{\varrho}(0)|^2_{2,0,\Omega} \leqslant C_3(1+\|a\|^2_{3,2,\Omega}+\|a\|^4_{3,2,\Omega}+\|\sigma\|^2_{2,2,\Omega}\|a\|^2_{3,2,\Omega} + \|\sigma\|^{2\varkappa}_{2,2,\Omega}+\|f(0)\|^2_{1,2,\Omega})\|\sigma\|^2_{2,2,\Omega}.$$

Proof. The existence of solutions of the problem (2.9) and (2.10) follows from the method of characteristics [13], [16], [19]. The estimate (3.12) is obtained from Eq. (3.1) by integrating with respect to time. The estimate (3.13) follows from Eqs. (2.4) and (2.5) for v in the initial moment where we used the initial values (1.4). In this case the assumption $|\sigma| \leqslant 1/2$ must be used. This ends the proof.

Let us introduce a new variable $\psi = \dfrac{1}{1+\overset{m}{\varrho}}$, then instead of Eqs. (2.9) and (2.10) we obtain

$$(3.14)\qquad \overset{m}{\psi}_t + \overset{m-1}{v}\cdot\nabla\overset{m}{\psi} - \overset{m}{\psi}\operatorname{div}\overset{m-1}{v} = 0,$$

$$(3.15)\qquad \overset{m}{\psi}|_{t=0} = \frac{1}{1+\sigma} \equiv \bar{\sigma}, \qquad \overset{m}{\psi}|_{\partial\Omega} = \frac{1}{1+b} \equiv \bar{b}.$$

Therefore we can formulate the following lemma:

LEMMA 3.3.

Let $\partial\Omega$ be of class C^2, $d \geqslant d_0 = \text{const}$, $\bar{b} \in \Pi^2_{0,2}(\partial\Omega^T)\cap\Pi^2_{0,\infty}(\partial\Omega^T)$, $\eta \in \Pi^2_{1,\infty}(\partial\Omega^T)$, $\overset{m-1}{v} \in \Pi^3_{1,2}(\Omega^T)\cap\Pi^2_{1,\infty}(\Omega^T)$, $\bar{\sigma} \in H^2(\Omega)$, $a \in H^3(\Omega)$, $f(0) \in H^1(\Omega)$ and $|\sigma| \leqslant 1/2$, then there exists a unique solution of the problem (3.14) and (3.15) such that $\overset{m}{\psi} \in \Pi^2_{0,\infty}(\Omega^T)$ and the following estimate is valid:

$$(3.16)\qquad |\overset{m}{\psi}(t)|^2_{2,0,\Omega} \leqslant \exp[C_2(T+T^{1/2}|\overset{m-1}{v}|_{3,1,2,\Omega^T})]\,[C_1(1+|\eta|^6_{2,1,\infty,\partial\Omega^T})|\bar{b}|^2_{2,0,2,\partial\Omega^T} + (1+|\eta|^4_{2,1,\infty,\partial\Omega^T})(1+|\bar{b}|_{2,0,\infty,\partial\Omega^T})|\overset{m-1}{v}|_{3,2,2,\Omega^T}(1+|\overset{m-1}{v}|^2_{2,1,\infty,\Omega^T}) + C_2|\overset{m-1}{v}|^2_{3,1,2,\Omega^T}+|\overset{m}{\psi}(0)|^2_{2,0,\Omega}],$$

for $t \in [0, T]$, where

$$(3.17)\qquad |\overset{m}{\psi}(0)|^2_{2,0,\Omega} \leqslant C_3(1+\|a\|^2_{3,2,\Omega}+\|a\|^4_{3,2,\Omega}+\|\bar{\sigma}\|^2_{2,2,\Omega}\|a\|^2_{3,2,\Omega} + \|\sigma\|^{2\varkappa}_{2,2,\Omega}+\|f(0)\|^2_{1,2,\Omega})\|\bar{\sigma}\|^2_{2,2,\Omega}.$$

The proof is the same as the proof of Lemma 3.2.

Let us assume

$$(3.18)\qquad R_1 = \|a\|^2_{3,2,\Omega}+\|\sigma\|^2_{2,2,\Omega}+\|f(0)\|^2_{1,2,\Omega}+|\eta|^2_{2,1,\infty,\partial\Omega^T}+|b|^2_{2,0,2,\partial\Omega^T}+|b|^2_{2,0,\infty,\partial\Omega^T},$$

$$(3.19)\qquad R_2 = \|a\|^2_{3,2,\Omega}+\|\bar{\sigma}\|^2_{2,2,\Omega}+\|f(0)\|^2_{1,2,\Omega}+|\eta|^2_{2,1,\infty,\partial\Omega^T}+|\bar{b}|^2_{2,0,2,\partial\Omega^T}+|\bar{b}|^2_{2,0,\infty,\partial\Omega^T},$$

$$(3.20)\qquad v_1 = |\overset{m-1}{v}|_{3,1,2,\Omega^T},\quad v_2 = |\overset{m-1}{v}|_{2,1,\infty,\Omega^T},$$

$$(3.21)\qquad \varrho_* = \min_{\Omega^T}(1+\overset{m}{\varrho}),\qquad \varrho^* = \max_{\Omega^T}(1+\overset{m}{\varrho}).$$

Using the above assumptions we obtain from Eqs. (3.13) and (3.16)

$$(3.22)\qquad \varrho^* \leqslant 1+\sup_t|\overset{m}{\varrho}(t)|_{2,0,\Omega} \leqslant \exp[C(T+T^{1/2}\overset{m-1}{v_1})]C[g_1(R_1) +g_2(R_1)\overset{m-1}{v_1}{}^2(1+\overset{m-1}{v_2}{}^2)]^{1/2}+1,$$

where $\lim\limits_{R_1\to 0} g_1(R_1) = 0$, $\lim\limits_{R_1\to 0} g_2(R_1) = 1$ and

$$(3.23)\qquad \frac{1}{\varrho_*} \leqslant \sup_t|\overset{m}{\psi}(t)|_{2,0,\Omega} \leqslant C\exp[C(T+T^{1/2}\overset{m-1}{v_1})][g_3(R_2) +g_4(R_2)\overset{m-1}{v_1}{}^2(1+\overset{m-1}{v_2}{}^2)]^{1/2}+1,$$

where $\lim\limits_{R_2\to 0} g_3(R_2) = 0$, $\lim\limits_{R_2\to 0} g_4(R_2) = 1$. We see that the functions g_i, $i = 1, \ldots, 4$, are ncreasin g.

4. The existence of local solutions of the problem (2.11) and (2.12)

At first we shall obtain an estimate for solutions of the problem (2.11) and (2.12).

LEMMA 4.1.

Let us assume that $\partial\Omega$ is of class C^3,

(1) $1+\sigma \geqslant \sigma_* > 0, \quad 1+b \geqslant b_* > 0,$

(2) $f\in \Pi^1_{0,2}(\Omega^T), \quad f_{tt} \in L_2(\Omega^T), \quad f(0) \in H^2(\Omega), \quad f_t(0) \in H^1(\Omega),$

(3) $\beta \in \Pi^3_{0,2}(\Omega^T), \quad \beta_{ttxx} \in L_2(\Omega^T), \quad \beta(0) \in H^2(\Omega), \quad \beta_t(0) \in H^1(\Omega), \quad \beta_{tt}(0) \in L_2(\Omega),$

(4) $a \in H^4(\Omega), \quad \sigma \in H^3(\Omega), \quad \bar{\sigma} \in H^3(\Omega),$

(5) $\overset{m}{\varrho} \in \Pi^2_{0,\infty}(\Omega^T), \quad \dfrac{1}{1+\overset{m}{\varrho}} \in \Pi^2_{0,\infty}(\Omega^T),$

(6) $\overset{m-1}{v} \in \Pi^2_{0,\infty}(\Omega^T)\cap \Pi^3_{1,2}(\Omega^T).$

Then for an arbitrary solution $\overset{m}{u} \in \Pi^2_{0,\infty}(\Omega^T)\cap\Pi^3_{1,2}(\Omega^T)$ of the problem (2.11) and (2.12) the following estimate is valid:

$$(4.1)\qquad \varrho_*|\overset{m}{u}|^2_{2,0,\Omega}+\mu|\overset{m}{u}|_{3,1,2,\Omega^t} \leqslant G_1(R)N+G_2(R)M,$$

for $t \leqslant T$, where

$$R = |\overset{m}{\varrho}|^2_{2,0,\infty,\Omega^t}+\left|\frac{1}{1+\overset{m}{\varrho}}\right|^2_{2,0,\infty,\Omega^t}+|\overset{m-1}{v}|^2_{2,0,\infty,\Omega^t}+|\beta|^2_{2,0,\infty,\Omega^t}+||\sigma||^2_{2,2,\Omega} +||\bar{\sigma}||^2_{2,2,\Omega}+||a||^2_{2,2,\Omega},$$

$$N = |\varrho|^2_{2,0,2,\Omega^t}+|f|^2_{1,0,2,\Omega^t}+||f_{tt}||^2_{2,\Omega^t}+||\beta_{xxtt}||^2_{2,\Omega^t}+|\beta|^2_{3,0,2,\Omega^t},$$

$$M = ||a||^2_{4,2,\Omega}+||\sigma||^2_{3,2,\Omega}+|\beta|_{2,0,\Omega}|_{t=0}+|f|^2_{2,1,\Omega}|_{t=0},$$

G_1, G_2 are positive, increasing functions such that $G_i \searrow$ const > 0 if $R \searrow \min R = \varrho_*^2 + \left(\frac{1}{\varrho^*}\right)^2 + \frac{1}{\sigma_*^2}$, $i = 1, 2$.

Proof. To simplify the proof we consider the equations

$$\varphi_t + \operatorname{div}(\varphi v) = 0, \tag{4.2}$$

$$\varphi u_t + \varphi v \cdot \nabla u - \mu \Delta u - \nu \nabla \operatorname{div} u = -\nabla p(\varphi) + \varphi f' - \varphi v \cdot \nabla \beta, \tag{4.3}$$

where $\varphi = 1 + \overset{m}{\varrho}$, $v = \overset{m-1}{v} = \overset{m-1}{u} + \beta$, $u = \overset{m}{u}$ and we assume that φ is a given solution of Eq. (4.2). Moreover, we introduce the following notations:

$$\begin{gathered} f_1 = |f|_{1,0,2,\Omega^t}, \quad f_2 = ||f_{tt}||_{2,\Omega^t}, \quad f_3 = |f|_{2,1,\Omega}|_{t=0}, \\ \beta_1 = |\beta|_{3,0,2,\Omega^t}, \quad \beta_2 = |\beta|_{2,0,\infty,\Omega^t}, \quad \beta_3 = |\beta|_{2,0,\Omega}|_{t=0}, \quad \beta_4 = ||\beta_{xxtt}||_{2,\Omega^t}, \\ \varphi_1 = |\varphi|_{2,0,2,\Omega^t}, \quad \varphi_2 = |\varphi|_{2,0,\infty,\Omega^t}, \quad \psi_1 = |\psi|_{2,0,2,\Omega^t}, \quad \psi_2 = |\psi|_{2,0,\infty,\Omega^t}, \\ a_1 = ||a||_{4,2,\Omega}, \quad a_2 = ||a||_{2,2,\Omega}, \quad \sigma_1 = ||\sigma||_{3,2,\Omega}, \quad \sigma_2 = ||\sigma||_{2,2,\Omega}, \\ \bar\sigma_1 = ||\bar\sigma||_{3,2,\Omega}, \quad \bar\sigma_2 = ||\bar\sigma||_{2,2,\Omega}, \\ v_1 = |v|_{3,1,2,\Omega^t}, \quad v_2 = |v|_{2,0,\infty,\Omega^t}, \quad \varrho_1 = |\overset{m}{\varrho}|_{2,0,2,\Omega^t}, \quad \varrho_2 = |\overset{m}{\varrho}|_{2,0,\infty,\Omega^t}. \end{gathered} \tag{4.4}$$

Using Eq. (4.4) we have

$$\begin{gathered} R = \varphi_2^2 + \psi_2^2 + v_2^2 + \beta_2^2 + \sigma_2^2 + \bar\sigma_2^2 + a_2^2, \quad N = \varrho_1^2 + f_1^2 + f_2^2 + \beta_1^2 + \beta_4^2, \\ M = a_1^2 + \sigma_1^2 + \beta_3^2 + f_3^2. \end{gathered} \tag{4.5}$$

Multiplying Eq. (4.3) by u, integrating the result over Ω and making use of the fact that $[\varphi_t + \operatorname{div}(\varphi v)]\frac{1}{2}u^2 = 0$, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega \varphi u^2 dx + \frac{\mu}{2}\int_\Omega u_x^2 dx + \nu \int_\Omega (\operatorname{div})^2 dx &\leqslant \frac{\alpha_2^2}{2\mu}(||p_x||_{2,\Omega}^2 + ||\varphi f'||_{2,\Omega}^2 \\ + ||\varphi v \cdot \nabla \beta||_{2,\Omega}^2) &\leqslant \frac{\alpha_2^2}{2\mu}[R^2(\sup_\Omega \varphi)^{2(\gamma-1)}||\varphi_x||_{2,\Omega}^2 + ||f||_{2,\Omega}^2 + |\beta|_{2,1,\Omega}^2 \\ &+ ((\sup_\Omega|\beta|)^2 + (\sup_\Omega|v|)^2)||\beta_x||_{2,\Omega}^2](\sup_\Omega \varphi)^2, \end{aligned} \tag{4.6}$$

where the Young inequality was used and α_2 is the constant from the inequality (2.15). Integrating Eq. (4.6) with respect to time, we obtain

$$\begin{aligned} \varrho_*||u||_{2,\Omega}^2 + \mu||u_x||_{2,\Omega^t}^2 &\leqslant \tilde C_1 \varphi_2^{2(\gamma-1)}\varrho_1^2 + \tilde C_2 \varphi_2^2[f_1^2 + \beta_1^2 + (\beta_2^2 + v_2^2)\beta_1^2] \\ &+ \varrho_*(a_1^2 + \beta_3^2) \equiv g_5(\varphi_2, \beta_2, v_2)N + \varrho_* M, \end{aligned} \tag{4.7}$$

where we used Eq. (2.12). From Eq. (4.7) we see that $g_5(\varphi_2, \beta_2, v_2) \searrow$ const > 0 if $\varphi_2 \searrow \min\varphi_2 = \varrho_*|\Omega|^{1/2}$, $\beta_2 \searrow 0$, $v_2 \searrow 0$.

Multiplying Eq. (4.3) by u_t and integrating over Ω, we obtain

$$\begin{aligned} \int_\Omega \varphi u_t^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega [\mu u_x^2 + \nu(\operatorname{div} u)^2]dx &\leqslant [\varrho^*(\sup_\Omega|v|(||u_x||_{2,\Omega} \\ ||\beta_x||_{2,\Omega}) + R\varrho^{*\gamma-1}||\varphi_x||_{2,\Omega} + ||f||_{2,\Omega} + ||\beta_t||_{2,\Omega} &+ \sup_\Omega|\beta|\, ||\beta_x||_{2,\Omega}) + ||\beta_{xx}||_{2,\Omega}]\, ||u_t||_{2,\Omega}. \end{aligned} \tag{4.8}$$

Using the Young inequality, from Eq. (4.8) we have

$$(4.9)\quad \varrho_*\|u_t\|^2_{2,\Omega}+2\frac{d}{dt}(\mu\|u_x\|^2_{2,\Omega}+\nu\|\operatorname{div}u\|^2_{2,\Omega}) \leqslant \frac{1}{\varrho_*}[\varrho^{*2}((\sup_\Omega|v|)^2(\|u_x\|^2_{2,\Omega} + \|\beta_x\|^2_{2,\Omega})+R^2\varrho^{*2(\gamma-1)}\|\varphi_x\|^2_{2,\Omega}+\|f\|^2_{2,\Omega}+\|\beta_t\|^2_{2,\Omega}+(\sup_\Omega|\beta|)^2\|\beta_x\|^2_{2,\Omega})+\|\beta_{xx}\|^2_{2,\Omega}].$$

Integrating Eq. (4.9) with respect to time we obtain

$$(4.10)\quad \varrho_*\|u_t\|^2_{2,\Omega t}+\mu\|u_x(t)\|^2_{2,\Omega}+\nu\|\operatorname{div}u(t)\|^2_{2,\Omega} \leqslant \tilde{C}_3\psi_2[\varphi_2^2v_2^2(\|u_x\|^2_{2,\Omega^t}+\beta_1^2) +\varphi_2^{2(\gamma-1)}\varrho_1^2+f_1^2+\beta_1^2+\beta_2^2\beta_1^2]+\beta_1^2+\tilde{C}_4\|u_x(0)\|^2_{2,\Omega} \leqslant g_6(\psi_2,\varphi_2,v_2,\beta_2)N+CM,$$

where $g_6(\psi_2,\varphi_2,v_2,\beta_2)\searrow$ const >0 if $\psi_2\searrow \min\psi_2 = \frac{1}{\varrho^*}|\Omega|^{1/2}$, $\varphi_2\searrow\varrho_*|\Omega|^{1/2}$, $v_2\searrow 0$, $\beta_2\searrow 0$ and C is a constant.

To estimate the second and third derivatives we introduce the partition of unity generated by $\xi^{(k)}(x)$, $\Omega^{(k)}=\sup\xi^{(k)}(x)$ and $w^{(k)}=\{x\in\Omega^{(k)}:\xi^{(k)}(x)=1\}$, where $\xi^k(x)$ are smooth functions [11]. We assume that $\bigcup_k w^{(k)}=\bigcup_k\Omega^{(k)}=\Omega$ and $\operatorname{diam}w^{(k)}= =\frac{1}{2}\operatorname{diam}\Omega^{(k)}=\lambda$. Moreover, we assume that only a finite number of subdomains $\Omega^{(k)}$ has nonempty cross-sections. Let us assume also that $k\in\mathfrak{N}_1$ if $w^{(k)}\cap\partial\Omega=\emptyset$ and $k\in\mathfrak{N}_2$ if $w^{(k)}\cap\partial\Omega\neq\emptyset$. Now we shall restrict our considerations to $x\in\Omega^{(k)}$, where $k\in\mathfrak{N}_2$. We introduce new variables $y=T(x)$, $x\in\Omega^{(k)}$, where T is a diffeomorphizm of class C^3, such that the boundary $T(\partial\Omega\cap\Omega^{(k)})$ is described by the equation $y^n=0$, where $n=2$ or $n=3$. We introduce the notation $\hat{f}(y)=f(x)|_{x=T^{-1}(y)}$, $\hat{\Omega}^{(k)}=T\Omega^{(k)}$ and $\hat{u}^{(k)}=\hat{u}\hat{\xi}^{(k)}$. For simplicity in the next considerations we assume that $\tilde{u}=\hat{u}^{(k)}$, $\tilde{f}=\hat{f}^{(k)}=\hat{f}\hat{\xi}^{(k)}$, and so on, and we omit the index k in $\hat{\Omega}^{(k)}$ and $\xi^{(k)}$. Then Eq. (4.3) for $y\in\hat{\Omega}$ and $\tilde{u}$ has the following form:

$$(4.11)\quad \hat{\varphi}\tilde{u}_t+\hat{\varphi}\hat{v}\cdot\hat{\nabla}\tilde{u}-(\mu\Delta\tilde{u}+\nu\nabla\operatorname{div}\tilde{u}) = -[\mu(\Delta-\hat{\Delta})\tilde{u}+\nu(\nabla\operatorname{div}-\hat{\nabla}\hat{\operatorname{div}})\tilde{u}] +\hat{\varphi}\hat{v}\hat{u}\hat{\nabla}\hat{\xi}-[2\mu\hat{\nabla}\hat{u}\hat{\nabla}\hat{\xi}+\nu\hat{\nabla}\hat{u}\hat{\nabla}\hat{\xi}+\nu\hat{\nabla}\hat{\xi}\hat{\operatorname{div}}\hat{u}+\mu\hat{u}\hat{\Delta}\hat{\xi}+\nu\hat{u}\hat{\nabla}\hat{\nabla}\hat{\xi}]+\hat{\nabla}\hat{p}\hat{\xi}+\hat{\varphi}\tilde{f}'-\hat{v}\hat{\varphi}\hat{\nabla}\hat{\beta}\hat{\xi},$$

where

$$\nabla_l=\frac{\partial}{\partial y^l},\quad \hat{\nabla}_l=\left.\frac{\partial y^k}{\partial x^l}\right|_{x=T^{-1}(y)}\frac{\partial}{\partial y^k}=\left.\frac{\partial y^k}{\partial x^l}\right|_{x=T^{-1}(y)}\nabla_k,$$

$$\hat{\Delta}=\hat{\nabla}_k\hat{\nabla}_k=\hat{\nabla}_k\left.\left(\frac{\partial y^l}{\partial x^k}\right)\right|_{x=T^{-1}(y)}\nabla_l+\left.\left(\frac{\partial y^i}{\partial x^k}\frac{\partial y^j}{\partial x^k}\right)\right|_{x=T^{-1}(y)}\nabla_i\nabla_j.$$

Differentiating Eq. (4.11) with respect to y', $y'\in\{y^1,\dots,y^{n-1}\}$, multiplying the result by $\tilde{u}_{y'}$, making use of the fact that $[\hat{\varphi}_t+\hat{\operatorname{div}}(\hat{\varphi}\hat{v})]\frac{1}{2}\tilde{u}^2_{y'}=0$ and integrating over $\hat{\Omega}$, we obtain

$$(4.12)\quad \frac{1}{2}\frac{d}{dt}\|\hat{\varphi}\hat{u}^2_{y'}\|_{1,\hat{\Omega}}+\frac{3}{4}\mu\|\nabla\tilde{u}_{y'}\|^2_{2,\hat{\Omega}}+\nu\|\operatorname{div}\tilde{u}_{y'}\|^2_{2,\hat{\Omega}}\leqslant \lambda C_1'\|\nabla\tilde{u}_{y'}\|^2_{2,\hat{\Omega}}+\|\hat{\varphi}_{y'}\tilde{}_t\tilde{u}_{y'}\|_{1,\hat{\Omega}} +\|(\hat{\varphi}\hat{v}\hat{\nabla})_{,y'}\tilde{u}\tilde{u}_{y'}\|_{1,\hat{\Omega}}+\|(\hat{\varphi}\hat{v}\hat{u}\hat{\nabla}\hat{\xi})_{,y'}\tilde{u}_{y'}\|_{1,\hat{\Omega}}+C_2'\|\hat{u}\|^2_{1,2,\hat{\Omega}}+C_3'(\|\hat{\nabla}\hat{p}\hat{\xi}\|^2_{2,\hat{\Omega}} +\|\hat{\varphi}\tilde{f}'\|^2_{2,\hat{\Omega}}+\|(\hat{v}\hat{\varphi}\hat{\nabla}\hat{\beta}\hat{\xi})_{,y'}\|^2_{2,\hat{\Omega}}).$$

Using the Young and Hölder inequalities, from Eq. (4.12) we get

$$(4.13)\quad \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}} \hat{\varphi}|\tilde{u}_{y'}|^2 dy+\frac{\mu}{2}||\nabla\tilde{u}_{y'}||^2_{2,\hat{\Omega}}+\nu||\operatorname{div}\tilde{u}_{y'}||^2_{2,\hat{\Omega}} \leqslant \varepsilon||\tilde{u}_{y't}||^2_{2,\hat{\Omega}}$$
$$+C_4'||\varphi||^2_{2,2,\Omega}(1+||\hat{u}||^2_{1,2,\hat{\Omega}})+C_5'||\varphi||^2_{2,2,\Omega}||v||^2_{2,2,\Omega}(||\hat{u}||^2_{1,2,\hat{\Omega}}+||\hat{\beta}||^2_{1,2,\hat{\Omega}})$$
$$+C_6'(||\nabla\hat{p}||^2_{2,\hat{\Omega}}+||\hat{\varphi}\tilde{f}'||^2_{2,\hat{\Omega}}),$$

where C_i', $i = 1, \ldots, 6$, are constants.

Multiplying Eq. (4.11) by $\tilde{u}_{zz}$, where $z = y^n$, $n = 2$ or $n = 3$, and integrating over $\hat{\Omega}$, we obtain

$$(4.14)\quad \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}} \hat{\varphi}|\tilde{u}_z|^2 dy+\mu||\tilde{u}_{zz}||^2_{2\hat{\Omega}}+\nu||\tilde{u}^n_{zz}||^2_{2,\hat{\Omega}} = -\int_{\hat{\Omega}} (\mu\Delta'\tilde{u}+\nu\nabla'\operatorname{div}\tilde{u})\tilde{u}_{zz}dy$$
$$-\nu\int_{\hat{\Omega}} \operatorname{div}'\tilde{u}_z\tilde{u}_{zz}dy+\int_{\hat{\Omega}} [\mu(\Delta-\hat{\Delta})\tilde{u}+\nu(\nabla\operatorname{div}-\hat{\nabla}\hat{\operatorname{div}})\tilde{u}]\tilde{u}_{zz}dy+\int_{\hat{\Omega}} [2\mu\hat{\nabla}\hat{u}\hat{\nabla}\hat{\xi}+\nu\hat{\nabla}\hat{u}\hat{\nabla}\hat{\xi}$$
$$+\nu\hat{\nabla}\hat{\xi}\hat{\operatorname{div}}\hat{u}+\mu\hat{u}\hat{\Delta}\hat{\xi}+\nu\hat{u}\hat{\nabla}\hat{\xi}]\tilde{u}_{zz}dy+\int_{\hat{\Omega}} [\hat{\varphi}\tilde{u}_t\tilde{u}_{zz}-\frac{1}{2}\hat{\operatorname{div}}(\hat{\varphi}\hat{v})\tilde{u}_z^2+\hat{\varphi}\hat{v}\hat{\nabla}\tilde{u}\tilde{u}_{zz}$$
$$-\hat{\varphi}\hat{v}\hat{u}\hat{\nabla}\hat{\xi}\tilde{u}_{zz}+\hat{\varphi}\tilde{u}_{zt}\tilde{u}_z]dy+\int_{\hat{\Omega}} [-\hat{\nabla}\hat{p}\hat{\xi}-\hat{\varphi}\hat{f}'+\hat{\varphi}\hat{v}\hat{\nabla}\hat{\beta}\hat{\xi}]\tilde{u}_{zz}dy,$$

where ∇', Δ', div' are operators in which the derivative $\partial/\partial z$ do not appear. From Eq. (4.14) we get

$$(4.15)\quad \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}} \hat{\varphi}\tilde{u}_z^2 dy+\frac{1}{2}\mu||\tilde{u}_{zz}||^2_{2,\hat{\Omega}}+\nu||\tilde{u}^n_{zz}||^2_{2,\hat{\Omega}} \leqslant \varepsilon||\tilde{u}_{yt}||^2_{2,\hat{\Omega}}+C_7'||\nabla\tilde{u}_{y'}||^2_{2,\hat{\Omega}}$$
$$+C_8'||\hat{u}||^2_{1,2,\hat{\Omega}}+C_9'||\varphi||^2_{2,2,\Omega}||\tilde{u}_t||^2_{2,\hat{\Omega}}+C_{10}'||\varphi||^2_{2,2,\Omega}||v||^2_{2,2,\Omega}(||\hat{u}||^2_{1,2,\hat{\Omega}}$$
$$+||\hat{\beta}||^2_{1,2,\hat{\Omega}})+C_{11}'(||\hat{\nabla}\hat{p}||^2_{2,\hat{\Omega}}+||\hat{\varphi}\tilde{f}'||^2_{2,\hat{\Omega}}),$$

where C_i', $i = 7, \ldots, 11$, are constants. Multiplying Eq. (4.13) by a constant C such that $\frac{1}{2}\mu C > C_7'$ and adding to Eq. (4.15), then, going back to old variables and summing over all k, we obtain

$$(4.16)\quad \frac{d}{dt}\int_{\Omega} \varphi u_x^2 dx+\mu||u_{xx}||^2_{2,\Omega} \leqslant \varepsilon||u_{xt}||^2_{2,\Omega}+\bar{C}_1||\varphi||^2_{2,2,\Omega}(||u_t||^2_{2,\Omega}+||u||^2_{1,2,\Omega})$$
$$+\bar{C}_2||u||^2_{1,2,\Omega}+\bar{C}_3||\varphi||^2_{2,2,\Omega}||v||^2_{2,2,\Omega}(||u||^2_{1,2,\Omega}+||\beta||^2_{1,2,\Omega})+\bar{C}_4(||p_x||^2_{2,\Omega}+||\varphi f'||^2_{2,\Omega}),$$

where $\bar{C}_i$, $i = 1, \ldots, 4$, are constants. Differentiating Eq. (4.3) with respect to t, multiplying the result by u_t and integrating over Ω, we get

$$(4.17)\quad \frac{1}{2}\frac{d}{dt}\int_{\Omega} \varphi u_t^2 dx+\mu||u_{xt}||^2_{2,\Omega}+\nu||\operatorname{div}u_t||^2_{2,\Omega} \leqslant ||\varphi_t u_t^2||_{1,\Omega}+||(\varphi v)_t\cdot\nabla u u_t||_{1,\Omega}$$
$$+||p_t||_{2,\Omega}||u_{xt}||_{2,\Omega}+||(\varphi f')_t||_{2,\Omega}||u_t||_{2,\Omega}+||(\varphi v\cdot\nabla\beta)_t||_{2\Omega}||u_t||_{2,\Omega}.$$

From Eq. (2.16) we have the following estimates:

(4.18) $$||\varphi_t u_t^2||_{1,\Omega} \leqslant ||\varphi_t||_{2,\Omega}||u_t||_{4,\Omega}^2 \leqslant \varepsilon||u_{xt}||_{2,\Omega}^2 + C||\varphi_t||_{2,\Omega}^{\frac{4}{4-n}}||u_t||_{2,\Omega}^2,$$

(4.19) $$||(\varphi v)_t \cdot \nabla u u_t||_{1,\Omega} \leqslant |\varphi|_{2,1,\Omega}|v|_{2,1,\Omega}(||u_x||_{4,\Omega}^2 + ||u_t||_{4,\Omega}^2)$$
$$\leqslant \varepsilon(||u_{xx}||_{2,\Omega}^2 + ||u_{xt}||_{2,\Omega}^2) + C|\varphi|_{2,1,\Omega}^{\frac{4}{4-n}}|v|_{2,1,\Omega}^{\frac{4}{4-n}}(||u_x||_{2,\Omega}^2 + ||u_t||_{2,\Omega}^2),$$

(4.20) $$||(\varphi v \nabla\beta)_t||_{2,\Omega}^2 \leqslant |\varphi|_{2,1,\Omega}^2|v|_{2,1,\Omega}^2|\beta|_{2,1,\Omega}^2.$$

Using (4.18) ÷ (4.20) in (4.17) we get

(4.21) $$\frac{d}{dt}\int_\Omega \varphi u_t^2\,dx + \mu||u_{xt}||_{2,\Omega}^2 + \nu||\operatorname{div} u_t||_{2,\Omega}^2 \leqslant \varepsilon||u_{xx}||_{2,\Omega}^2 + \overline{C}_5|\varphi|_{2,1,\Omega}^{\frac{4}{4-n}}(1 + |v|_{2,1,\Omega}^{\frac{4}{4-n}})$$
$$\cdot(||u_x||_{2,\Omega}^2 + ||u_t||_{2,\Omega}^2) + \overline{C}_6|\varphi|_{2,1,\Omega}^2|v|_{2,1,\Omega}^2|\beta|_{2,1,\Omega}^2 + \overline{C}_7(||p_t||_{2,\Omega}^2 + ||(\varphi f')_t||_{2,\Omega}^2),$$

where $\overline{C}_i$, $i = 5, \ldots, 7$, are constants.

From Eqs. (4.16) and (4.21), for sufficiently small ε, we have

(4.22) $$\frac{d}{dt}\int_\Omega \varphi(u_x^2 + u_t^2)\,dx + \mu(||u_{xx}||_{2,\Omega}^2 + ||u_{xt}||_{2,\Omega}^2) \leqslant C_1||u||_{1,2,\Omega}^2 + C_2||\varphi||_{2,2,\Omega}^2$$
$$\cdot(1 + ||v||_{2,2,\Omega}^2)(||u||_{1,2,\Omega}^2 + ||u_t||_{2,\Omega}^2) + C_3|\varphi|_{2,0,\Omega}^{\frac{4}{4-n}}(1 + |v|_{2,0,\Omega}^{\frac{4}{4-n}})(||u_x||_{2,\Omega}^2 + ||u_t||_{2,\Omega}^2)$$
$$+ C_4|\varphi|_{2,0,\Omega}^2|v|_{2,0,\Omega}^2|\beta|_{2,1\Omega}^2 + C_5(||p_x||_{2,\Omega}^2 + ||p_t||_{2,\Omega}^2 + ||\varphi f'||_{2,\Omega}^2 + ||(\varphi f')_t||_{2,\Omega}^2).$$

Integrating Eq. (4.22) with respect to t and using Eq. (4.4), we obtain

(4.23) $$\varrho_*(||u_x||_{2,\Omega}^2 + ||u_t||_{2,\Omega}^2) + \mu(||u_{xx}||_{2,\Omega^t}^2 + ||u_{xt}||_{2,\Omega^t}^2) \leqslant \tilde{C}_5||u_x||_{2,\Omega^t}^2$$
$$+ \tilde{C}_6[\varphi_2^2(1 + v_2^2) + \varphi_2^{\frac{4}{4-n}}(1 + v_2^{\frac{4}{4-n}})]||u||_{1,2,\Omega^t}^2 + \tilde{C}_7(\varphi_2^2 v_2^2\beta_1^2 + \varphi_2^{2(\gamma-1)}\varrho_1^2$$
$$+ ||\varphi f'||_{2,\Omega^t}^2 + ||(\varphi f')_t||_{2,\Omega^t}^2) + \varrho_*(||u_x(0)||_{2,\Omega}^2 + ||u_t(0)||_{2,\Omega}^2).$$

From Eq. (2.13) it follows that

(4.24) $$||\varphi f'||_{2,\Omega^t}^2 \leqslant \beta_1^2 + [(1 + \beta_2^2)\beta_1^2 + f_1^2]\varphi_2^2,$$

(4.25) $$||(\varphi f')_t||_{2,\Omega^t} \leqslant \varphi_2^2\psi_2^2\beta_1^2 + \beta_1^2 + [(1 + \beta_2^2)\beta_1^2 + f_1^2]\varphi_2^2.$$

Moreover, we have

(4.26) $$||u_t(0)||_{2,\Omega}^2 \leqslant ||v_t(0)||_{2,\Omega}^2 + ||\beta_t(0)||_{2,\Omega}^2 \leqslant \frac{1}{\varrho_*^2}||a||_{2,2,\Omega}^2 + ||a||_{2,2,\Omega}^4 + ||f(0)||_{2,\Omega}^2$$
$$+ (1 + ||\sigma||_{2,2,\Omega})^{2(\gamma-1)}||\sigma||_{2,2,\Omega}^2 + ||\beta_t(0)||_{2,\Omega}^2 \leqslant \psi_2^2 a_2^2 + a_2^4 + f_3^2 + (1 + \sigma_2)^{2(\gamma-1)}\sigma_2^2 + \beta_3^2.$$

Using Eqs. (4.7), (4.10), (4.24) ÷ (4.26) in Eq. (4.23) we obtain

(4.27) $$\varrho_*(||u_x||_{2,\Omega}^2 + ||u_t||_{2,\Omega}^2) + \mu(||u_{xx}||_{2,\Omega^t}^2 + ||u_{xt}||_{2,\Omega^t}^2) \leqslant g_7(\varphi_2, \psi_2, v_2, \beta_2)N$$
$$+ g_8(\varphi_2, \psi_2, v_2, \beta_2, \sigma_2, a_2)M,$$

where $g_7(\varphi_2, \psi_2, v_2, \beta_2) \searrow \text{const} > 0$, $g_8(\varphi_2, \psi_2, v_2, \beta_2, \sigma_2, a_2) \searrow \text{const} > 0$ if $\varphi_2 \searrow \min\varphi_2$, $\psi_2 \searrow \min\psi_2$, $v_2 \searrow 0$, $\beta_2 \searrow 0$, $\sigma_2 \searrow 0$, $a_2 \searrow 0$.

Now we shall estimate the third derivatives. Differentiating Eq. (4.11) two times with respect to y', multiplying the result by $\tilde{u}_{y'y'}$, using the fact that $[\hat{\varphi}_t+\hat{\operatorname{div}}(\hat{\varphi}\hat{v})]\frac{1}{2}\tilde{u}^2_{y'y'} = 0$, then after integration over $\hat{\Omega}$ we get

$$(4.28)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\hat{\Omega}}\hat{\varphi}u^2_{y'y'}dy+\frac{1}{2}\mu||\nabla\tilde{u}_{y'y'}||^2_{2,\hat{\Omega}}+\nu||\operatorname{div}\tilde{u}_{y'y'}||^2_{2,\hat{\Omega}} \leqslant C'_{12}||\hat{u}||^2_{2,2,\hat{\Omega}}$$

$$+C'_{13}||\varphi||^2_{2,2,\Omega}||v||^2_{2,2,\Omega}(||\hat{u}||^2_{2,2\hat{\Omega}}+||\hat{\beta}||^2_{2,2,\hat{\Omega}})+C'_{14}(||\hat{p}_{y'y'}||^2_{2,\hat{\Omega}}+||(\hat{\varphi}f')_{y'}||)^2_{2,\hat{\Omega}}.$$

Differentiating Eq. (4.11) with respect to y', multiplying by $\tilde{u}_{y'zz}$ and integrating over $\hat{\Omega}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\hat{\Omega}}\hat{\varphi}\tilde{u}_{y'z}dy+\mu||\tilde{u}_{y'zz}||^2_{2,\hat{\Omega}}+\nu||\tilde{u}^n_{y'zz}||^2_{2,\hat{\Omega}} = \int\limits_{\hat{\Omega}}[\hat{\varphi}\tilde{u}_t+\hat{\varphi}\hat{v}\hat{\nabla}\tilde{u}]_{,y'}\tilde{u}_{y'zz}dy$$

$$+\int\limits_{\hat{\Omega}}\left(\frac{1}{2}\hat{\varphi}_t\tilde{u}^2_{y'z}+\hat{\varphi}\tilde{u}_{y'z}\tilde{u}_{y'zt}\right)dy+\int\limits_{\hat{\Omega}}\left(\mu\Delta'+\nu\nabla'\operatorname{div}'+\nu\frac{\partial}{\partial z}\operatorname{div}'+\nu\nabla'\frac{\partial}{\partial z}\right)\tilde{u}_{y'}\tilde{u}_{y'zz}dy$$

$$+\int\limits_{\hat{\Omega}}[\mu(\Delta-\tilde{\Delta})\tilde{u}+\nu(\nabla\operatorname{div}-\hat{\nabla}\hat{\operatorname{div}})\tilde{u}]_{,y'}\tilde{u}_{y'zz}dy+\int\limits_{\hat{\Omega}}[2\mu\hat{\nabla}\hat{u}\hat{\nabla}\hat{\xi}+\nu\hat{\nabla}\hat{u}\hat{\nabla}\hat{\xi}+\nu\hat{\nabla}\hat{\xi}\hat{\operatorname{div}}\hat{u}$$

$$+\mu\hat{u}\hat{\Delta}\hat{\xi}+\nu\hat{u}\hat{\nabla}\hat{\nabla}\hat{\xi}]_{,y'}\tilde{u}_{y'zz}dy-\int\limits_{\hat{\Omega}}[\hat{\varphi}\hat{v}\hat{u}\hat{\nabla}\hat{\xi}+\hat{\nabla}\hat{p}\hat{\xi}+\hat{\varphi}\tilde{f}'-\hat{v}\hat{\varphi}\hat{\nabla}\hat{\beta}\hat{\xi}]_{,y'}\tilde{u}_{y'zz}dy.$$

For a sufficiently small diameter of $\hat{\Omega}$, using the Young and Hölder inequalities, we have

$$(4.29)\quad \frac{1}{2}\frac{d}{dt}\int\limits_{\hat{\Omega}}\hat{\varphi}\tilde{u}_{y'z}dy+\frac{\mu}{2}||\tilde{u}_{y'zz}||^2_{2,\hat{\Omega}}+\nu||\tilde{u}^n_{y'zz}||^2_{2,\hat{\Omega}} \leqslant \varepsilon||\tilde{u}_{yyt}||^2_{2,\hat{\Omega}}+C'_{15}||\tilde{u}_{yy'y'}||^2_{2,\hat{\Omega}}$$

$$+C'_{16}||\hat{u}||^2_{2,2,\hat{\Omega}}+C'_{17}||\varphi||^2_{2,2,\Omega}|\tilde{u}|^2_{2,1,\hat{\Omega}}+C'_{18}||\varphi||^{\frac{4}{4-n}}_{2,2,\Omega}||v||^{\frac{4}{4-n}}_{2,2,\Omega}||\tilde{u}_{yy}||^2_{2,\hat{\Omega}}$$

$$+C'_{19}||\varphi||^2_{2,2,\Omega}||v||^2_{2,2,\Omega}(||\hat{u}||^2_{2,2,\Omega}+||\hat{\beta}||^2_{2,2,\Omega})+C'_{20}(||\hat{p}_{yy}||^2_{2,\hat{\Omega}}+||(\hat{\varphi}\tilde{f}')_{,y}||^2_{2,\hat{\Omega}}).$$

Differentiating Eq. (4.11) with respect to z, multiplying by $\tilde{u}_{zzz}$ and integrating over $\hat{\Omega}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\hat{\Omega}}\hat{\varphi}\tilde{u}^2_{zz}dy+\mu||\tilde{u}_{zzz}||^2_{2,\hat{\Omega}}+||\nu\tilde{u}^n_{zzz}||^2_{2,\hat{\Omega}} = \int\limits_{\hat{\Omega}}\Bigg[\frac{1}{2}\hat{\varphi}_t\tilde{u}^2_{zz}+\hat{\varphi}\tilde{u}_{zz}\tilde{u}_{zzt}$$

$$+(\hat{\varphi}\hat{u}_t+\hat{\varphi}\hat{v}\hat{\nabla}u)_{,z}\tilde{u}_{zzz}\Bigg]dy+\int\limits_{\hat{\Omega}}\left(\mu\Delta'+\nu\nabla'\operatorname{div}'+\nu\frac{\partial}{\partial z}\operatorname{div}'+\nu\nabla'\frac{\partial}{\partial z}\right)\tilde{u}_z\tilde{u}_{zzz}dy$$

$$+\int\limits_{\hat{\Omega}}[\mu(\Delta-\hat{\Delta})\tilde{u}+\nu(\nabla\operatorname{div}-\hat{\nabla}\hat{\operatorname{div}})\tilde{u}]_{,z}\tilde{u}_{zzz}dy$$

$$+\int\limits_{\hat{\Omega}}[2\mu\hat{\nabla}\hat{u}\hat{\nabla}\hat{\xi}+\nu\hat{\nabla}\hat{u}\hat{\nabla}\hat{\xi}+\nu\hat{\nabla}\hat{\xi}\hat{\operatorname{div}}\hat{u}+\mu\hat{u}\hat{\Delta}\hat{\xi}+\nu\hat{u}\hat{\nabla}\hat{\nabla}\hat{\xi}]_{,z}u_{zzz}dy$$

$$-\int\limits_{\hat{\Omega}}[\hat{\varphi}\hat{v}\hat{u}\hat{\nabla}\hat{\xi}+\hat{\nabla}\hat{p}\hat{\xi}+\hat{\varphi}\tilde{f}'-\hat{v}\hat{\varphi}\hat{\nabla}\hat{\beta}\hat{\xi}]_{,z}\tilde{u}_{zzz}dy.$$

For a sufficiently small diameter of $\hat{\Omega}$, using the Young and Hölder inequalities, we have

$$(4.30)\quad \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}} \hat{\varphi}\tilde{u}_{zz}^2\,dy+\frac{\mu}{2}\|\tilde{u}_{zzz}\|_{2,\hat{\Omega}}^2+\nu\|\tilde{u}_{zzz}^n\|_{2,\hat{\Omega}}^2 \leqslant \varepsilon\|\tilde{u}_{yyt}\|_{2,\hat{\Omega}}^2+C'_{21}\|\tilde{u}_{yyy'}\|_{2,\hat{\Omega}}^2$$
$$+C'_{22}\|\hat{u}\|_{2,2,\hat{\Omega}}^2+C'_{23}\|\varphi\|_{2,2,\Omega}^2|\tilde{u}|_{2,1\hat{\Omega}}^2+C'_{24}\|\varphi\|_{2,2,\Omega}^{\frac{4}{4-n}}\|v\|_{2,2,\Omega}^{\frac{4}{4-n}}\|\tilde{u}\|_{2,2,\hat{\Omega}}^2$$
$$+C'_{25}\|\varphi\|_{2,2,\Omega}^2\|v\|_{2,2,\Omega}^2(\|\hat{u}\|_{2,2,\hat{\Omega}}^2+\|\hat{\beta}\|_{2,2,\hat{\Omega}}^2)+C'_{26}(\|\hat{p}_{yy}\|_{2,\hat{\Omega}}^2+\|(\hat{\varphi}\tilde{f}')_{,y}\|_{2,\hat{\Omega}}^2).$$

We multiply Eq. (4.28) by a constant C such that $\frac{1}{4}\mu C \geqslant C'_{15}$ and add to Eq. (4.29). We multiply the result by a constant C' such that $\frac{1}{4}\mu C' \geqslant C'_{21}$ and add to Eq. (4.30). Then after going back to old variables and summing over all k we obtain

$$(4.31)\quad \frac{1}{2}\frac{d}{dt}\int_{\Omega} \varphi u_{xx}^2\,dx+\frac{1}{4}\mu\|u_{xxx}\|_{2,\Omega}^2 \leqslant \varepsilon\|u_{xxt}\|_{2,\Omega}^2+\overline{C}_8\|\varphi\|_{2,2,\Omega}^2\|v\|_{2,2,\Omega}^2(\|u\|_{2,2,\Omega}^2$$
$$+\|\beta\|_{2,2,\Omega}^2)+\overline{C}_9(1+\|\varphi\|_{2,2,\Omega}^2)|u|_{2,1,\Omega}^2+\overline{C}_{10}\|\varphi\|_{2,2,\Omega}^{\frac{4}{4-n}}\|v\|_{2,2,\Omega}^{\frac{4}{4-n}}\|u\|_{2,2,\Omega}^2$$
$$+\overline{C}_{11}(\|p_{xx}\|_{2,\Omega}^2+\|(\varphi f')_{,x}\|_{2,\Omega}^2).$$

Differentiating Eq. (4.11) with respect to y' and t, multiplying by $\tilde{u}_{y't}$ and integrating the result over $\hat{\Omega}$, we have

$$(4.32)\quad \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}} \hat{\varphi}\tilde{u}_{y't}^2\,dy+\frac{1}{2}(\mu\|\tilde{u}_{yy't}\|_{2,\hat{\Omega}}^2+\nu\|\operatorname{div}\tilde{u}_{y't}\|_{2,\hat{\Omega}}^2) \leqslant \varepsilon\|\tilde{u}_{zzt}\|_{2,\hat{\Omega}}^2$$
$$+C'_{27}\|\hat{u}_t\|_{1,2,\hat{\Omega}}^2+C'_{28}\|\varphi\|_{2,2,\Omega}^{\frac{4}{4-n}}\|\tilde{u}_{y't}\|_{2,\hat{\Omega}}^2+C'_{29}|\varphi|_{2,1,\Omega}^2|\tilde{u}|_{2,0,\hat{\Omega}}^2$$
$$+C'_{30}|\varphi|_{2,1,\Omega}^2|v|_{2,1,\Omega}^2(|\hat{u}|_{2,1,\hat{\Omega}}^2+|\hat{\beta}|_{2,1,\hat{\Omega}}^2)+C'_{31}(\|\hat{p}_{yt}\|_{2,\hat{\Omega}}^2+\|(\hat{\varphi}\tilde{f}')_{,t}\|_{2,\hat{\Omega}}^2).$$

Differentiating Eq. (4.11) with respect to t, multiplying the result by $\hat{u}_{zzt}$ and integrating over $\hat{\Omega}$, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}} \hat{\varphi}\tilde{u}_{zt}^2\,dy+\mu\|\tilde{u}_{zzt}\|_{2,\hat{\Omega}}^2+\nu\|\tilde{u}_{zzt}^n\|_{2,\hat{\Omega}}^2 = \int_{\hat{\Omega}}\left(\frac{1}{2}\hat{\varphi}_t\tilde{u}_{zt}^2+\hat{\varphi}\tilde{u}_{zt}\tilde{u}_{ztt}\right)dy$$
$$+\int_{\hat{\Omega}}(\hat{\varphi}\tilde{u}_t+\hat{\varphi}\hat{v}\hat{\nabla}\tilde{u})_{,t}\tilde{u}_{zzt}\,dy+\int_{\hat{\Omega}}\left(\mu\Delta'+\nu\nabla'\operatorname{div}'+\nu\frac{\partial}{\partial z}\operatorname{div}'+\nu\nabla'\frac{\partial}{\partial z}\right)\tilde{u}_t\tilde{u}_{zzt}$$
$$+\int_{\hat{\Omega}}[\mu(\Delta-\hat{\Delta})\tilde{u}_t+\nu(\nabla\operatorname{div}-\hat{\nabla}\widehat{\operatorname{div}})\tilde{u}_t]\tilde{u}_{zzt}\,dy$$
$$+\int_{\hat{\Omega}}[2\mu\hat{\nabla}\hat{u}_t\hat{\nabla}\hat{\xi}+\nu\hat{\nabla}\hat{u}_t\hat{\nabla}\hat{\xi}+\nu\hat{\nabla}\hat{\xi}\,\widehat{\operatorname{div}}\hat{u}_t+\mu\hat{u}_t\hat{\Delta}\hat{\xi}+\nu\hat{u}_t\hat{\nabla}\hat{\nabla}\hat{\xi}]\tilde{u}_{zzt}$$
$$+\int_{\hat{\Omega}}[\hat{\varphi}\hat{v}\hat{u}\hat{\nabla}\hat{\xi}-\hat{\varphi}\hat{v}\hat{\nabla}\hat{\beta}\hat{\xi}+\hat{\nabla}\hat{p}\,\hat{\xi}+\hat{\varphi}\tilde{f}']_{,t}\tilde{u}_{zzt}.$$

Using the Hölder and Young inequalities, we obtain

$$(4.33)\quad \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}} \hat{\varphi}\tilde{u}_{zt}^2\,dy+\frac{1}{2}(\mu\|\tilde{u}_{zzt}\|^2_{2,\hat{\Omega}}+\nu\|\tilde{u}^n_{zzt}\|^2_{2,\hat{\Omega}}) \leqslant \varepsilon\|\tilde{u}_{ztt}\|^2_{2,\hat{\Omega}}+C'_{32}\|\tilde{u}_{yy't}\|^2_{2,\hat{\Omega}}$$

$$+C'_{33}\|\varphi_t\|^{\frac{4}{4-n}}_{1,2,\Omega}\|\tilde{u}_{zt}\|^2_{2,\hat{\Omega}}+C'_{34}|\varphi|^2_{2,1,\Omega}|\tilde{u}|^2_{2,0,\hat{\Omega}}+C'_{35}|\hat{u}_t|^2_{1,2,\Omega}$$

$$+C'_{36}|\varphi|^2_{2,1,\Omega}|v|^2_{2,1\Omega}(|\hat{u}|^2_{2,1,\hat{\Omega}}+|\hat{\beta}|^2_{2,1,\hat{\Omega}})+C'_{37}(\|\hat{p}_{yt}\|^2_{2,\hat{\Omega}}+\|(\hat{\varphi}\tilde{f}')_t\|^2_{2,\hat{\Omega}}).$$

We multiply Eq. (4.32) by a constant C'' such that $\frac{1}{4}\mu C''\geqslant C'_{32}$ and add to Eq. (4.33). Then after going back to old variables and summing over all k, we obtain

$$(4.34)\quad \frac{d}{dt}\int_{\Omega}\varphi u_{xt}^2\,dx+\frac{1}{2}\mu\|u_{xxt}\|^2_{2,\Omega}\leqslant \varepsilon\|u_{xtt}\|^2_{2,\Omega}+\overline{C}_{12}(1+|\varphi|^{\frac{4}{4-n}}_{2,1,\Omega})|u|^2_{2,0,\Omega}$$

$$+\overline{C}_{13}|\varphi|^2_{2,1,\Omega}|u|^2_{2,0,\Omega}+\overline{C}_{14}|\varphi|^2_{2,1,\Omega}|v|^2_{2,1,\Omega}(|u|^2_{2,1,\Omega}+|\beta|^2_{2,1,\Omega})$$

$$+\overline{C}_{15}(\|p_{xt}\|^2_{2,\Omega}+\|(\varphi f')_t\|^2_{2,\Omega}.$$

Differentiating Eq. (4.3) two times with respect to t, multiplying by u_{tt} and integrating over Ω, we obtain

$$(4.35)\quad \frac{1}{2}\frac{d}{dt}\int_{\Omega}\varphi u_{tt}^2\,dx+\mu\|u_{xtt}\|^2_{2,\Omega}+\nu\|\operatorname{div}u_{tt}\|^2_{2,\Omega}\leqslant \|\varphi_{tt}u_tu_{tt}\|_{1,\Omega}$$

$$+\|\varphi_t u_{tt}^2\|_{1,\Omega}+\|(\varrho v)_{tt}u_tu_{tt}\|_{1,\Omega}+\|(\varrho v)_t u_{xt}u_{tt}\|_{1,\Omega}+\|p_{tt}\|_{2,\Omega}\|u_{xtt}\|_{2,\Omega}$$

$$+\|(\varphi f')_{tt}\|_{2,\Omega}\|u_{tt}\|_{2,\Omega}+\|(\varphi v\beta_x)_{tt}u_{tt}\|_{1,\Omega}.$$

Using the following estimates

$$\|\varphi_t u_{tt}^2\|_{1,\Omega}\leqslant \varepsilon\|u_{xtt}\|^2_{2,\Omega}+C\|\varphi_t\|^{\frac{4}{4-n}}_{2,\Omega}\|u_{tt}\|^2_{2,\Omega},$$

$$\|\varphi_{tt}u_tu_{tt}\|_{2,\Omega}\leqslant \varepsilon\|u_{xtt}\|^2_{2,\Omega}+C\|\varphi_{tt}\|_{2,\Omega}\|u_t\|^2_{1,2,\Omega}+C\|\varphi_{tt}\|^{\frac{4}{4-n}}_{2,\Omega}\|u_{tt}\|^2_{2,\Omega},$$

$$\|(\varrho v)_{tt}u_xu_{tt}\|_{1,\Omega}+\|(\varrho v)_tu_{xt}u_{tt}\|_{1,\Omega}\leqslant \varepsilon(\|u_{xtt}\|^2_{2,\Omega}+\|u_{xxt}\|^2_{2,\Omega})$$

$$+C|\varphi|^2_{2,0,\Omega}|v|^2_{2,0,\Omega}|u|^2_{1,0,\Omega}+C|\varphi|^{\frac{4}{4-n}}_{2,0,\Omega}|v|^{\frac{4}{4-n}}_{2,0,\Omega}(\|u_{tt}\|^2_{2,\Omega}+\|u_{xt}\|^2_{2,\Omega}),$$

$$\|(\varphi v\beta_x)_{tt}u_{tt}\|_{1,\Omega}\leqslant C|\varphi|^2_{2,0,\Omega}|v|^2_{2,0,\Omega}|\beta|^2_{3,1,\Omega}+C\|u_{tt}\|^2_{2,\Omega},$$

we have

$$(4.36)\quad \frac{1}{2}\frac{d}{dt}\int_{\Omega}\varphi u_{tt}^2\,dx+\frac{1}{2}\mu\|u_{xtt}\|^2_{2,\Omega}\leqslant \varepsilon\|u_{xxt}\|^2_{2,\Omega}+\overline{C}_{16}|\varphi|^2_{2,0,\Omega}|v|^2_{2,0,\Omega}(|u|^2_{2,1,\Omega}$$

$$+|\beta|^2_{3,1,\Omega})+\overline{C}_{17}|\varphi|^{\frac{4}{4-n}}_{2,0,\Omega}(1+|v|^{\frac{4}{4-n}}_{2,0,\Omega})(\|u_{tt}\|^2_{2,\Omega}+\|u_{xt}\|^2_{2,\Omega})+\overline{C}_{18}\|u_{tt}\|^2_{2,\Omega}$$

$$+\overline{C}_{19}(\|p_{tt}\|^2_{2,\Omega}+\|(\varphi f')_{tt}\|^2_{2,\Omega}.$$

From Eqs. (4.31), (4.34) and (4.36) we have

$$(4.37)\quad \frac{d}{dt}\int_{\Omega}\varphi(u_{xx}^2+u_{xt}^2+u_{tt}^2)\,dx+\mu(\|u_{xxx}\|^2_{2,\Omega}+\|u_{xxt}\|^2_{2,\Omega}+\|u_{xtt}\|^2_{2,\Omega}$$

$$\leqslant C_7|u|^2_{2,0,\Omega}+C_8|\varphi|^2_{2,0,\Omega}(1+|v|^2_{2,0\Omega})(|u|^2_{2,0,\Omega}+|\beta|^2_{3,1,\Omega})$$
$$+C_9|\varphi|^{\frac{4}{4-n}}_{2,0,\Omega}(1+|v|^{\frac{4}{4-n}}_{2,0,\Omega})|u|^2_{2,0,\Omega}+C_{10}(||p_{xx}||^2_{2,\Omega}+||p_{xt}||^2_{2,\Omega}+||p_{tt}||^2_{2,\Omega}$$
$$+||(\varphi f')_x||^2_{2,\Omega}+||(\varphi f')_t||^2_{2,\Omega}+||(\varphi f')_{tt}||^2_{2,\Omega}).$$

To estimate $||u_{tt}||_{2,\Omega}$ we integrate Eq. (4.37) with respect to time without the second term on the left-hand side. Using Eq. (4.4), from Eq. (2.13) we have

$$||(\varphi f')_{tt}||^2_{2,\Omega^t} \leqslant \varphi_2^2(f_1^2+f_2^2)+\beta_4^2+\varphi_2^2\beta_1^2(1+\beta_2^2). \tag{4.38}$$

Moreover, from Eq. (1.2) we have

$$\begin{gathered} ||u_{xx}(0)||^2_{2,\Omega} \leqslant a_2^2+\beta_3^2, \\ ||u_{xt}(0)||^2_{2,\Omega} \leqslant \bar{\sigma}_2^4(1+\sigma_2^2)a_2^2+\bar{\sigma}_2^2a_1^2+a_2^2+(1+\sigma_2)^{2(\gamma-2)}\sigma_2^2+f_3^2+\beta_3^2, \\ ||u_{tt}(0)||^2_{2,\Omega} \leqslant g_9(a_1,\bar{\sigma}_2,\sigma_2)(a_1^2+\sigma_1^2+f_3^2)+\beta_3^2, \end{gathered} \tag{4.39}$$

where $g_9(a_1,\bar{\sigma}_2,\sigma_2)\searrow$ const >0 if $a_1\searrow 0$, $\sigma_2\searrow 0$. Integrating Eq. (4.37) with respect to time and using Eqs. (4.38), (4.39), (4.24), (4.25) and (4.4), we obtain

$$\begin{aligned} &\varrho_*(||u_{xx}||^2_{2,\Omega}+||u_{xt}||^2_{2,\Omega}+||u_{tt}||^2_{2,\Omega})+\mu(||u_{xxx}||^2_{2,\Omega^t}+||u_{xxt}||^2_{2,\Omega^t}+||u_{xtt}||^2_{2,\Omega^t} \\ &\qquad\leqslant g_{10}(\varphi_2,\psi_2,v_2,\beta_2,\sigma_2,a_2)N+g_{11}(\bar{\sigma}_2,\varphi_2,\psi_2,v_2,\beta_2,\sigma_2,a_2)M. \end{aligned} \tag{4.40}$$

Adding Eqs. (4.7), (4.10), (4.27) and (4.40), we obtain Eq. (4.1). This completes the proof.

LEMMA 4.2.

Let the assumptions of Lemma 4.1 be satisfied. Then there exists a unique solution of the problem (2.11) and (2.12) such that $\overset{m}{u}\in \mathfrak{M}(\Omega^T)\equiv \Pi^2_{0,\infty}(\Omega^T)\cap\Pi^3_{1,2}(\Omega^T)$ and the estimate (4.1) is valid.

Proof. We use the results of [17]. Let us assume that there exists a sequence of problems

$$\overset{k}{u}{}^i_t=\frac{\mu}{\overset{k}{\varphi}}\overset{k}{u}{}^i_{x^jx^j}+\frac{\nu}{\overset{k}{\varphi}}\overset{k}{u}{}^j_{x^ix^j}-\overset{k}{v}{}^j\overset{k}{u}{}^i_{x^j}-\overset{k}{v}{}^j\overset{k}{\beta}{}^i_{x^j}-\frac{R[\overset{k}{\varphi}{}^\gamma]}{\overset{k}{\varphi}},{}_{x^i}+\overset{k}{f}{}', \tag{4.41}$$

$$\overset{k}{u}|_{t=0}=\overset{k}{a}-\overset{k}{\beta}|_{t=0},\quad \overset{k}{u}|_{\partial\Omega}=0, \tag{4.42}$$

where the notations of Lemma 4.1 are used and $\overset{k}{\varphi}$, $\overset{k}{\beta}$, $\overset{k}{f}$, $\overset{k}{a}$, $\overset{k}{v}$, are functions in spaces $C^{4+\alpha,2+\alpha/2}(\Omega^T)$, $C^{6+\alpha,3+\alpha/2}(\Omega^T)$, $C^{4+\alpha,2+\alpha/2}(\Omega^T)$, $C^6(\Omega)$, $C^{4+\alpha,2+\alpha/2}(\Omega^T)$, $0<\alpha<1$, respectively. Then [17] implies that there exists a unique solution of Eqs. (4.41) and (4.42) such that $\overset{k}{u}\in C^{6+\alpha,3+\alpha/2}(\Omega^T)$ and Lemma (4.1) follows that $\overset{k}{u}$ satisfies the estimate (4.1). Let us assume that the sequences $\{\overset{k}{\varphi}\}$, $\{\overset{k}{\beta}\}$, $\{\overset{k}{f}\}$, $\{\overset{k}{a}\}$, $\{\overset{k}{v}\}$ constitute Cauchy sequences in spaces given in the assumptions of Lemma 4.1 convergent to the elements φ, β, f, a, v, respectively. Hence $\{\overset{k}{u}\}$ constitutes the Cauchy sequence in $\mathfrak{M}$ also. The existence of solutions of the limit problem (4.41) and (4.42) follows from the fact that the limit function u satisfies an integral indentity which is a limit of the sequence of integral indentities corresponding to the problems (4.41) and (4.42) [16]. This concludes the proof.

5. The existence of solutions of the problem (2.4) and (2.8)

Using the method of successive approximations described in Sect. 2 and the existence of unique solutions of the linear problems (2.9) and (2.10), independently, we can prove the following theorem:

THEOREM 5.1. *Let us assume that*

$a \in H^4(\Omega), \quad \sigma \in H^2(\Omega),$

$f(0) \in \Gamma_1^2(\Omega), \quad f \in \Pi^1_{0,2}(\Omega^T), \quad f_{tt} \in L_2(\Omega^T),$

$\eta \in \Pi^{2+1/2}_{0,\infty}(\partial\Omega^T), \quad \eta \in \Pi^{3+1/2}_{0,2}(\partial\Omega^T), \quad \eta(0) \in \Gamma_0^{2+1/2}(\partial\Omega), \quad \eta_{tt} \in \Pi^{2+1/2}_{2,2}(\partial\Omega^T),$

$b \in \Pi^2_{0,\infty}(\partial\Omega^T) \cap \Pi^2_{0,2}(\partial\Omega^T).$

Moreover, we assume that T is sufficiently small (see the comments following Eq. (5.14)). *Then there exists a unique solution of the problem* (2.4) ÷ (2.8) *such that* $u \in \mathfrak{M}$, $\varrho \in \Pi^2_{0,\infty}(\Omega^T)$.

Proof. From Eqs. (3.12), (3.16) and (4.1) it follows that

$$\|\overset{m}{u}\| \leqslant G(t, \mathsf{R}, \mathsf{N}, \mathsf{M}, \|\overset{m-1}{u}\|), \tag{5.1}$$

where

$$\|u\|^2 = \sup_t |u|^2_{2,0,\Omega} + |u|^2_{3,1,2,\Omega^t},$$

and

$$G(t, \mathsf{R}, \mathsf{N}, \mathsf{M}, \chi) = \bar{G}_1(t, \mathsf{R}, \chi)\mathsf{N} + \bar{G}_2(t, \mathsf{R}, \chi)\mathsf{M} + \bar{G}_3(t, \mathsf{R}, \chi)\chi, \tag{5.2}$$

where

$$\mathsf{R} = \|a\|^2_{2,2,\Omega} + \|\sigma\|^2_{2,2,\Omega} + |b|^2_{2,0,\infty,\partial\Omega^t} + |\eta|^2_{2+1/2,0,\infty,\partial\Omega^t},$$

$$\mathsf{N} = |b|^2_{2,0,2,\partial\Omega^t} + |f|^2_{1,0,2,\Omega^t} + \|f_{tt}\|^2_{2,\Omega^t} + |\eta_{tt}|^2_{2+1/2,2,2,\partial\Omega^t} + |\eta|^2_{3+1/2,0,2,\partial\Omega^t},$$

$$\mathsf{M} = \|a\|^2_{4,2,\Omega} + \|\sigma\|^2_{3,2,\Omega} + |\eta|^2_{2+1/2,0,\partial\Omega}|_{t=0} + |f|^2_{2,1,\Omega}|_{t=0}$$

and $\bar{G}_3(t, R, \chi) \searrow$ if $t \searrow 0$. Therefore, there exist M_0, N_0, t_0 for which

$$\inf\{\chi > 0 : G(t_0, \mathsf{R}, \mathsf{N}_0, \mathsf{M}_0, \chi) < \chi\} = \chi_0 > 0. \tag{5.3}$$

Then from Eqs. (5.2), (5.3) and (2.14) it follows that

$$\|\overset{m}{u}\| \leqslant \chi_0. \tag{5.4}$$

Therefore, from the estimate (3.12) we have

$$|\overset{m}{\varrho}|_{2,0,\infty,\Omega^t} \leqslant \bar{\chi}_0, \tag{5.5}$$

where $\bar{\chi}_0$ is a constant which is not dependent on m.

To prove the existence of solutions of the problem (2.4) ÷ (2.8) we use the method of successive approximations. Therefore, from Eqs. (2.9) and (2.11) we obtain

$$\begin{aligned}(1+\overset{m}{\varrho})\overset{m}{U}{}^i_t + \overset{m}{P}\,\overset{m-1}{u}{}^i_t + (1+\overset{m}{\varrho})(\overset{m-1}{u}+\beta)^j \overset{m}{U}{}^i_{x^j} + [\overset{m}{P}(\overset{m-1}{u}+\beta)^j + (1+\overset{m-1}{\varrho})\overset{m-1}{U}{}^j]\,\overset{m-1}{u}{}^i_{x^j} \\ + [(1+\overset{m}{\varrho})\overset{m-1}{U}{}^j + \overset{m}{P}\,\overset{m-2}{u}{}^j]\beta^i_{x^j} - \mu \overset{m}{U}{}^i_{x^j x^j} - \nu \overset{m}{U}{}^j_{x^i x^j} + R[(1+\overset{m}{\varrho})^\gamma_{,x^i} - (1+\overset{m-1}{\varrho})^\gamma_{,x^i}] = \overset{m}{P}(f^i - \beta^i_t - \beta^j \beta^i_{x^j})\end{aligned} \tag{5.6}$$

and

$$\overset{m}{P}_t = -\operatorname{div}[\overset{m-1}{U} + \overset{m-1}{U}\,\overset{m-1}{\varrho} + \overset{m-1}{v}\,\overset{m}{P}], \tag{5.7}$$

where $\overset{m}{U} = \overset{m}{u} - \overset{m-1}{u}$, $\overset{m}{P} = \overset{m}{\varrho} - \overset{m-1}{\varrho}$ which satisfy the following conditions:

$$\overset{m}{U}|_{t=0} = 0, \quad \overset{m}{U}|_{\partial\Omega} = 0, \quad \overset{m}{P}|_{t=0} = 0, \quad \overset{m}{P}|_{\partial\Omega} = 0. \tag{5.8}$$

Multiplying Eq. (5.6) by $\overset{m}{U}{}^i$, integrating the result over Ω, using the fact that $[\overset{m}{\varrho}_t + \operatorname{div}[\overset{m-1}{v}(1+\overset{m}{\varrho})]]\frac{1}{2}\overset{m}{U}{}^2 = 0$ and Eq. (5.8), we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (1+\overset{m}{\varrho})\overset{m}{U}{}^2 dx + \mu\|\overset{m}{U}_x\|^2_{2,\Omega} + \nu\|\operatorname{div}\overset{m}{U}\|^2_{2,\Omega} = -\int_\Omega \overset{m}{P}\,\overset{m-1}{u}{}^i_t\,\overset{m}{U}{}^i dx \tag{5.9}$$

$$-\int_\Omega [\overset{m}{P}(\overset{m-1}{u}+\beta)^j + (1+\overset{m-1}{\varrho})\overset{m-1}{U}{}^j]\overset{m-1}{u}{}^i_{x^j}\,\overset{m}{U}{}^j dx - \int_\Omega [(1+\overset{m}{\varrho})\overset{m-1}{U}{}^j + \overset{m}{P}\,\overset{m-2}{u}{}^j]\beta^i_{x^j}\overset{m}{U}{}^i dx$$

$$+R\int_\Omega [(1+\overset{m}{\varrho})^\gamma - (1+\overset{m-1}{\varrho})^\gamma]\operatorname{div}\overset{m}{U}dx + \int_\Omega \overset{m}{P}[f^i - \beta^i_t - \beta^j\beta^i_{x^j}]\overset{m}{U}{}^i dx.$$

Multiplying Eq. (5.7) by $\overset{m}{P}$, integrating the result over Ω, using Eq. (5.8), we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \overset{m}{P}{}^2 dx = -\frac{1}{2}\int_\Omega \operatorname{div}\overset{m-1}{v}\,\overset{m}{P}{}^2 - \int_\Omega \operatorname{div}(\overset{m-1}{U} + \overset{m-1}{U}\,\overset{m-1}{\varrho})\overset{m}{P}. \tag{5.10}$$

Using the Hölder inequality, from Eq. (5.9) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (1+\overset{m}{\varrho})\overset{m}{U}{}^2 dx + \mu\|\overset{m}{U}_x\|^2_{2,\Omega} + \nu\|\operatorname{div}\overset{m}{U}\|^2_{2,\Omega} \leqslant [\|\overset{m-1}{u}_t\|_{4,\Omega} + \max_\Omega|\overset{m-1}{u}+\beta|\,\|\overset{m-1}{u}_x\|_{4,\Omega} \tag{5.11}$$

$$+\max_\Omega|\overset{m-2}{u}|\,\|\beta_x\|_{4,\Omega} + \|f\|_{4,\Omega} + \|\beta_t + \beta\beta_x\|_{4,\Omega}]\|\overset{m}{U}\|_{4,\Omega}\|\overset{m}{P}\|_{2,\Omega}$$

$$+\max_\Omega(1+\overset{m-1}{\varrho})(\|\overset{m-1}{u}_x\|_{4,\Omega} + \|\beta_x\|_{4,\Omega})\|\overset{m-1}{U}\|_{2,\Omega}\|\overset{m}{U}\|_{4,\Omega} + \gamma\max_\Omega(1+\overset{m}{\varrho})^{\gamma-1}\|\operatorname{div}\overset{m}{U}\|_{2,\Omega}\|\overset{m}{P}\|_{2,\Omega},$$

and from Eq. (5.10) we have

$$\frac{1}{2}\frac{d}{dt}\|\overset{m}{P}\|^2_{2,\Omega} \leqslant \max_\Omega|\operatorname{div}\overset{m-1}{v}|\,\|\overset{m}{P}\|^2_{2,\Omega} + (\max_\Omega|\overset{m-1}{\varrho}|\,\|\overset{m-1}{U}_x\|_{2,\Omega} \tag{5.12}$$

$$+\|\operatorname{div}\overset{m-1}{U}\|_{2,\Omega} + \|\overset{m-1}{\varrho}_x\|_{4,\Omega}\|\overset{m-1}{U}\|_{4,\Omega})\|\overset{m}{P}\|_{2,\Omega}.$$

Adding Eq. (5.12) to Eq. (5.11), using the Young and Poincaré inequalities, $|\varrho| \leqslant 1/2$ and the Sobolev imbedding theorems, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega [(1+\overset{m}{\varrho})\overset{m}{U}{}^2 + \overset{m}{P}{}^2]dx + \frac{\mu}{2}\|\overset{m}{U}_x\|^2_{2,\Omega} + \frac{\nu}{2}\|\operatorname{div}\overset{m}{U}\|^2_{2,\Omega} \leqslant C[|\overset{m-1}{u}|^2_{2,1,\Omega} \tag{5.13}$$

$$+(\|\overset{m-1}{u}\|^2_{2,2,\Omega} + \|\beta\|^2_{2,2,\Omega})\|\overset{m-1}{u}\|^2_{2,2,\Omega} + \|\overset{m-2}{u}\|^2_{2,2,\Omega}\|\beta\|^2_{2,2,\Omega} + \|f\|^2_{1,2,\Omega} + \|\overset{m-1}{\varrho}\|^2_{2,2,\Omega}$$

$$+|\beta|^2_{2,1,\Omega}(1+\|\beta\|^2_{2,2,\Omega}) + 1 + \|\overset{m-1}{v}\|_{3,2,\Omega}]\|\overset{m}{P}\|^2_{2,\Omega} + C(\|\overset{m-1}{u}\|^2_{2,2,\Omega} + \|\beta\|^2_{2,2,\Omega})\|\overset{m-1}{U}\|^2_{2,\Omega}$$

$$+\varepsilon(\|\overset{m-1}{U}_x\|^2_{2,\Omega} + \|\operatorname{div}\overset{m-1}{U}\|^2_{2,\Omega}).$$

Using the estimates (5.4), (5.5), $|\varrho| \leqslant 1/2$, and integrating Eq. (5.13) with respect to time, we get

$$(5.14)\quad \frac{1}{4}\sup_t(\|\overset{m}{U}\|^2_{2,\Omega}+\|\overset{m}{P}\|^2_{2,\Omega})+\frac{\mu}{2}\|\overset{m}{U}_x\|^2_{2,\Omega^t}+\frac{\nu}{2}\|\operatorname{div}\overset{m}{U}\|^2_{2,\Omega^t}$$
$$\leqslant Ct\sup_t\|\overset{m}{P}\|^2_{2,\Omega}+Ct\sup_t\|\overset{m-1}{U}\|^2_{2,\Omega}+\varepsilon(\|\overset{m-1}{U}_x\|^2_{2,\Omega}+\|\operatorname{div}\overset{m-1}{U}\|^2_{2,\Omega}).$$

Introducing the notation $J_m = \sup_t(\|\overset{m}{P}\|^2_{2,\Omega}+\|\overset{m}{U}\|^2_{2,\Omega})+\|\overset{m}{U}_x\|^2_{2,\Omega^t}+\|\operatorname{div}\overset{m}{U}\|^2_{2,\Omega^t}$ and assuming that t and ε are sufficiently small, we have $J_m \leqslant \frac{1}{2}J_{m-1}$, which implies that the sequences $\{\overset{m}{\varrho}\}$, $\{\overset{m}{u}\}$ converge in $L_{2,\infty}(\Omega^t)$, $t \leqslant T$. Using the weak compactness in the spaces $\mathfrak{M}$ and $\Pi^2_{0,\infty}(\Omega^T)$ there exists limits $u \in \mathfrak{M}$ of the sequence $\{\overset{m}{u}\} \in \mathfrak{M}$ and $\varrho \in \Pi^2_{0,\infty}(\Omega^T)$ of $\{\overset{m}{\varrho}\} \in \Pi^2_{0,\infty}(\Omega^T)$. To show that u and ϱ corresponding to the problem (2.4) and (2.8) is a solution of the problem (2.4) ÷ (2.8), we must consider integral identities instead of Eqs. (2.9) and (2.11), see [16]. Passing with m to infinity in these identities, we obtain that the limit functions are solutions of the problem (2.4) ÷ (2.8). This concludes the proof.

6. Remarks

The existence and uniqueness of solutions of the considered equations (1.1) and (1.2) can be shown also in the case of other boundary conditions: vanishing velocity on the boundary or an outflow from the domain. The considered boundary conditions (1.5) and (1.7) are necessary to prove the existence of global in time solutions, because only in this case we know how to obtain the global a priori estimate. From the nonlinearity of our problem we proved the existence of solutions in H^s spaces with the smallest possible smoothness.

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