Arch. Mech., 35, 4, pp. 443–456, Warszawa 1983

On approximate structures in mechanics

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THE NOTION of approximate models considered in mechanics is defined in terms of mathematical objects, and the method is outlined of comparing the solutions of the boundary-value problems formulated for the approximate models with the unknown solution concerning the accurate model.

W pracy formalizuje się, tzn. definiuje za pomocą obiektów matematycznych, rozważane w mechanice pojęcie modelu przybliżonego oraz podaje metodę porównywania rozwiązań zagadnień brzegowych modeli przybliżonych z nieznanym rozwiązaniem modelu ścisłego.

В работе формализуется, т.зн. определяется при помощи математических объектов, рассматриваемое в механике, понятие приближенной модели, а также приводится метод сравнивания решений краевых задач приближенных моделей с неизвестным решением точной модели.

1. Introduction

THE NOTION of models of real bodies are frequently used in mechanics, cf. [13, 15]. Various problems are then formulated for such models (e.g. the boundary value problems [12, 16]), and the corresponding solutions are sought for.

Some of the real body models are called approximate (or simplified), e.g. [16]. These models are approximate in comparison to other models called accurate or exact; the shells may serve as an example of such models which approximate the model of a three-dimensional body.

Using the above notions of the accurate and approximate models and the solution of the problem, let us formulate the following questions:

(i) How the solutions of the approximate model problems should be compared with the unknown solutions concerning the accurate models?

(ii) How a prescribed model may be found to approximate a given accurate model?

This paper is aimed at defining the notion of an approximate model in terms of certain mathematical objects so as to enable us to answer the questions posed above.

Not all models used in mechanics will be dealt with in this paper. The considerations will be confined to such models which enable the separation of structures in the sense of relation systems [2, 4] having the form

$$\langle X, F, M\rangle. \tag{1.1}$$

Here X, F are sets, and M is a multifunction([1])

$$M: X \to 2^F; \tag{1.2}$$

([1]) Notation $f: X \to Y$ implies that f is defined on X, and hence dom f may differ from X.

in what follows, only the structures of the form (1.1) are considered. The choice of such structures is connected with the following problem:

Problem (a)

Let f_0 be a fixed element of F. Determine the set of such $x_0 \in X$ that

$$f_0 \in M(x_0). \tag{1.3}$$

Problem (a) represents the motivation of assuming the object of our investigation in the form of the relation system (1.1). Such structures may thus be called the structures of problems (a).

Numerous problems of the form (1.3) are being considered in mechanics; obviously, these are not all the structures of the problems of mechanics.

The set of structures (1.1) will be denoted by $\mathfrak{M}$. Each element x_0 satisfying Eq. (1.3) will be called a solution corresponding to f_0. Evidently, the solution of problem (a) constitutes the set of all solutions corresponding to f_0.

The relations between the theory, model and structure are not analyzed here. The corresponding remarks may be found in [10].

2. Examples of structures

The most frequently considered structures (1.1) have the form $\langle X, F, M\rangle$ in which every multifunction M assignes to each $x \in \operatorname{dom} M$ a single-element set $M(x) = \{f\}$ and $f \in F$. Hence the multifunction determines a certain operator A: $\operatorname{dom} M \to F$ such that $A(x) = f \Leftrightarrow \{f\} = M(x)$. Structure $\langle X, F, M\rangle$ will be called the operator structure. An example of such a structure is the structure of boundary value problems of continuum mechanics.

Let Ω be a regular domain in R^3, and τ — an interval $(t_0, t_1) \subset R^1$. Denote by X the set of all vector functions x: $\Omega \times \tau \to R^3$ of class C^2 in $\Omega \times \tau$ and of class C^1 in $\overline{\Omega \times \tau}$.

In order to determine the domain of operator A let us denote by D a certain non-empty set X such that all elements D for each fixed $t \in \tau$ are inversible. Let

$$\operatorname{dom} A = \{x;\, x = i \circ d,\, i \in I,\, d \in D\}, \tag{2.1}$$

where I is the set of all isomeries of space R^3 reduced to codom d.

Also considered will be sets dom A of the form

$$\operatorname{dom} A = \{x;\, x = i \circ (d-1),\, i \in I,\, d \in D\}, \tag{2.2}$$

where 1 is the identity in X.

Let us define space F as a Cartesian product $B \times P \times V \times V$, where B, P are spaces of almost continuous vector functions b: $\Omega \times \tau \to R^3$, p: $\partial_1 \Omega \times \tau \to R^3$, $U = \{u; u = x|_{\partial_2 \Omega \times \tau}\}$ and V is a space of continuous functions v: $\Omega \times \{t_0\} \to R^3$. Elements of F are denoted by (b, p, u, w, v). It will be assumed that $\partial_1 \Omega \cup \overline{\partial_2 \Omega} = \partial\Omega$, $\partial_1 \Omega \cap \partial_2 \Omega = \emptyset$.

In order to determine the operator A let us assume that the operator. $T: X \to S$ is given, where S is a space of symmetric tensor functions s: $\Omega \times \tau \to R^{3\times 3}$

$$s = T(x) \tag{2.3}$$

and that a scalar function $\varrho: \Omega \times \tau \to R^1$ is also prescribed.

Operator A is assumed in the form

$$A(x) = \{\operatorname{div} T(x) - \varrho\ddot{x},\ T(x)|_{\partial_1\Omega\times\tau} n,\ x|_{\partial_2\Omega\times\tau},\ x|_{\Omega\times\{t_0\}},\ \dot{x}|_{\Omega\times\{t_0\}}\}, \tag{2.4}$$

where the vector function $n: \partial_1 \Omega \times \tau \to R^3$ equals the outer unit vector normal to $\partial_1\Omega$.

The structure described above may be used in solving the boundary value problems of continuum dynamics by assuming that Ω — reference configuration of the material medium, τ — time interval, $x \in \operatorname{dom} A$ given by Eq. (2.1) — deformation, $x \in \operatorname{dom} A$ given by Eq. (2.2) — displacement, $-b \in B$ — body forces, $p \in P$ — surface loads, S — space of stresses, $s \in S$ — Piola–Kirchhoff stress tensor, ϱ — mass density, Eqs. (2.4) — constitutive equations.

The structure specified here is general enough, though it does not embrace all possible descriptions of continua. Several modifications are possible: instead of the C^n spaces, spaces H^n may be considered, operator A may be replaced with the multifunction determined by the Lagrange functional, etc.

In addition to the dynamic structures, static structures are also considered. In the particular case when the operator T is defined in the form

$$T(x) = \frac{1}{2} C(\nabla x + \nabla x^T), \tag{2.6}$$

where

$$C = (C_{ijkl}), \qquad C_{ijkl} = C_{jikl} = C_{klij} = C_{lkij}, \qquad i, j, k, l = 1, 2, 3$$

are the scalar functions defined in Ω (elastic constants), problem (1.3) for the operator (2.1) consists in determining such $x \in \operatorname{dom} A$ that

$$s = \frac{1}{2} C(\nabla x + \nabla x^T),$$

$$(\operatorname{div} s - \varrho\ddot{x},\ s|_{\partial_1\Omega\times\tau} n) = (b, p), \tag{2.7}$$

$$x|_{\partial_2\Omega\times\tau} = u, \qquad x|_{t=t_0} = w, \qquad \dot{x}|_{t=t_0} = v.$$

Let us consider another structure. Let X be the function space

$$x^\alpha: \pi \times \tau \to R^3, \qquad \alpha = 1, 2, \ldots, N,$$

where π is a regular region in R^2, and τ — time interval in R^1. Functions x^α are assumed to be of the class C^2 in $\pi \times \tau$, and of class C^1 in $\overline{\pi \times \tau}$.

Let the space F be the product $B \times P \times U \times V \times V$ of the functions

$$\begin{aligned} &b^\alpha: \pi \times \tau \to R^3, \\ &p^\alpha: \partial_1\pi \times \tau \to R^3, \\ &u^\alpha: \partial_2\pi \times \tau \to R^3, \\ &w^\alpha: \pi \times \{t_0\} \to R^3, \\ &v^\alpha: \pi \times \{t_0\} \to R^3, \qquad \alpha = 1,2, \ldots, N. \end{aligned} \tag{2.8}$$

In order to determine A let us assume that the regular functionals are given $\sigma: X \times X \to R^1$, $\varkappa: X \times X \to R^1$, their domains satisfying the conditions $(x_1^\alpha, x_2^\alpha) \in \operatorname{dom}\sigma \Leftrightarrow x_2^\alpha = \nabla x_1^\alpha$, $(x_1^\alpha, x_2^\alpha) \in \operatorname{dom}\varkappa \Leftrightarrow x_2^\alpha = \dot{x}_1^\alpha$.

Let us assume

$$(2.9)\quad A(x^\alpha) = \left[\operatorname{div}\frac{\partial\sigma}{\partial\nabla x^\alpha} - \frac{\partial\sigma}{\partial x^\alpha} - \frac{d}{dt}\frac{\partial\varkappa}{\partial\dot{x}^\alpha} + \frac{\partial\varkappa}{\partial x^\alpha},\ \frac{\partial\sigma}{\partial\nabla x^\alpha}\Big|_{\partial_1\pi\times\tau} n,\ x^\alpha|_{\partial_2\pi\times\tau},\ x^\alpha|_{\pi\times\{t_0\}},\ \dot{x}^\alpha|_{\pi\times\{t_0\}}\right],$$

where n is the unit outer vector normal to $\partial_1\pi$.

Applicability of the structure to solving the boundary value problems of solid body mechanics depends on the method of interpretation of x^α as the functions describing the motion, functionals σ and $\varkappa$ as the functionals describing the internal and kinetic energies of the body, and elements of F as the functions describing the body forces, surface loads and the initial and boundary conditions. Such an interpretation will be presented in Sect. 7.

3. Formal considerations

Let us procede now according to the following scheme. Let the two structures $m = \langle X, F, M\rangle$ and $n = \langle Y, G, N\rangle$ be given. Structure m is extended to include new elements making it possible to define the multifunction M_ε which assignes to the unknown accurate solution the elements "close" to it (in a definite sense).

The set of structures in which the definition of such a multifunction M_ε is possible will be denoted by $\mathfrak{M}_\varepsilon$. Furthermore, the structure $m_\varepsilon \in \mathfrak{M}_\varepsilon$ (structure $m \in \mathfrak{M}$ extended to include M_ε) and the structure $n \in \mathfrak{M}$ are interrelated by the operators

$$\Phi: Y \to \operatorname{dom} M_\varepsilon, \qquad \Psi: \operatorname{codom} M_\varepsilon \to G,$$

so as to make, for instance, space Y — the space of generalized coordinates for X, and G — the space of generalized forces for F, (Fig. 1).

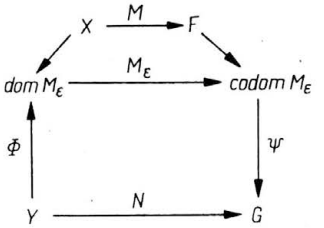

FIG. 1.

The system $\langle m_\varepsilon, n, \Phi, \Psi\rangle$, $m_\varepsilon \in \mathfrak{M}_\varepsilon$, $n \in \mathfrak{M}$ satisfying certain conditions (to be determined later) will be called the approximate structure. It may also be expressed in the form: "n approximates the structure m_ε". The set of approximate structures will be denoted by $\mathfrak{M}_a$.

Before passing to more precise definition of the notion of approximate structures, let us decide that the domain of our formal considerations [3] is the system

$$\langle \mathfrak{M}, \mathfrak{M}_s, \mathfrak{M}_a \rangle. \tag{3.1}$$

It is the relation methasystem [2] for the structures belonging to $\mathfrak{M}$. The questions (i), (ii) of Sect. 1 may now be formulated in the form:

(i') Let the structure n approximate m_s. How the solutions of the problems n should be compared with the unknown solution of the structure m_s?

(ii') How to verify whether the given structure $n \in \mathfrak{M}$ approximates the given structure $m \in \mathfrak{M}$?

In the following sections the sets $\mathfrak{M}_s$, $\mathfrak{M}_a$ of the system (3.1) will be defined.

4. The "closeness" operator

Let us now define a certain functional ε determining the "closeness" within the product $X \times F$. Let us fix two non-empty subsets $X_0 \subset \operatorname{dom} M$, $\tilde{X} \subset \operatorname{dom} M$.

Moreover, let the operator

$$W: X \to Z \tag{4.1}$$

be given which, with $X_0 \in \operatorname{dom} W$, assignes to the unknown solution $x_0 \in X_0$ the elements from a certain space Z. The role of the space Z characterizing the unknown solution x_0 will be explained later(2)

Operator W will be assumed to satisfy the condition

$$(\forall f_0 \in \bigcup_{x_0 \in X_0} M(x_0))[(x_1 \neq x_2, x_1 \in M^{-1}(f_0), x_2 \in M^{-1}(f_0)) \Rightarrow (W(x_1) \neq W(x_2))]. \tag{4.2}$$

Denote

$$R = \{(z_0, f_0); (\exists x_0 \in X_0)[z_0 = W(x_0), f_0 \in M(x_0)]\}.$$

Let the sets X and F be the metric spaces, and let us define the functional

$$\delta: X \times Z \to \bar{R}_+ \tag{4.3}$$

fulfilling the conditions

$$\tilde{X} \times Z_0 \subset \operatorname{dom} \delta, \quad Z_0 \equiv W(X_0),$$

$$(\forall x_0 \in X_0)(\forall \tilde{x} \in \tilde{X})[\varrho(\tilde{x}, x_0) \leqslant \delta(\tilde{x}, z_0), z_0 = W(x_0), (x_0 \in \tilde{X} \Rightarrow \delta(x_0, z_0) = 0)]. \tag{4.4}$$

Functional (4.3) is used to estimate the unknown distance $\varrho, (\tilde{x}, x_0)$ (x_0 being unknown) within the space X.

Spaces X and F were assumed to be metric, hence the metric in the Cartesian product $X \times F$ may also be defined by assuming, for instance,

$$\varrho_{X \times F}[(x_1, f_1), (x_2, f_2)] = \max(\varrho_X(x_1, x_2), \varrho_F(f_1, f_2)). \tag{4.5}$$

The metric (4.5) may be estimated from above by the functional

$$\varepsilon: X \times F \times Z \times F \to R^1$$

(2) Elements of Z may be e.g. the forces corresponding to the solution x_0, the statically admissible stresses etc.

defined as

$$\varepsilon(x, f, z_0, f_0) = \max\big(\delta(x, z_0), \varrho_F(f, f_0)\big). \tag{4.6}$$

Hence the functional ε bounds the unknown distance between the solution x_0 corresponding to f_0 and an arbitrary pair (x, f), $x \in \tilde{X}$, $f \in M(x)$.

Let us now present some examples of functionals ε.

Let $\langle X, F, L\rangle$ be a linear structure, i.e. such in which operator L is linear ([3])

$$L \in (X \to F). \tag{4.7}$$

It will be moreover assumed that operator (4.7) is bounded and inversible, while X, F are Banach spaces. If f_0 is fixed like before, then $x_0 = L^{-1}(f_0)$. Let us assume that $Z \equiv F$ and $W \equiv L$.

In order to determine the functional (4.3) let us observe that from the definition of the operator norm we have

$$||x - x_0|| \leqslant ||L^{-1}||\,||L(x) - f_0||,$$

so it may be assumed [9] that

$$\delta(x, f_0) = m||L(x) - f_0||,$$

where m is given and $m \geqslant ||L^{-1}||$. This functional is defined in the entire space $X \times F$ and does not explicitly depend on the unknown solution x_0 corresponding to f_0.

Functional (4.6) will be of the form

$$\varepsilon\big(x, L(x), f_0, f_0\big) = m_0||L(x) - f_0||, \tag{4:8}$$

where $m_0 \geqslant \max(m_1, m_1)$, $m \geqslant ||L^{-1}||$, and m_1 is the weight of the norm in the space F.

Another example of the functional ε is defined in the Hilbert spaces [11].

Let in X for each $x_0 \in X_0$ be determined two linear manifolds X_1, X_2 such that for all $x_1 \in X_1$ and $x_2 \in X_2$

$$(x_1 - x_0)(x_2 - x_0) = 0. \tag{4.9}$$

Each element $x \in X$ may uniquely be represented in the form $x = (x_1 + x_2)/2$, where x_1, x_2 satisfy Eq. (4.9).

Equation (4.9) is equivalent to the equation

$$||x - x_0|| = \frac{1}{2}||x_1 - x_2|| \tag{4.10}$$

with $x = (x_1 + x_2)/2$.

Let $Z = 2^{X \times X}$ and let the operator W have the form

$$W(x_0) = (x_1, x_2) = z_0,$$

where x_0 is fixed and x_0, x_1, x_2 satisfy Eqs. (4.9).

Making use of Eq. (4.10), functional (4.3) will be defined as

$$\delta(x, z_0) = \frac{1}{2}||x_1 - x_2||,$$

where $(x_1, x_2) = z_0$ and $x = (x_1 + x_2)/2$.

([3]) Notation $f \in (X \to Y)$ means that f is defined in the entire X.

Functional ε may then be assumed to have the form

$$\varepsilon\left(\frac{1}{2}(x_1+x_2), f, z_0, f_0\right) = \max\left(\frac{1}{2}||x_1-x_2||, \varrho_F(f, f_0)\right) \tag{4.11}$$

and

$$f \in M\left(\frac{1}{2}(x_1+x_2)\right).$$

Space X may be decomposed into manifolds X_1, X_2 without the knowledge of x_0. An example of such decomposition in linear elasticity (into the statically and kinematically admissible stress spaces) is given in [14].

Let us consider the functional ε in the operator structures $\langle X, F, A\rangle$ with the operator $A \equiv L^*A_0L$, where the operators $L: X \to Y$, $A_0: Y \to Y^*$ are given and $F \equiv X^*$. Operator $L^*: Y^* \to X^*$ is conjugate to L.

Assume L to be a linear, and A_0 — a strictly monotonic, coercive and potential operator. Potential A_0 is denoted by F.

Making use of the *a posteriori* estimates for the norm in space X given in [1] we obtain

$$||x-x_0|| \leqslant \frac{2}{m}\left(F(L(x))-\langle f_0, x\rangle + F^*(y^*)\right), \tag{4.12}$$

where F^* is the conjugate potential, m — coercivity constant, and y^* satisfies the conditions

$$y^* \in Y_0^* = \{y^*; L^*(y^*) = f_0\}.$$

Here $f_0 = A(x_0)$.

Assume that space $Z = F \times Y^*$ and operator W assign to x_0 the pairs (f_0, y_0^*), where y_0^* is a fixed element of Y^* (e.g. such that $F^*(Y_0^*) = \min\limits_{y^* \in Y_0^*} F^*(Y^*)$).

Functional δ is defined, in accordance with Eq. (4.12), as

$$\delta(x, f_0, y_0^*) = \frac{2}{m}\left(F(L(x))-\langle f_0, x\rangle + F^*(y_0^*)\right). \tag{4.13}$$

Two components in the functional (4.13) may be singled out:

$$V_p(x, f_0) = F(L(x))-\langle f_0, x\rangle$$

called the generalized potential energy, and

$$V_k(y^*) = F^*(y^*)$$

playing the role of the complementary energy.

Functional ε is written in the form

$$\varepsilon(x, A(x), f_0, y_0^*, f_0) = \max\left(\frac{2}{m}\left(V_p(x, f_0)+V_k(y_0^*), m_1||A(x)-f_0||\right)\right), \tag{4.14}$$

m_1 being the weight of the norm in X^*.

5. The approximating multifunction

Functional ε will be used to define the multifunction M_ε

$$M_\varepsilon : X \to 2^{Z \times F} \tag{5.1}$$

in the form

$$(z_0, f_0) \in M_\varepsilon(\tilde{x}) \Leftrightarrow [(z_0, f_0) \in R,\ \tilde{x} \in \tilde{X},\ \exists \tilde{f} \in M(\tilde{x}),\ \varepsilon(\tilde{x}, \tilde{f}, z_0, f_0) \in \langle \varepsilon_0 - a,\ \varepsilon_0 + a \rangle]$$

a being a fixed non-negative real number, and $\varepsilon_0 = \inf\limits_{\substack{x \in \tilde{X} \\ f \in M(x)}} \varepsilon(x, f, z_0, f_0)$.

Multifunction M_ε will be called the approximating multifunction. Structures $\langle X, F, Z, M, \varepsilon \rangle$ will be denoted by m_ε, $m_\varepsilon \in \mathfrak{M}_\varepsilon$.

Let us observe that, in the particular case of $a = 0$, multifunction (5.1) asignes to a pair (z_0, f_0) the pairs $(\tilde{x}, \tilde{f})$, $\tilde{x} \in X$, $\tilde{f} \in M(\tilde{x})$ according to the rule

$$\varepsilon(\tilde{x}, \tilde{f}, z_0, f_0) = \inf_{\substack{x \in \tilde{X} \\ f \in M(x)}} \varepsilon(x, f, z_0, f_0) \tag{5.2}$$

which is the minimization principle.

Let us investigate the particular cases of principle (5.2). Assume the closeness potential ε to be differentiable. The Gateaux derivative is denoted by ∂_G, the Frechet derivative — by ∂_F.

Let the set $\tilde{X}$ be defined by the condition

$$B(x) = 0, \tag{5.3}$$

where $B \in (X \to Q)$ and Q is a certain Banach space.

If both B and ε possess continuous Gateaux derivatives in the neighbourhood of a certain point $\tilde{x} \in \tilde{X}$ satisfying Eq. (5.2), $\tilde{f} \in A(\tilde{x})$ and the image $\partial_G B^*(\tilde{x}) : X \to Q$ is closed, then, according to the Lagrange theorem [5] there exists a number λ_0 and a linear functional q^* (called Lagrange multipliers) defined on Q (simultaneously non-vanishing) such that

$$\lambda_0 \partial_G \varepsilon(\tilde{x}, A(\tilde{x}), z_0, f_0) + [\partial_G B(\tilde{x})]^* q^* = 0, \tag{5.4}$$

where $[\partial_G B(\tilde{x})]^*$ is conjugate to $\partial_G B(\tilde{x})$.

Let us assume that in the operator structure $\langle X, F, A \rangle$ spaces X and F are the Hilbert spaces, operator $A : X \to F$ being differentiable.

Assume the functional ε in the form (4.8)

$$\varepsilon(\tilde{x}, A(\tilde{x}), f_0, f_0) = m \|A(\tilde{x}) - f_0\|^2. \tag{5.5}$$

Functional (5.5) is Frechet differentiable as a combination of the operator $A(\tilde{x}) - f_0$ and functional $\|f\|^2$. The derivative of $A(\tilde{x}) - f_0$ equals the operator $\partial_F A(\tilde{x})$, and the derivative $\partial_F \|f\|^2 = 2f$ since

$$\|f + h\| - \|f\| = 2(f, h) + \|h\|^2.$$

Hence

$$\partial_F \|A(x) - f_0\| = 2\big(A(\tilde{x}) - f_0,\ \partial_F A(\tilde{x})\big)_F. \tag{5.6}$$

Let now the set $\tilde{X}$ be defined by the condition (5.3); then Eq. (5.4) yields

$$2\lambda_0 \big(A(\tilde{x}) - f_0,\ \partial_F A(\tilde{x})\big)_F + [\partial_G B(\tilde{x})]^* q^* = 0. \tag{5.7}$$

Relation (5.7) holds true for all $x \in X$; under additional conditions the relation may be reduced to the virtual works principle.

To this end let us consider the structure of linear elasticity as given by Eqs. (2.7).

Operator B which determines the set $\tilde{X}$ will be defined by the condition [18].

$$h_\nu(x, \nabla x) = 0, \quad \nu = 1, 2, \ldots, r. \tag{5.8}$$

Here h_ν are functions of class C^2 in their arguments.

According to [17], the vector field $\delta x \in C^1(\tilde{\Omega})$ represents the virtual displacements provided

$$\frac{\partial h_\nu}{\partial x}\delta x + \frac{\partial h_\nu}{\partial \nabla x}\nabla(\delta x) = 0, \quad \nu = 1, 2, \ldots, r. \tag{5.9}$$

The set of all virtual displacements for $\tilde{x}$ is denoted by $V_{\tilde{x}}$. Taking into account Eqs. (5.7) and (5.9) and assuming A to be the inversible linear operator, the principle (5.7) may be written as

$$(\forall x \in X)\left[2\lambda_0\big(L(\tilde{x})-f_0, L(x)\big)_F + \int_\Omega \lambda_\nu\left(\frac{\partial h_\nu}{\partial \tilde{x}} + \frac{\partial h_\nu}{\partial \nabla \tilde{x}}x\right)dv = 0\right]. \tag{5.10}$$

In the case of virtual displacements relation (5.10) takes the form

$$(\forall \delta x \in V_{\tilde{x}})[2\lambda_0\big(L(\tilde{x})-f_0, L(\delta)\big)_F = 0]. \tag{5.11}$$

Let us now determine in X the scalar product(4) in the form

$$(x_1, x_2) = \int_\Omega A^{ijkl}x^1_{i,j}x^2_{k,l}\,dv$$

and in F — in the form

$$(f_1, f_2)_F = \big(L^{-1}(f_1), L^{-1}(f_2)\big)_X.$$

Then we have

$$\big(L(\tilde{x})-f_0, L(x)\big)_F = \int_\Omega \big(T(\tilde{x})-T(x_0)\big)\nabla x\,dv = \int_\Omega \big(T(\tilde{x})\nabla x + \operatorname{div}T(x_0)x\big)dv$$

$$- \int_{\partial\Omega} T(x_0)nx\,da = \int_\Omega T(\tilde{x})\nabla x\,dv - \int_\Omega bx\,dv - \int_{\partial\Omega} px\,da.$$

Making use of this relation, Eq. (5.10) is transformed into virtual works principle

$$(\forall \delta x \in V_{\tilde{x}})\left[\int_\Omega T(\tilde{x})\nabla(\delta x)\,dv - \int_\Omega b\delta x\,dv - \int_{\partial\Omega} p\delta x\,da = 0\right] \tag{5.12}$$

equivalent to the perfect constraints principle

$$(\forall \delta x \in V_{\tilde{x}})\left[\int_\Omega r\delta x\,dv + \int_{\partial\Omega} s\delta x\,da = 0\right] \tag{5.13}$$

with $r = -b - \operatorname{div}T(\tilde{x})$, $s = T(\tilde{x})n - p$.

Let us now consider the closeness functional ε in the form (4.14) with the condition

$$\frac{2}{m}\big(V_p(x, f_0) + V_k(y_0^*)\big) \geqslant m_1 \|A(x) - f_0\|.$$

(4) Space X is now the space of abstraction classes differing by rigid displacements.

Then we have

$$\varepsilon(\tilde{x}, A(x), f_0, f_0) = \frac{2}{m}\left(V_p(\tilde{x}, f_0) + V_k(y_0^*)\right). \tag{5.14}$$

The complementary energy being independent of x, the principle (5.2) assumes the form

$$\min_{x \in \tilde{X}} V_p(x, f_0), \quad V_k(y_0^*) = \min_{y^* \in Y_0^*} V_k(y^*). \tag{5.15}$$

Relation (5.15) expresses the principle of minimum potential and complementary energy.

A similar expression (in the form of minimum potential and complementary energy) is obtained in the structure of linear elasticity with the closeness functional (4.11), [10].

6. Approximate structures

Let us now return to the situation shown in Fig. 1, i.e. to the case of two structures related by means of the operators Φ and Ψ.

Due to the definition of multifunction M_ε, the operators are defined as

$$\Phi: Y \to X \times F, \quad \Psi: Z \times F \to G. \tag{6.1}$$

Once the system $\langle m_\varepsilon n, \Phi, \Psi \rangle$ satisfies the conditions

$$\begin{aligned} &\emptyset \neq R \subset \operatorname{dom}\Psi, \quad \operatorname{codom}\Psi \subset \operatorname{codom} N, \\ &\operatorname{dom} N \subset \operatorname{dom}\Phi, \quad \operatorname{codom}\Phi \subset \tilde{X}, \\ &(\forall y \in \operatorname{dom}\Phi)\,[\Phi(y) = (\tilde{x}, \tilde{f}), \tilde{f} \in M(\tilde{x})], \\ &(\forall (z_0, f_0) \in R)\Big(\exists (\tilde{x}, \tilde{f}) \in \Phi\big(N^{-1}(\Psi(z_0, f_0))\big)\Big)\,[(z_0, f_0) \in M_\varepsilon(\tilde{x}, \tilde{f})] \end{aligned} \tag{6.2}$$

it will be called the approximate structure. The following problem may be formulated within the structure:

Problem (a')

Let f_0 be a fixed element of F, and x_0 — an unknown solution of Eq. (1.3) *corresponding to f_0. Determine such $y \in N^{-1}(\Psi(z_0, f_0))$, with $z_0 = W(x_0)$, that*

$$(z_0, f_0) \in M_\varepsilon(\Phi(y)). \tag{6.3}$$

Each element y fulfilling Eq. (6.3) will be called an approximate solution close to the accurate one x_0.

The approximate structure $\langle m_\varepsilon, n, \Phi, \Psi \rangle$ may then be called the structure of problem a', and problem (6.3) may be said to approximate the problem (1.3).

Let us now present the answers to the questions (i'), (ii') posed before.

(i') The approximate solution, that is the solution of problem (a') in the approximate structure $\langle m_a = m_\varepsilon, n, \Phi, \Psi \rangle$ is compared with the unknown solution of (a) in the structure m_ε by means of the functional ε.

In the case when two approximate structures are given, $m_{a_1} = \langle m_\varepsilon, n_1, \Phi_1, \Psi_1 \rangle$ and $m_{a_2} = \langle m_\varepsilon, n_2, \Phi_2, \Psi_2 \rangle$ and x_0 is the unknown accurate solution corresponding

to f_0 in the structure m_ε, y_1 is an approximate solution in m_{a_1} and y_2 — an approximate solution in m_{a_2}, and if

$$\varepsilon(\Phi_1(y_1), z_0, f_0) \leqslant \varepsilon(\Phi_2(y_2), z_0, f_0) \tag{6.4}$$

then it may be stated that y_1 is a better approximate solution than y_2 when compared with the unknown value of x_0.

An example of such comparison of solutions is found in the controlled discretization of elastic bodies [6, 7].

(ii') In order to learn whether the given structure $n = \langle Y, G, N\rangle$ approximates the structure $m_\varepsilon = \langle X, F, Z, M_\varepsilon\rangle$ it should be verified if m_ε, n and the given Φ and Ψ satisfy the conditions (6.2).

In the case when only one operator (6.1) is given, the other one must satisfy the conditions (6.2). Namely, if:

(a) operator Φ is given, Ψ must fulfill the condition

$$g = \Psi(z_0, f_0) \Leftrightarrow \big(\exists(\tilde{x}, \tilde{f}) \in \Phi\big(N^{-1}(g)\big)\big)\,[(z_0, f_0) \in M_\varepsilon(\tilde{x}, \tilde{f})]. \tag{6.5}$$

If

$$\big(\forall(z_0, f_0) \in R\big)[\Psi(z_0, f_0) \neq \emptyset]$$

then the structure n approximates m_ε (in the set R);

(b) if operator Ψ is given, Φ must fulfill the conditions

$$(\tilde{x}, \tilde{f}) = \Phi(y) \Leftrightarrow \big(\exists(z_0, f_0) \in \Psi^{-1}\big(N(y)\big)\big)\,[(z_0, f_0) \in M_\varepsilon(\tilde{x}, \tilde{f})]. \tag{6.6}$$

If

$$\big(\forall(z_0, f_0) \in R\big)\big[\Phi\big(N^{-1}\big(\Psi(z_0, f_0)\big)\big) \neq \emptyset\big]$$

then the structure n approximates m_ε (in the set R).

An example of the approximate structure satisfying the above conditions is presented in the following section.

7. Structures of layered bodies

The starting point is represented by the two-dimensional structure described in Sect. 2 by the conditions (2.19). This structure will be combined with the structure of linear elasticity by constructing the operators (6.1).

Let the region Ω, the reference configuration of a three-dimensional elastic body, be of the form $\pi \times (a, b)$, where the points belonging to π have coordinates z^k, $k = 1, 2$, and the points (a, b) have the coordinate y.

Let us construct the set of surfaces $\pi_\alpha \in \Theta$, $\alpha = 1, 2, \ldots, l$, $l \geqslant 1$, where

$$\Theta = \big\{\pi_\alpha;\ \pi_\alpha = \varphi(\Pi \times \{y_\alpha\}),\quad a = y_0 < y_1, < \ldots < y_l = b\big\}. \tag{7.1}$$

Transformation φ occurring in Eq. (7.1) is the diffeomorphic transformation of region Ω into the physical space. The surfaces π_α uniquely divide the body into the subregions denoted by Ω_c, $c = 1, 2, \ldots, l$, $\Omega_c = \varphi\big(\pi \times (y_{c-1}, y_c)\big)$. Sets Ω_c will be called the layers.

Assume the operator Φ in the form

$$x(z, y, t) = \Phi\big(q^\alpha(z, t), y\big) = q^\alpha(z, t)\xi_\alpha(y), \quad \alpha = 0, 1, \ldots, l \tag{7.2}$$

with $q^\alpha(z,t) \equiv x(z, y_\alpha, t)$. Functions q^α are here the generalized coordinates sought for which describe the motion of the separation surfaces π_α, functions ξ_α being known.

The subdivision of the shell presented here may be treated as the process of discretization of an elastic body, the individual layers representing the finite elements Ω_c. However, contrary to that method, infinite number of nodal points is obtained here since they include not only the points y_α but also all the points of the region π. Polynomials in y may serve as examples of the functions ξ_α.

Let us consider Φ in the following form

$$x(z,y,t) = w_j(q^\alpha(z,t))y^j, \quad j = 0, 1, \ldots, j_0, \tag{7.3}$$

w_j being known functions of the generalized coordinates. Functions (7.3) constitute a particular case of internal constraints, [15, 18].

Since q^α were defined as the deformations of the separation surfaces π_α, the following conditions should be satisfied.

$$w_j(y)^j|_{y=y_\alpha} = q^\alpha. \tag{7.4}$$

Conditions (7.4) form a linear set of equations for the coefficients w_j. In the case of $j_0 = l$, Eq. (7.4) reduces to the Cramer system for w_j which may easily and uniquely be solved provided the principal matrix is not singular. The solutions are

$$w_\alpha = a_{\alpha\beta}q^\beta. \tag{7.5}$$

On substituting Eqs. (7.5) into (7.3) we obtain

$$x(z,y,t) = a_{\alpha\beta}y^\alpha q^\beta(z,t). \tag{7.6}$$

In the simplest case of the first degree polynomial functions, Eqs. (7.6) assume the form

$$x(z,y,t) = \frac{b-y}{h}q^0 + \frac{y-a}{h}q^1,$$

where $h = b-a$. This is the case when no internal separation surfaces exist in the shell, i.e. Θ is the set of two (upper and lower) surfaces.

Polynomials of higher degrees were considered in [8].

Let us now pass to the determination of the operator occurring in Eq. (6.1). The postulate of virtual works principle (5.12) in linear elasticity yields [8]

$$\begin{aligned} &(b_\alpha, p_\alpha) \in \Psi(b,p) \Leftrightarrow b_\alpha = \int_a^b \varrho b\xi_\alpha dy + p\xi_\alpha|_{y=a} + p\xi_\alpha|_{y=b}, \quad z \in \pi, \\ &p_\alpha = \int_a^b p\xi_\alpha dy, \quad z \in \partial\pi. \end{aligned} \tag{7.7}$$

$(b_\alpha, \varrho_\alpha)$ and $\partial_1\pi = \partial\pi$ were assumed as the only functions in the space F of the statical structure (2.8).

The operator defined by Eq. (2.9) yields in the case of Eqs. (7.2) and (7.7) the following equations:

$$\begin{aligned} &\varrho_{\alpha\beta}(\mu q^\beta_{K,LL} + (\lambda+\mu)q^\beta_{L,LK}) + (\lambda\varrho'_{\alpha\beta} - \varrho'_{\beta\alpha})q^\beta_{3,K} - \mu\varrho''_{\alpha\beta}q^\beta_K + b_{K\alpha} = i_{K\alpha}, \\ &\mu\varrho_{\alpha\beta}q^\beta_{3,KK} + (\mu\varrho'_{\alpha\beta} - \lambda\varrho'_{\beta\alpha})q^\beta_{K,K} - (2\mu+\lambda)\varrho''_{\alpha\beta}q^\beta_3 + b_{3\alpha} = i_{3\alpha}, \quad z \in \pi, \end{aligned} \tag{7.8}$$

$$\frac{\partial \varepsilon}{\partial \nabla q_\alpha} n = p_\alpha, \quad z \in \partial\pi.$$

Here $K, L = 1, 2$ and

$$\varepsilon = \int_a^b \varrho e\big(\nabla(\xi_\alpha q^\alpha)\big) dy,$$

$$\varrho_{\alpha\beta} = \int_a^b \varrho \xi_\alpha \xi_\beta dy,$$

$$\varrho'_{\alpha\beta} = \int_a^b \varrho \xi_\alpha \xi'_\beta dy,$$

$$\varrho''_{\alpha\beta} = \int_a^b \varrho \xi'_\alpha \xi'_\beta dy, \quad \xi'_\alpha = \frac{d\xi_\alpha}{dy}.$$

In Eqs. (7.8) the magnitudes b_α are given by Eqs. (7.7), and

$$i_\alpha = \frac{d}{dt}\frac{\partial \varkappa}{\partial \dot{q}_\alpha} - \frac{\partial \varkappa}{\partial q^\alpha}, \quad \varkappa = \frac{1}{2}\int_a^b \varrho |\dot{\overline{\xi_\alpha q^\alpha}}|^2 dy.$$

8. Concluding remarks

The subject of our considerations were the mathematical structures consisting of two sets and multifunctions applied to solving the problems of mechanics in the form (1.3); the aim of our considerations consisted in answering the question when two structures m, n may be said to represent the solution and its approximation. This aim has been achieved by means of the notion of an approximate structure.

The effectiveness of the proposed method depends on the definition of the closeness functional (4.6). In Sect. 4 certain examples of such constructions were presented for a broad class of structures (linear and defined in Hilbert and conjugate spaces). Also the postulated minimization principle (5.2) determines the effectiveness of the method depending on the proper choice of minimization methods.

In the paper such cases were considered in which the closeness functional was differentiable (in the weak or strong sense) then it followed that (under some additional conditions) the principle (5.2) contains the virtual works principle, the principles of minimum of potential and complementary energies, and the principle of perfect constraints. The latter result is of particular importance in mechanics of the bodies with constraints since it may be utilized in proving that all structures built up on this basis (fulfilling certain additional conditions) represent the approximation structures provided the closeness functional depends on the reaction forces.

The theoretical considerations were illustrated by an example of the structure of multilayer bodies approximating the theory of linear elasticity.

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Received April 19, 1982.