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Remarks on Rayleigh-type waves in viscoelastic incompressible fluids

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THIS PAPER is a supplement to our previous considerations [6] where the surface- and interface-type waves in viscoelastic fluids were discussed in greater detail. Now it is shown that certain properties of viscoelastic Rayleigh-type waves in two-layer incompressible fluids essentially differ from their counterparts for compressible fluids.

Niniejsza praca stanowi uzupełnienie naszych poprzednich rozważań [6], w których szczegółowo przedyskutowano fale typu powierzchniowego i międzyfazowego w cieczach lepkosprężystych. Obecnie pokazano, że niektóre własności lepkosprężystych fal typu Rayleigha w dwuwarstwowych cieczach nieściśliwych różnią się istotnie od odpowiednich własności dla cieczy ściśliwych.

Настоящая работа составляет дополнение наших предыдущих рассуждений [6], в которых подробно обсуждены волны поверхностного и межфазного типов в вязкоупругих жидкостях. Сейчас показано, что некоторые свойства вязкоупругих волн типа Рэлея в двухслоистых несжимаемых жидкостях отличаются существенно от соответствующих свойств для сжимаемых жидкостей.

1. Introduction

THE PROPERTIES of viscoelastic Rayleigh waves in homogeneous media were discussed in a series of papers (cf. [1–5]). In our recent contribution [6], devoted to the surface- and interface-type waves in homogeneous and two-layer viscoelastic fluids, we intentionally omitted the case of Rayleigh-type waves in incompressible fluids; similar waves were discussed for more general cases of compressible fluids. In the present note, being a supplement to paper [6], we briefly consider viscoelastic Rayleigh waves in two-layer incompressible fluids. It is shown that many properties of such waves essentially differ from those observed in two-layer compressible fluids.

2. Basic equations for homogeneous fluids

In paper [6] we considered the following linearized equations (cf. Appendix):

$$(2.1)\qquad \left(\nabla^2-\frac{\varrho}{\lambda^*+2\eta^*}\frac{\partial}{\partial t}\right)\Phi_1=0,\quad \left(\nabla^2-\frac{\varrho}{\eta^*}\frac{\partial}{\partial t}\right)\Phi_2=0,\quad \nabla^2 p=0,$$

where p is the hydrodynamic pressure, ϱ — the density of a fluid, λ^* and η^* denote the frequency-dependent dynamic second (dilatational) and shear viscosities, respectively. For plane flows the scalar potentials Φ_i $(i = 1, 2)$ determine the velocity components in the form

$$\text{(2.2)} \qquad u = \frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial z}, \quad v = 0, \quad w = \frac{\partial \Phi_1}{\partial z} - \frac{\partial \Phi_2}{\partial x}.$$

We analysed certain solutions of Eqs. (2.1) in the form of harmonic waves, viz.

$$\text{(2.3)} \qquad \begin{aligned} \Phi_i &= (A_i e^{\nu_i z} + B_i e^{-\nu_i z}) \exp(\mu x + i\omega t), \\ p &= p_0(z) \exp(\mu x + i\omega t), \end{aligned}$$

where capital letters denote integration constants, and

$$\text{(2.4)} \qquad \mu^2 = -k_x^2, \quad \nu_i^2 = -k_{iz}^2 = -\left(\frac{\varrho\omega^2}{G_i^*} + \mu^2\right), \quad i = 1, 2$$

are simply related to the components of waves vectors. $G_1^* = i\omega(\lambda^* + 2\eta^*)$, $G_2^* = i\omega\eta^*$ denote the dynamic complex moduli. The waves considered may propagate along the x-axis if $\mathrm{Re}\,\mu^2 < 0$, being simultaneously damped in the z-direction if also $\mathrm{Re}\,\nu_i^2 > 0$.

For incompressible fluids $\nabla^2 \Phi_1 = 0$, and Eqs. (2.1) remain valid under the assumption that $\lambda^* \to \infty$. This assumption implies that also $G_1^* \to \infty$, and $\nabla^2 \Phi_1 = 0$ gives $\nu_1^2 = -\mu^2$. The same results could be rediscovered if we started directly from the constitutive equation of an incompressible fluid.

For homogeneous compressible fluids contained in the lower half-space $z \leqslant 0$ (with the z-axis directed upwards), the relevant boundary conditions at the free surface ($T^{13} = T^{33} = 0$) lead to the following secular equation (cf. [6, 5, 3]):

$$\text{(2.5)} \qquad -4\mu^2 \nu_1 \nu_2 = (\nu_2^2 - \mu^2)^2.$$

On denoting

$$\text{(2.6)} \qquad n = -\frac{\omega^2 \varrho}{\mu^2 G_2^*}, \quad \vartheta = \frac{G_2^*}{G_1^*},$$

we also have

$$\text{(2.7)} \qquad n^3 - 8n^2 + (24 - 16\vartheta)n - 16(1 - \vartheta) = 0.$$

For incompressible fluids, for which $G_1^* \to \infty$ and $\vartheta \to 0$, we arrive at

$$\text{(2.8)} \qquad n^3 - 8n^2 + 24n - 16 = 0.$$

The same equation can be obtained on the basis of the corresponding constitutive equation of an incompressible fluid. To this end, however, one has to substitute into the boundary conditions at the free surface $p_0(0) = -i\omega\varrho A_1$ instead of $p_0(0) = 0$ (cf. [6]).

Numerical solutions of Eqs. (2.7) and (2.8) were extensively discussed by CURRIE *et al.* [3, 4]. It has been proved, among other things, that only two roots of Eq. (2.8) are admissible; these are

$$\text{(2.9)} \qquad n_1 = 0.9126, \quad n_2 = 3.5437 - i2.2303.$$

The real root n_1 describes quasi-elastic waves, also occurring in purely elastic and elastic-like media, while the complex root n_2 characterizes new viscoelastic waves. Such viscoelastic waves may be observed in incompressible fluids if $\tan\delta > 0.159$, where δ denotes the frequency-dependent loss angle (cf. [4]). It is worth noting that the speeds of propagation corresponding to the roots n_1 and n_2 are less and greater than C_2, respectively.

For homogeneous compressible fluids with rigid outer surfaces, the relevant boundary conditions ($u = w = 0$) lead to the secular equation (cf. [6])

$$\mu^2+\nu_1\nu_2 = 0 \qquad \left(n = \frac{1+\vartheta}{\vartheta}\right), \tag{2.10}$$

where n and ϑ has been defined by Eqs. (2.6). After passing to a limit for incompressible fluids ($\vartheta \to 0$), we obtain

$$n \to +\infty, \quad \text{or} \quad \mu^2 \to 0^-. \tag{2.11}$$

Thus the Rayleight-type waves have an infinite speed of propagation along the x-axis, without any damping effects in that direction. Since

$$\nu_2^2 = -\frac{\omega^2}{C_2^2}\cos^2\delta_2(1-i\tan\delta_2), \tag{2.12}$$

where

$$C_2^2 = \frac{\operatorname{Re} G_2^*}{\varrho}, \qquad \tan\delta_2 = \frac{\operatorname{Im} G_2^*}{\operatorname{Re} G_2^*}, \tag{2.13}$$

the waves also propagate (with a finite speed) in the z-direction, being simultaneously damped if only $\tan\delta_2 \neq 0$.

3. Rayleigh waves in two-layer incompressible fluids

We shall briefly discuss certain cases of two-layer incompressible fluids in which thin layers of thickness h are superposed on bulk fluids contained in the lower half-space $z \leqslant 0$ (cf. [6]). Apart from the boundary conditions at the outer surface of a fluid ($T^{13} = T^{33} = 0$ or $u = w = 0$), the boundary conditions at the interface between two immiscible fluids should be taken into account. If the layers can slide freely at the interface, we have

$$T^{33}(0) = \overline{T}^{33}(0), \qquad w(0) = \overline{w}(0), \tag{3.1}$$

where the overbars refer to the lower fluid. If the layers fully adhere at the interface, we have, moreover,

$$T^{13}(0) = \overline{T}^{13}(0), \qquad u(0) = \overline{u}(0). \tag{3.2}$$

The above conditions, after substituting from Eqs. (2.3) and taking into account the fact that far away from the interface the amplitudes must vanish, lead to systems of homogeneous linear equations. Their nontrivial solutions essentially depend on the number of constants available and the assumed values of $p_0(h)$, $p_0(0)$ and $\overline{p}_0(0)$ (cf. [6]).

For example, in more general cases of compressible fluids with the free outer surfaces and the layers sliding freely at the interface, we obtained four equations involving eight quantities to be determined. If the waves are only transmitted from the upper fluid to the lower one, the number of constants can be reduced (cf. [6]) and the secular equation is expressed in the form

$$\left(4\mu^2\nu_1\nu_2-(\nu_2^2-\mu^2)^2\right)\left(2\overline{\nu}_1\overline{\nu}_2-(\overline{\nu}_2^2-\mu^2)\right) = 0, \tag{3.3}$$

where the overbars refer to the lower fluid. Therefore, two types of waves are possible for incompressible fluids: the waves described by Eq. (2.8) may appear in the upper layer, and those for which

$$\bar{n} = \frac{4\bar{\vartheta}}{4\bar{\vartheta}-1} \to 0^- \quad \text{or} \quad \mu^2 \to \infty \tag{3.4}$$

in the lower one. They do not propagate at all along the x-axis; the damping is full in that direction.

Similarly, for compressible fluids with the rigid outer surfaces and the layers sliding freely at the interface, we obtain, instead of Eq. (3.3),

$$\big(\mu^2-\nu_1\nu_2\big)\big(2\bar{\nu}_1\nu_2-(\bar{\nu}_2^2-\mu^2)\big) = 0. \tag{3.5}$$

Therefore two types of waves are possible for incompressible fluids: waves with infinite speeds of propagation along the x-axis in the upper layer, described by Eq. (2.11), and those described by Eq. (3.4) in the lower fluid.

Now let us consider the case of waves fully reflected in the upper layer. If the outer surface is free, we arrive at the secular equation (cf. [6])

$$\big(4\mu^2\nu_1\nu_2+(\nu_2^2-\mu^2)^2\big)\big(2\nu_1\nu_2+(\nu_2^2-\mu^2)\big) = 0. \tag{3.6}$$

Therefore two types of waves are possible for incompressible fluids: the waves described by Eq. (2.8) and those for which

$$n = \frac{4\vartheta}{4\vartheta-1} \to 0^- \quad \text{or} \quad \mu^2 \to \infty; \tag{3.7}$$

both waves occur only in the upper layer. If the outer surface is rigid, we obtain

$$\big(\mu^2-\nu_1\nu_2\big)\big(2\nu_1\nu_2-(\nu_2^2-\mu^2)\big) = 0. \tag{3.8}$$

Therefore two types of waves are possible for incompressible fluids: waves with infinite speeds of propagation along the x-axis, described by Eq. (2.11), and those described by Eq. (3.7); both waves occur only in the upper layer. Since there are no waves in the lower bulk fluids, the upper layers act as waveguides for Rayleigh waves of the types considered. Of course, the case of full reflection at the interface between two incompressible fluids requires that

$$\frac{\bar{C}_2^2}{C_2^2} > \frac{1+\tan^2\bar{\delta}_2}{1+\tan\bar{\delta}_2\tan\delta_2}, \tag{3.9}$$

where the overbars refer to the lower bulk fluid.

The above analysis shows that for incompressible two-layer fluids sliding freely at the interface, the Rayleigh waves of finite or infinite speeds of propagation may appear only in the upper layers of fluids if any disturbances arising in these layers either are transmitted to the lower bulk fluids without reflection or are fully reflected at the interface. This conclusion is opposed to that drawn in the cases of two-layer compressible fluids for which the Rayleigh waves with finite speeds of propagation could occur in the lower bulk fluids (cf. [6]).

For further illustration we also consider the case of two-layer incompressible fluids with free outer surfaces and the layers fully adhering at the interface. We additionally assume that only shear waves can propagate in the upper layer. The system of equations resulting from the relevant boundary conditions (six in number) can be satisfied if (cf. [6])

$$\mu^2 = \nu_2^2 = -\frac{\omega^2}{2C_2^2}\cos^2\delta_2(1-i\tan\delta_2). \tag{3.10}$$

Thus the waves considered always propagate along the x-axis. The damping conditions in the z-direction require that

$$\frac{\omega^2}{2C_2^2}\cos^2\delta_2 > 0, \quad \frac{\overline{C}_2^2}{C_2^2} > 2\frac{\cos^2\overline{\delta}_2}{\cos^2\delta_2}. \tag{3.11}$$

This means that the dilatational waves ($\overline{\nu}_1^2 = -\mu^2$) are always damped, while any damping of shear waves in the lower fluid is determined by the inequality $(3.11)_2$. If, moreover,

$$\frac{\overline{C}_2^2}{C_2^2} = 2\frac{\sin 2\overline{\delta}_2}{\sin 2\delta_2} \tag{3.12}$$

for certain discrete values of frequency, the damping of shear waves is full, i.e. there is no propagation in the z-direction.

In a similar way, other more complex cases of two-layer or multi-layer incompressible fluids can be analysed. The question still remains as to experimental evidence for Rayleigh waves in real, practically incompressible fluids.

Appendix

It seems that the form of Eqs. (2.1) deserves some additional comments. Starting from the linearized constitutive equation of a viscoelastic compressible fluid, viz.

$$\mathbf{T} = (-p+\lambda^*\mathrm{tr}\mathbf{D})\mathbf{1}+2\eta^*\mathbf{D}, \quad \mathbf{D} = \frac{1}{2}(\nabla\mathbf{v}+(\nabla\mathbf{v})^T), \tag{A.1}$$

where λ^*, η^* denote the frequency-dependent dynamic viscosities, and denoting

$$\mathbf{v} = \mathrm{grad}\phi_1+\mathrm{rot}\boldsymbol{\Phi}_2, \quad \mathrm{div}\boldsymbol{\Phi}_2 = 0, \tag{A.2}$$

we arrive rather at

$$\nabla^2\left(\nabla^2-\frac{\varrho}{\eta^*}\frac{\partial}{\partial t}\right)\boldsymbol{\Phi}_2 = 0, \quad -\frac{\nabla^2 p}{\lambda^*+2\eta^*}+\nabla^2\left(\nabla^2-\frac{\varrho}{\lambda^*+2\eta^*}\frac{\partial}{\partial t}\right)\Phi_1 = 0 \tag{A.3}$$

than at Eqs. (2.1). The condition $(A.2)_2$ leads directly to Eq. $(2.1)_1$, where for plane flows the vector potential $\boldsymbol{\Phi}_2$ has been replaced by the scalar one Φ_2 (cf. Eq. (2.2)). If p is an undetermined hydrodynamic pressure such that $\nabla^2 p = 0$, we immediately obtain the remaining Eqs. $(2.1)_{2,3}$, since for compressible fluids $\nabla^2\Phi_1 \neq 0$. In particular, for isochoric flows $\nabla^2\Phi_1 = 0$ and Eq. $(A.3)_2$ gives $\nabla^2 p = 0$. On the other hand, if p is a barotropic pressure, we have either

$$p(\varrho) = C\varrho \quad \text{or} \quad p(\varrho) = C_1\varrho^k, \tag{A.4}$$

for isothermal or adiabatic flows, respectively. Since the equation of continuity leads to

$$\varrho = \varrho_0 \exp\left(-\int_0^t \nabla^2 \Phi_1 \, dt\right) = \varrho_0(1 - \nabla^2 \Phi_1 + \ldots), \tag{A.5}$$

where ϱ_0 is a constant density at rest, we have

$$\nabla^2 p = -\varrho_0 \frac{dp}{d\varrho} \nabla^2 \nabla^2 \Phi_1 + \ldots, \tag{A.6}$$

where only linear terms in Φ_1 have been retained. It also results from Eqs. (A.4) that either

$$\varrho_0 \frac{dp}{d\varrho} = C\varrho_0 \quad \text{or} \quad \varrho_0 \frac{dp}{d\varrho} = C_1 k \varrho_0 \varrho^{k-1} = C_1 k \varrho_0^k (1 - \nabla^2 \phi_1)^{k-1}, \tag{A.7}$$

respectively. Thus the linearized expressions (A.7) may be treated as quantities independent of time. After substituting from Eq. (A.6) into Eq. $(A.3)_2$, we again arrive at Eq. $(2.1)_2$ if $\lambda^* + 2\eta^*$ is replaced by $\lambda^* + 2\eta^* + \varrho_0 / i\omega(dp/d\varrho)$.

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