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On stability and symmetry conditions in time-independent plasticity

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RESTRICTIONS imposed on the general form of constitutive rate equations by the energy-type postulate of stability of quasi-static plastic deformation processes are investigated. It is shown that the principal symmetry of the moduli relating actual rates of work-conjugate stress and strain measures is necessary for stability of a deformation process. Moreover, it is shown that a non-normality flow rule for elastic-plastic solids is excluded by the less restrictive postulate of stability of equilibrium. The obtained restrictions are related to the possibility of cyclic instability.

Badane są ograniczenia nakładane na przyrostowe równania konstytutywne przez energetyczny postulat stabilności quasi-statycznych procesów deformacji plastycznych. Wykazano, że symetria główna tensora modułów, wiążącego aktualne przyrosty sprzężonych miar naprężeń i odkształceń, jest konieczna dla stabilności procesu deformacji. Wykazano także, że słabszy postulat stateczności stanu równowagi wyklucza niestowarzyszone prawo płynięcia dla materiałów sprężysto-plastycznych. Uzyskane ograniczenia związane są z możliwością cyklicznej utraty stabilności.

Исследуются ограничения накладываемые на определяющие уравнения в приростах энергетическим постулатом стабильности квазистатических процессов пластических деформаций. Показано, что главная симметрия тензора модулей, связывающего актуальные приросты сопряженных мер напряжений и деформаций, необходима для стабильности процесса деформаций. Показано также, что более слабый постулат устойчивости состояния равновесия исключает неассоциированный закон течения для упруго-пластических материалов. Полученные ограничения связаны с возможностью циклической потери стабильности.

1. Introduction

IN THE PAPER [16], an energy-type postulate of stability was proposed for quasi-static isothermal processes of deformation of a continuous body subject to varying external loading: in the postulate the stability of equilibrium is considered as a particular case only. It has been shown [17, 18] that the postulate is sufficiently universal to have relation to various plastic instability phenomena such as buckling, necking, snap-through and localization of deformation. The postulated definition of stability is in the spirit of the general theory of stability of motion; however, the usually investigated perturbations of initial conditions have been replaced by more general persistent disturbances. An intuitive assumption has been introduced claiming that the strength of a small disturbance is sufficiently characterized by the additional amount of energy supplied to the system due to the disturbance. This is the basic assumption whose limits of acceptability are yet not known.

In the stability postulate the constitutive law for the material is assumed to be time-independent but is otherwise arbitrary. For conventional elastic-plastic solids obeying the normality flow rule, or more generally, for solids whose constitutive rate-equations admit a potential, the criteria for velocity fields derived in [17, 18] from the stability postulate are consistent with the properties of solutions to the first-order rate boundary value problem, established by HILL [3, 4, 5]. For such solids the onset of instability calculated from these energy criteria coincides in many practical cases with the instant of primary bifurcation [18]. However, for time-independent solids with a nonsymmetric tensor of incremental moduli, an inconsistency appears between the obtained minimum principle for velocities and the lack of self-adjointness of the rate problem. This shows that the stability postulate imposes certain restrictions on the general form of constitutive relations. Such restrictions are investigated in the present paper.

The objective of this paper is twofold. First, it is to prove that lack of the principal symmetry of the actual moduli in piecewise-linear constitutive rate-equations implies instability of a deformation process in the assumed energy sense, regardless of the actual boundary conditions or material inhomogeneity. Secondly, it is to demonstrate that plastic instability may appear not only as a simple branching of the deformation path but also in a more complicated way, modelled theoretically by a sequence of nonuniform deformation cycles of increasing amplitude. The amplitude of subsequent cycles is increased on account of an additional amount of energy taken from the deforming material over each cycle.

The possibility of cyclic instability indicates that the criteria based merely on consideration of velocity fields at the initial instant of possible branching of the deformation path are in general not sufficient for stability in the present energy sense unless appropriate restrictions on material properties are imposed. This concerns the criteria excluding bifurcation in velocities [3, 4, 5, 19], Hill's criterion for stability of equilibrium [3, 4] and the criteria examined in [17, 18]. This is not surprising since difficulties in finding a rigorous proof of the second-order energy conditions for stability of continua are known even in the theory of elasticity (cf. [15]) where no problems arise with path-dependence, being essential here.

2. Constitutive relations

2.1. General assumptions

We are concerned with isothermal, arbitrarily large deformations of solids whose mechanical properties do not depend on a natural time, neither explicitly nor through a rate sensitivity. The constitutive equations for such materials are assumed in the rate form

(2.1) $$\dot{t}_{ij} = L_{ijkl}\dot{e}_{kl},$$

where

(2.2) $$L_{ijkl} = \tilde{L}_{ijkl}(\dot{\mathbf{e}}, \mathbf{H}).$$

The following notation is used. $\mathbf{t}$ and $\mathbf{e}$ are symmetric second order tensors forming a work-conjugate pair of objective stress and strain measures, respectively [8, 10], in the sense that

(2.3) $$\dot{w} = t_{ij}\dot{e}_{ij}$$

is the work-rate per unit volume in a fixed reference configuration. The lower case Latin indices range from 1 to 3 and denote always vector or tensor components relative to a fixed rectangular basis. A summation convention for repeated indices is adopted. The dot over a symbol of a quantity denotes the rate of change of the quantity at a fixed material element with respect to a scalar variable θ which parametrizes the deformation path in place of a natural time. Since the letter does not appear at all in this paper, the parameter θ will, for simplicity, be referred to as time. The rate is understood as the right-hand time derivative. A tilde over a symbol is used to distinguish a function from its value, when needed.

The moduli L_{ijkl} are, in general, dependent on the direction of the actual strain rate but not on its magnitude. L_{ijkl} may depend also on the prior strain history, symbolized by $\mathbf{H}$. Within an internal variable representation of strain history dependence, $\mathbf{H}$ may be a collection of internal variables and stress or strain components. The considered material body may be inhomogeneous with the restriction that the moduli vary from point to point in a piecewise smooth manner.

We assume that for a given $\mathbf{H}$ the strain-rate space is divided into a number of open domains (necessarily cones with vertices at $\dot{\mathbf{e}} = \mathbf{0}$), called constitutive cones, such that within each domain the moduli L_{ijkl} are constant; the relations (2.1) are thus piecewise linear. This corresponds to certain models of elastoplastic material response at a vertex on the yield surface (cf. [10, 20]), e.g. to those for crystals deformed by multislip [6, 12, 2]. In elastic range there is of course only one constitutive cone coinciding with the whole strain-rate space, while in the conventional formulation of elastoplasticity equations at a regular point on the yield surface, two half-spaces play the role of constitutive cones [7]. The latter particular case will be examined in Sect. 7. The moduli are well defined or not by a strain rate if the corresponding point in $\dot{\mathbf{e}}$-space lies in a constitutive cone or on a cone boundary, respectively.

A strain path $\tilde{\mathbf{e}}$ is defined as a directed, continuous and piecewise smooth curve in strain space. We will say that a strain path *corresponds to one constitutive cone* if the moduli are well defined along the path and vary continuously and piecewise — smoothly with strain. It is assumed that if at a certain $\mathbf{H}$ the moduli are well defined by a strain rate, then every sufficiently short, smooth strain path of bounded curvature, starting from $\mathbf{H}$ with this strain rate, corresponds to one constitutive cone. Moreover, we assume that every strain path $\tilde{\mathbf{e}}$, which starts from the same $\mathbf{H}$ as a strain path $\tilde{\mathbf{e}}^0$ corresponding to one constitutive cone, also corresponds to one (and the same as $\tilde{\mathbf{e}}^0$) constitutive cone, provided that $\sup_\theta |\dot{\tilde{\mathbf{e}}}(\theta) - \dot{\tilde{\mathbf{e}}}^0(\theta)| < \varepsilon$, where ε is a small positive number and $|\dot{\mathbf{e}}| = (\dot{e}_{ij}\dot{e}_{ij})^{1/2}$. These formal though rather natural assumptions are needed to make the further considerations sufficiently rigorous.

We assume that

$$L_{ijkl} = L_{jikl} = L_{ijlk} \tag{2.4}$$

but the principal symmetry of the moduli under the interchange $ij \leftrightarrow kl$ is not assumed in general. However, if $L_{ijkl} = L_{klij}$ for one choice of a work-conjugate pair of stress and strain measures, then the principal symmetry is preserved under transformation to a different conjugate pair [8, 9], though the moduli change.

2.2. Nonobjective and generalized measures

When nonuniform deformations of a material body are considered, it is convenient to use the constitutive rate equations (2.1) reformulated in terms of rates of the nominal stress **s** and deformation gradient **F** which are work-conjugate though nonobjective measures of stress and strain. Denote by x_i and ξ_i the components of the position vector of a material element in the current and reference configuration, respectively ($\boldsymbol{\xi}$ shall be identified with a material point). We denote $\mathbf{F} = \nabla\mathbf{x}$, $F_{ij} = x_{i,j}$ and $\dot{F}_{ij} = v_{i,j}$, where the comma denotes partial differentiation with respect to ξ_j and $\mathbf{v} = \dot{\mathbf{x}}$ is the velocity. We have $\dot{w} = s_{ij}v_{j,i}$ (cf. Eq. (2.3)), while Eqs. (2.1) can be rewritten as

$$\dot{s}_{ij} = C_{ijkl}v_{l,k}. \tag{2.5}$$

To obtain a direct link between the moduli C_{ijkl} and L_{ijkl}, one can take as $(\mathbf{t}, \mathbf{e})$ the particular pair $(\mathbf{T}, \mathbf{E})$, where $\mathbf{T}$ is the second (symmetric) Piola–Kirchoff stress tensor and $\mathbf{E}$ is the Green strain tensor. By differentiating with respect to θ the formulae

$$s_{ij} = T_{ik}F_{jk}, \tag{2.6}$$

$$E_{ij} = \frac{1}{2}(F_{ki}F_{kj} - \delta_{ij}), \tag{2.7}$$

we obtain [10]

$$C_{ijkl} = F_{jp}F_{lq}\hat{L}_{ipkq} + T_{ik}\delta_{jl}, \tag{2.8}$$

where δ_{ij} is the Kronecker symbol and $\hat{L}_{ipkq}$ denote the moduli associated with the pair $(\mathbf{T}, \mathbf{E})$. On account of the symmetry of $\mathbf{T}$, we have

$$C_{ijkl} - C_{klij} = F_{jp}F_{lq}(\hat{L}_{ipkq} - \hat{L}_{kqip}), \tag{2.9}$$

so that $C_{ijkl} = C_{klij}$ if and only if $L_{ijkl} = L_{klij}$, provided $\mathbf{F}$ is nonsingular.

To simplify the formulae, we shall use, following Hill [9, 10], a generalized notation and denote by $(\mathbf{p}, \mathbf{q})$ a pair of generalized measures of stress and strain such that the work-rate per unit reference volume is expressed as

$$\dot{w} = p_\alpha \dot{q}_\alpha. \tag{2.10}$$

$\dot{w}$ is proportional to the work-rate per unit mass since the reference configuration is held fixed. The pair $(\mathbf{p}, \mathbf{q})$ can replace here either $(\mathbf{t}, \mathbf{e})$ or $(\mathbf{s}, \mathbf{F})$; we will not discuss the possibility of nontensorial measures of strain. $\mathbf{p}$, $\mathbf{q}$ are regarded as nine-dimensional vectors of components p_α, q_α. The Greek indices range from 1 to 9 and replace in a prescribed manner a pair of adjacent Latin indices, with the exception that for $\mathbf{F}$ and $\dot{\mathbf{F}}$ the order of Latin indices is reversed. A summation convention for repeated Greek indices is adopted. The generalized moduli are denoted by $c_{\alpha\beta}$. For instance, $p_\alpha \dot{q}_\alpha = t_{ij}\dot{e}_{ij} = s_{ij}v_{j,i}$, $c_{\alpha\beta}\dot{q}_\beta \to$ $\to L_{ijkl}\dot{e}_{kl}$ or $C_{ijkl}v_{l,k}$, etc.

Equations (2.1) and (2.5) are written jointly as

$$\dot{p}_\alpha = c_{\alpha\beta}\dot{q}_\beta. \tag{2.11}$$

Principal symmetry of the moduli L_{ijkl} or C_{ijkl} corresponds to symmetry of $c_{\alpha\beta}$ under interchange of Greek indices.

3. Stability postulate

The reader is referred to the previous papers [16, 17, 18] for a more detailed exposition of the concept of stability of a deformation process in the energy sense. Here we recall only the basic definition, following [18].

Consider a continuous body of a time-independent material, occupying in an arbitrarily chosen, fixed reference configuration a space domain V bounded by a piecewise smooth surface S. On a part S_u of the body surface the displacement history is prescribed, while on the remaining part S_T the nominal tractions $\mathbf{T}$, referred to a surface element in the reference configuration, are assumed to be derivable from a given time-dependent potential ω, $T_i = -\dfrac{\partial\omega(\mathbf{x}, \boldsymbol{\xi}, \theta)}{\partial x_i}$. The boundary conditions vary in time so that the body undergoes deformations. A real process of deformation is idealized by an isothermal, quasi-static process, called *the fundamental process*, the stability of which is to be studied.

A deformation process is described by (and shall be identified with) a function $\boldsymbol{\chi}$, $\mathbf{x} = \boldsymbol{\chi}(\boldsymbol{\xi}, \theta)$, $\xi \in V$, $\theta \in [\theta_1, \theta_2]$. We consider only the kinematically admissible processes which start at $\theta = \theta_1$ from the same given initial state as the fundamental process, are compatible with the kinematical constraints on S_u and satisfy appropriate regularity conditions. The fundamental process (also belonging to this class) and the corresponding quantities are distinguished by the superscript "0".

In absence of strong discontinuities in velocities, the internal work done in a kinematically admissible process $\mathbf{x}$ in a time interval $[\theta_1, \theta] \subset [\theta_1, \theta_2]$ is expressed as (cf. Eq. (2.3))

$$W(\boldsymbol{\chi}, \theta) = \int_V \int_{\theta_1}^{\theta} \dot{w}\, d\theta\, d\xi, \tag{3.1}$$

where $d\xi$ is an infinitesimal volume element in the reference configuration ([1]). The potential energy of *the loading device* which corresponds to the assumed boundary conditions (body forces are neglected) is defined by

$$W^e(\boldsymbol{\chi}, \theta) = \int_{S_T} \omega\big(\boldsymbol{\chi}(\boldsymbol{\xi}, \theta), \boldsymbol{\xi}, \theta\big) dS; \tag{3.2}$$

without loss of generality we may assume that $W^e(\boldsymbol{\chi}, \theta_1) \equiv 0$.

Introduce the energy functional E,

$$E = W + W^e; \tag{3.3}$$

its value $E(\boldsymbol{\chi}, \theta)$ is the amount of energy which has to be supplied in the time interval $[\theta_1, \theta]$ from external sources to the *system* consisting of the body and the loading device, in order to realize the process $\boldsymbol{\chi}$ in a quasi-static manner.

We are concerned with the stability of the fundamental process $\boldsymbol{\chi}^0$ against *persistent* disturbances from a class wide enough to give in effect any kinematically-admissible process, called *a perturbed process*, sufficiently close to $\boldsymbol{\chi}$. The measure d of the distance between $\boldsymbol{\chi}$ and $\boldsymbol{\chi}^0$ need not be specified here; it suffices to assume that $\boldsymbol{\chi}(\boldsymbol{\xi}, \theta) \neq \boldsymbol{\chi}^0(\boldsymbol{\xi}, \theta)$

([1]) Volume $\int_B d\xi$ of a finite region B will be denoted by $|B|$.

for $\boldsymbol{\xi} \in B \subset V$ implies $d(\boldsymbol{\chi}, \boldsymbol{\chi}^0, \theta) > 0$. On the other hand, the choice of the measure ϱ of disturbance is very essential here. ϱ is assumed in the form

$$\varrho(\boldsymbol{\chi}, \boldsymbol{\chi}^0, \theta) = \sup_{\hat{\theta} \in [\theta_1, \theta]} \{E(\boldsymbol{\chi}, \hat{\theta}) - E(\boldsymbol{\chi}^0, \hat{\theta})\} \tag{3.4}$$

which is equivalent to that assumed in [16] if inertia forces are included into perturbing forces (cf. [17]).

DEFINITION 1 [16]. *A fundamental process* $\boldsymbol{\chi}^0$ *is stable in a time interval* $[\theta_1, \theta_2]$ *if and only if for every number* $\varepsilon > 0$ *there is another number* $\delta > 0$ *such that for every perturbed process* $\boldsymbol{\chi}$ *and all* $\theta \in [\theta_1, \theta_2]$

$$\varrho(\boldsymbol{\chi}, \boldsymbol{\chi}^0, \theta) < \delta \quad \textit{implies} \quad d(\boldsymbol{\chi}, \boldsymbol{\chi}^0, \theta) < \varepsilon.$$

The process is called unstable if it is not stable.

This definition jointly with the definitions of a process $\boldsymbol{\chi}$ and measures ϱ and d form a stability postulate.

One of the consequences of the stability postulate is the minimum principle for velocities [17, 18]. A smooth velocity field $\tilde{\mathbf{v}}^0$ corresponding to a stable process assigns to the functional

$$\frac{1}{2}\int_V \dot{s}_{ij} v_{j,i}\, d\xi + \int_{S_T} \left(\frac{\partial^2 \omega}{\partial \theta\, \partial x_i} + \frac{1}{2}\frac{\partial^2 \omega}{\partial x_i \partial x_j} v_j\right) v_i\, dS \tag{3.5}$$

its absolute minimum value. Consequently, the first (weak) variation of the functional (3.5) should vanish at $\tilde{\mathbf{v}} = \tilde{\mathbf{v}}^0$, which can be shown to imply

$$\left((C^0_{ijkl} + C^0_{klij}) v^0_{j,i}\right)_{,k} = 0 \tag{3.6}$$

at any regular point. The second term appearing in Eq. (3.6) vanishes if the rate equilibrium equations $\dot{s}^0_{ij,i} = 0$ are satisfied but the first *in general* need not vanish unless $C^0_{ijkl} = C^0_{klij}$. However, we cannot infer that the minimum principle for the functional (3.5) implies the principal symmetry of the moduli; for instance, Eq. (3.6) is satisfied trivially in the case of a uniformly deformed homogeneous body while the value of the functional (3.5) is unaffected by the unsymmetric part $\frac{1}{2}(C_{ijkl} - C_{klij})$ of the moduli. We shall prove the principal symmetry of the moduli C^0_{ijkl} by assuming the stability postulate in its general form.

4. Path-dependence of deformation work

4.1. General strain paths

Consider a homogeneous material element subject to a uniform deformation process represented by a piecewise-smooth strain path. The path is parametrized by θ varying from $\bar{\theta}$ to $\bar{\bar{\theta}}$; the quantities evaluated at $\bar{\theta}$ or $\bar{\bar{\theta}}$ are distinguished by one or two bars over the symbol, respectively. The parametrization is always chosen such that the strain rate, $\dot{\mathbf{q}} = (\partial \mathbf{q}/\partial \theta)$ in the generalized notation, is uniformly bounded for all paths.

We start from the expression for the density of work done on a strain path, per unit reference volume, in the form (cf. Eq. (2.10))

$$w = \int_{\bar{\theta}}^{\bar{\bar{\theta}}} p_\alpha \dot{q}_\alpha d\theta. \tag{4.1}$$

By integrating the constitutive rate equations (2.11), we obtain

$$p_\alpha|_\theta = \bar{p}_\alpha + \int_{\bar{\theta}}^{\theta} c_{\alpha\beta}\dot{q}_\beta d\hat{\theta} \tag{4.2}$$

which, after substituting in Eq. (4.1), gives

$$w = \bar{p}_\alpha(\bar{\bar{q}}_\alpha - \bar{q}_\alpha) + \int_{\bar{\theta}}^{\bar{\bar{\theta}}} \dot{q}_\alpha \Big(\int_{\bar{\theta}}^{\theta} c_{\alpha\beta}\dot{q}_\beta d\hat{\theta}\Big) d\theta. \tag{4.3}$$

We can rearrange Eq. (4.3) as

$$w = \bar{p}_\alpha(\bar{\bar{q}}_\alpha - \bar{q}_\alpha) + \bar{c}_{\alpha\beta}\int_{\bar{\theta}}^{\bar{\bar{\theta}}} \dot{q}_\alpha(q_\beta - \bar{q}_\beta)d\theta + \int_{\bar{\theta}}^{\bar{\bar{\theta}}} \dot{q}_\alpha\Big\{\int_{\bar{\theta}}^{\theta}(c_{\alpha\beta} - \bar{c}_{\alpha\beta})\dot{q}_\beta d\hat{\theta}\Big\} d\theta, \tag{4.4}$$

where the moduli $\bar{c}_{\alpha\beta}$ correspond to the strain rate $\dot{\mathbf{q}}$ at $\theta = \bar{\theta}$. On decomposing the matrix of moduli $\bar{c}_{\alpha\beta}$ into the symmetric and antisymmetric parts, and integrating by parts the first integral in Eq. (4.4) multiplied by the symmetric part of $\bar{c}_{\alpha\beta}$, we obtain

$$\begin{aligned} w = {} & \bar{p}_\alpha(\bar{\bar{q}}_\alpha - \bar{q}_\alpha) + \frac{1}{2}\bar{c}_{\alpha\beta}(\bar{\bar{q}}_\alpha - \bar{q}_\alpha)(\bar{\bar{q}}_\beta - \bar{q}_\beta) \\ & + \frac{1}{2}(\bar{c}_{\alpha\beta} - \bar{c}_{\beta\alpha})\int_{\bar{\theta}}^{\bar{\bar{\theta}}} \dot{q}_\alpha(q_\beta - \bar{q}_\beta)d\theta + \int_{\bar{\theta}}^{\bar{\bar{\theta}}} \dot{q}_\alpha\Big\{\int_{\bar{\theta}}^{\theta}(c_{\alpha\beta} - \bar{c}_{\alpha\beta})\dot{q}_\beta d\hat{\theta}\Big\} d\theta. \end{aligned} \tag{4.5}$$

This is an exact formula for the density of work done on an arbitrary piecewise-smooth strain path. It is seen that path-dependence of work is due to the lack of principal symmetry of the initial moduli or due to variation of the moduli with the deformation.

4.2. Approximate expressions for short strain paths

Suppose that the length of a strain path is small so that $\overset{\triangle}{\theta} = \bar{\bar{\theta}} - \bar{\theta}$ is a small positive number and $\overset{\triangle}{\mathbf{q}} = \bar{\bar{\mathbf{q}}} - \bar{\mathbf{q}}$ is a small deformation increment. We consider now only the paths corresponding to one constitutive cone so that the increments of moduli along a path are, at the most, of order $\overset{\triangle}{\theta}$. The paths need not be smooth; they may be piecewise smooth no matter how small $\overset{\triangle}{\theta}$ is. Introduce the usual order symbol $O(\overset{\triangle}{\theta})$ which denotes a quantity such that $|O(\overset{\triangle}{\theta})/\overset{\triangle}{\theta}|$ is bounded when $\overset{\triangle}{\theta} \to 0$ [2]. From Eq. (4.5) we obtain

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[2] Note that the path need not be fixed when $\overset{\triangle}{\theta} \to 0$, but the strain rate must be uniformly bounded

$$(4.6)\qquad w = \bar{p}_\alpha \overset{\triangle}{q}_\alpha + \frac{1}{2}\bar{c}_{\alpha\beta}\overset{\triangle}{q}_\alpha \overset{\triangle}{q}_\beta + \frac{1}{2}(\bar{c}_{\alpha\beta} - \bar{c}_{\beta\alpha}) \int\limits_{\bar{\theta}}^{\bar{\bar{\theta}}} \dot{q}_\alpha (q_\beta - \bar{q}_\beta)\, d\theta + O(\overset{\triangle}{\theta}{}^3).$$

The unsymmetric part of $\bar{c}_{\alpha\beta}$ may have an influence on w of order $\overset{\triangle}{\theta}{}^2$ for nonproportional strain paths only since the term with $(\bar{c}_{\alpha\beta} - \bar{c}_{\beta\alpha})$ vanishes if $\dot{q}_\alpha$ is proportional to $(q_\alpha - \bar{q}_\alpha)$ at each θ. For the same reason, for a smooth path with bounded curvature the term with $(\bar{c}_{\alpha\beta} - \bar{c}_{\beta\alpha})$ is, at the most, of order $\overset{\triangle}{\theta}{}^3$ and the formula (4.6) reduces for such paths to the "trapezoid rule of quadrature" ([10], Eq. (2.5)):

$$(4.7)\qquad w = \bar{p}_\alpha \overset{\triangle}{q}_\alpha + \frac{1}{2}\bar{c}_{\alpha\beta}\overset{\triangle}{q}_\alpha \overset{\triangle}{q}_\beta + O(\overset{\triangle}{\theta}{}^3).$$

For other paths, however, the path-independent expression (4.7) need not be true unless $\bar{c}_{\alpha\beta} = \bar{c}_{\beta\alpha}$ since the term with $(\bar{c}_{\alpha\beta} - \bar{c}_{\beta\alpha}) \neq 0$ in Eq. (4.6) can be made nonzero and of order $\overset{\triangle}{\theta}{}^2$ (see the next subsection). Hence, the work done on a short strain path corresponding to one constitutive cone is path-independent to second order if and only if the initial moduli associated with work-conjugate stress and strain measures have the principal symmetry property. In essence, this is a standard result (see also [10], p. 30).

4.3. Superposition of a triangular cycle on a strain path

Consider a short, smooth strain path $\tilde{\mathbf{q}}^0$ of bounded curvature leading from $\bar{\mathbf{q}}$ to $\mathbf{q}$ and corresponding to one constitutive cone. We compare the work done on this path with that done on the other path $\tilde{\mathbf{q}}$ leading also from $\bar{\mathbf{q}}$ to $\bar{\bar{\mathbf{q}}}$. For a certain choice of work-conjugate measures, $\tilde{\mathbf{q}}$ is defined by

$$(4.8)\qquad \dot{\tilde{\mathbf{q}}}(\theta) = \dot{\tilde{\mathbf{q}}}^0(\theta) + \begin{cases} 3\mu\mathbf{m} & \text{if}\quad \bar{\theta} < \theta < \bar{\theta} + \dfrac{\vartheta}{3}, \\ 3\mu(\mathbf{n} - \mathbf{m}) & \text{if}\quad \bar{\theta} + \dfrac{\vartheta}{3} < \theta < \bar{\theta} + \dfrac{2\vartheta}{3}, \\ -3\mu\mathbf{n} & \text{if}\quad \bar{\theta} + \dfrac{2\vartheta}{3} < \theta < \bar{\theta} + \vartheta, \\ \mathbf{0} & \text{if}\quad \bar{\theta} + \vartheta < \theta < \theta. \end{cases}$$

$\mathbf{m}$, $\mathbf{n}$ are arbitrary constant vectors from $\dot{\mathbf{q}}$-space, $\vartheta > 0$ is arbitrary but not greater than $\overset{\triangle}{\theta}$, and μ is a positive constant chosen to be sufficiently small so that both paths $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{q}}^0$ correspond to one (and the same) constitutive cone. The path $\tilde{\mathbf{q}}$ may be regarded as a result of superposition of the triangular strain cycle from $\mathbf{0}$ to $\vartheta\mu\mathbf{m}$ to $\vartheta\mu\mathbf{n}$ to $\mathbf{0}$ on the path $\tilde{\mathbf{q}}^0$. The straight segments of this triangular cycle for the work-conjugate measures chosen are transformed into smooth curved segments of bounded curvature under change of measures. On substituting Eq. (4.8) in Eq. (4.6), the density of work done on the path $\tilde{\mathbf{q}}$ is found to be

$$(4.9)\qquad w = \bar{p}_\alpha \overset{\triangle}{q}_\alpha + \frac{1}{2}\bar{c}_{\alpha\beta}\overset{\triangle}{q}_\alpha \overset{\triangle}{q}_\beta + (\bar{c}_{\alpha\beta} - \bar{c}_{\beta\alpha})\left(\frac{1}{3}\bar{\dot{q}}{}^0_\alpha (m_\beta + n_\beta) + \frac{1}{2}\mu n_\alpha m_\beta\right)\mu\vartheta^2 + O(\overset{\triangle}{\theta}{}^3).$$

In obtaining Eq. (4.9), we have used the assumed regularity of the path $\tilde{\mathbf{q}}^0$ to substitute the expression $(\theta-\bar{\theta})\dot{\bar{\mathbf{q}}}^0+O(\overset{\Delta}{\theta^2})$ in place of $\tilde{\mathbf{q}}^0(\theta)-\bar{\mathbf{q}}$, where $\dot{\bar{\mathbf{q}}}^0 = (\partial\tilde{\mathbf{q}}^0/\partial\theta)\,(\bar{\theta})$. The density w^0 of work done on the path $\tilde{\mathbf{q}}^0$ is expressed by Eq. (4.7). The paths $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{q}}^0$ are, of course, assumed to correspond to the same initial values $\bar{p}_\alpha$, $\bar{c}_{\alpha\beta}$ of the stress and moduli. The difference of respective work densities is thus expressed as

$$w-w^0 = (\bar{c}_{\alpha\beta}-\bar{c}_{\beta\alpha})\left(\frac{1}{3}\dot{\bar{q}}^0_\alpha(m_\beta+n_\beta)+\frac{1}{2}\mu n_\alpha m_\beta\right)\mu\vartheta^2+O(\overset{\Delta}{\theta^3}). \tag{4.10}$$

Evidently, the difference can be of order ϑ^2 and of either sign, unless $\bar{c}_{\alpha\beta} = \bar{c}_{\beta\alpha}$.

5. Stability for local disturbances and the cycles of constrained deformation

To obtain restrictions imposed by the stability postulate on the mechanical properties at a given material element, regardless of the boundary conditions and of the material properties elsewhere, we will consider a special kind of disturbances which perturb the fundamental deformation process in the neighbourhood $\hat{B}$ of that element only. Such disturbances will be called *local disturbances.* A physical nature of such disturbances may be diverse but it is irrelevant here. Once the precise definition of stability has been formulated, we may proceed further quite formally and study the implications of the stability postulate with the help of arbitrary kinematically admissible perturbed processes, however artificial they might seem to be.

While a perturbed deformation path at any internal point of the body may be chosen arbitrarily (no internal constraints are assumed here), the deformations at neighbouring points are then not arbitrary since on the boundary of $\hat{B}$ the displacements in the perturbed and fundamental processes in the case of local disturbances coincide. For our present purposes it suffices to consider a class of the perturbed processes satisfying this requirement, obtained by superposition of a *cycle of constrained deformation* (or of a number of such cycles) on the fundamental process. The cycle of constrained deformation is, by definition, described by the continuous displacement function $\Delta\boldsymbol{\chi}$, defined on the product $B\times[\bar{\theta},\bar{\bar{\theta}}]$, $B\subset\hat{B}\subset V$, and satisfying the conditions

$$\Delta\boldsymbol{\chi}(\boldsymbol{\xi},\theta) = \mathbf{0} \quad \text{for} \quad \begin{cases}\boldsymbol{\xi}\in B & \text{if} \quad \theta=\bar{\theta} \quad \text{or} \quad \theta=\bar{\bar{\theta}},\\ \boldsymbol{\xi}\in\partial B & \text{at any} \quad \theta\in[\bar{\theta},\bar{\bar{\theta}}].\end{cases} \tag{5.1}$$

Note that the nonuniform deformation process described by $\Delta\boldsymbol{\chi}$ is compatible with rigid constraints on the boundary ∂B of B, which motivates the terminology. The superposition is meant in a kinematic sense: the resulting perturbed process is described by the function $\boldsymbol{\chi}$, defined as

$$\boldsymbol{\chi}(\boldsymbol{\xi},\theta) = \begin{cases}\boldsymbol{\chi}^0(\boldsymbol{\xi},\theta)+\Delta\boldsymbol{\chi}(\boldsymbol{\xi},\theta) & \text{for} \quad \boldsymbol{\xi}\in B \quad \text{and} \quad \theta\in[\bar{\theta},\bar{\bar{\theta}}],\\ \boldsymbol{\chi}^0(\boldsymbol{\xi},\theta) & \text{otherwise}.\end{cases} \tag{5.2}$$

Denote by ΔW the difference of work due to superposition of a cycle of constrained deformation,

$$\Delta W|_\theta = \int_B (w-w^0)|_\theta d\xi, \tag{5.3}$$

where w and w^0 are the work densities corresponding to the processes χ (defined by Eq. (5.2)) and χ^0, respectively. Note that $\Delta W|_\theta$ is just the difference of values of the energy functional (3.3) at time θ for the processes χ and χ^0, appearing in the formula (3.4), since the motion of the body surface and therefore also the potential energy of loading device have not been perturbed. This suggests that the negative sign of ΔW may have a direct influence on stability of the process χ^0. This can be proved under certain additional conditions, stated in the following lemma.

LEMMA 1. Let $\hat{B}$ be a ball contained in V, and let $[\theta_1, \theta_2]$ be a time interval. Suppose that for every ball $B \subset \hat{B}$ and for every time interval $[\bar{\theta}, \bar{\bar{\theta}}] \subset [\theta_1, \theta_2]$ a cycle of constrained deformation defined on $B \times [\bar{\theta}, \bar{\bar{\theta}}]$ can be found such that superposition of that cycle on the fundamental process results in a work difference ΔW estimated by

$$\frac{1}{|B|}\Delta W|_\theta < \begin{cases} c_1(\bar{\bar{\theta}}-\bar{\theta})^r & \text{for} \quad \theta \in [\bar{\theta}, \bar{\bar{\theta}}], \\ -c_2(\bar{\bar{\theta}}-\bar{\theta})^r + c_3(\theta-\bar{\theta})^{r+1} & \text{for} \quad \theta \in [\bar{\bar{\theta}}, \theta_2], \end{cases} \tag{5.4}$$

where c_1, c_2, c_3, r are positive constants.

Then the fundamental process is unstable in the time interval $[\theta_1, \theta_2]$ in the sense of Definition 1.

P r o o f. Let the assumptions of the lemma be satisfied; we show that this contradicts the definition of stability. Let φ be a constant such that $1 < \varphi^{r+3} < ((c_1+c_2)/c_1)$. Then, a positive constant $c < \theta_2-\theta_1$ can be found such that another constant $c_4 = cc_3/(1-1/\varphi)^{r+3}$ satisfies the inequalities $c_4 < c_2$ and $(\varphi^{r+3}-1)(c_1+c_4) < c_2$. Define a geometric sequence (ϑ_K), $\vartheta_K = \vartheta_1\varphi^{K-1}$, $K = 1, 2, \ldots, N$, with $\vartheta_1 = c(\varphi-1)(1/\varphi)^N$, so that for any N we have $\sum\limits_{K=1}^{N}\vartheta_K < c$. Then, there exists a sequence of balls $B_1, B_2, \ldots, B_N$ of the volumes $|B_K|$ forming also a geometric sequence defined by $|B_K| = |\hat{B}|\vartheta_K^3/c^3$, such that $B_K \subset \hat{B}$ for every K and $B_K \cap B_L = \emptyset$ for $K \neq L$. Now, for every positive number δ we take N sufficiently large such that $|B_1|\vartheta_1^r c_2/(\varphi^{r+3}-1) < \delta$, and construct a perturbed process in the time interval $[\theta_1, \theta_c]$, $\theta_c = \theta_1 + c$ by superposing on the fundamental process the sequence of the increasing cycles of constrained deformation defined on the products $B_K \times [\bar{\theta}_K, \bar{\theta}_K+\vartheta_K]$, $\bar{\theta}_K = \bar{\theta}_{K-1}+\vartheta_{K-1}$, $\bar{\theta}_1 = \theta_1$. At time θ_c the configurations of the body in the perturbed and fundamental processes coincide but the respective work done in these processes is different. Since the balls B_K do not overlap each other, the work differences due to superposition of the particular cycles can be calculated independently of each other. By the assumptions of the lemma, we can define the cycles such that each of the corresponding work differences ΔW_K is estimated by

$$\Delta W_K|_\theta < |B_K| \cdot \begin{cases} c_1\vartheta_K^r & \text{for} \quad \theta \in [\bar{\theta}_K, \bar{\theta}_K+\vartheta_K], \\ -c_2\vartheta_K^r + c_3(\theta-\theta_K)^{r+1} & \text{for} \quad \theta \in [\bar{\theta}_K+\vartheta_K, \theta_c]; \end{cases}$$

of course, $\Delta W_K|_\theta = 0$ for $\theta < \bar{\theta}_K$. On using these estimates and elementary properties of the geometric sequences defined, we find that the difference of the work done in the perturbed and fundamental processes is estimated by

$$\Delta W|_\theta = \sum_{K=1}^{N}\Delta W_K|_\theta < \begin{cases} \delta & \text{for} \quad \theta \in [\theta_1, \theta_c], \\ -\varepsilon_1 & \text{for} \quad \theta = \theta_c, \end{cases}$$

where $\varepsilon_1 = |\hat{B}|(c_2 - c_4)c^r\left(1 - \frac{1}{\varphi}\right)^{r+3}$ is a positive number independent of N and therefore also of δ.

The continuation of the perturbed process for $\theta > \theta_c$ is assumed independently of δ; it differs from the fundamental process but is otherwise arbitrary. Since $\Delta W|_{\theta_c}$ is negative, there is a time interval $[\theta_c, \theta_\varepsilon]$, $\theta_\varepsilon < \theta_2$, such that for any $\theta \in [\theta_c, \theta_\varepsilon]$ we have $\Delta W|_\theta < 0$. Hence, from the formula (3.4) we obtain that $\varrho(\boldsymbol{\chi}, \boldsymbol{\chi}^0, \theta_\varepsilon) < \delta$. Denote the value of d at $\theta = \theta_\varepsilon$ by ε; ε is a positive number independent of δ.

It is seen that for every positive number δ we have constructed a perturbed process such that $\varrho|_{\theta_\varepsilon} < \delta$ and $d|_{\theta_\varepsilon} = \varepsilon$. This contradicts the stability definition, and the lemma has been proved.

6. Symmetry of moduli and stability

In this section we show that the stability postulate implies the principal symmetry of the actually operating moduli provided they are associated with work-conjugate measures. To this purpose for nonsymmetric moduli we prove the existence of cycles of constrained deformation of the properties stated in the lemma from the preceding section.

Consider a fundamental process $\boldsymbol{\chi}^0$ such that the moduli $c^0_{\alpha\beta}$ are well-defined and nonsymmetric at a certain regular point $(\hat{\boldsymbol{\xi}}, \theta_1)$ of the process $\boldsymbol{\chi}^0$.

A regular point of a process $\boldsymbol{\chi}^0$ is defined as a pair $(\hat{\boldsymbol{\xi}}, \theta_1)$ such that in a sufficiently small time interval $[\theta_1, \theta_2]$ and in a sufficiently small neighbourhood $\hat{B}$ of $\hat{\boldsymbol{\xi}}$, the velocity gradient $\nabla \mathbf{v}^0$ and the corresponding moduli $c^0_{\alpha\beta}$ are well defined and are continuously differentiable with respect to $\boldsymbol{\xi}$ and θ. Let $\hat{B} = \{\boldsymbol{\xi} : |\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}| < \hat{R}\} \subset V$ and $\theta_2 - \theta_1 = \hat{\theta}$, with $\hat{R}$ and $\hat{\theta}$ sufficiently small. Then, the variations of $c^0_{\alpha\beta}$ and $\nabla \mathbf{v}^0$ in $\hat{B} \times [\theta_1, \theta_2]$ are of order $O(\hat{R}) + O(\hat{\theta})$. Let $B = \{\boldsymbol{\xi} : |\boldsymbol{\xi} - \bar{\boldsymbol{\xi}}| < R\}$ be any ball contained in $\hat{B}$ (in general, $\bar{\boldsymbol{\xi}} \neq \hat{\boldsymbol{\xi}}$), and $[\bar{\theta}, \bar{\bar{\theta}}]$ be any time interval contained in $[\theta_1, \theta_2]$. Consider a cycle of constrained deformation defined on $B \times [\bar{\theta}, \bar{\bar{\theta}}]$ and described by the particular $\Delta \boldsymbol{\chi}$, satisfying the conditions (5.1) and defined by

$$
(6.1) \qquad \Delta \dot{\boldsymbol{\chi}}(\boldsymbol{\xi}, \theta) = \begin{cases} 3\mu \tilde{\mathbf{u}}(\boldsymbol{\xi}) & \text{for} \quad \bar{\theta} < \theta < \bar{\theta} + \frac{\vartheta}{3}, \\ 3\mu(\tilde{\mathbf{w}}(\boldsymbol{\xi}) - \tilde{\mathbf{u}}(\boldsymbol{\xi})) & \text{for} \quad \bar{\theta} + \frac{\vartheta}{3} < \theta < \bar{\theta} + \frac{2\vartheta}{3}, \\ -3\mu \tilde{\mathbf{w}}(\boldsymbol{\xi}) & \text{for} \quad \bar{\theta} + \frac{2\vartheta}{3} < \theta < \bar{\theta} + \vartheta, \\ \mathbf{0} & \text{for} \quad \theta \notin [\bar{\theta}, \bar{\theta} + \vartheta], \end{cases}
$$

where $\tilde{\mathbf{u}}$, $\tilde{\mathbf{w}}$ are at the moment arbitrary smooth vector fields defined on B which vanish on ∂B and have uniformly bounded gradients for all B, $\vartheta = \bar{\bar{\theta}} - \bar{\theta}$, and μ is a positive constant independent of B. Let such a cycle be superimposed on the fundamental process; then the gradients $\nabla \tilde{\mathbf{u}}(\boldsymbol{\xi})$, $\nabla \tilde{\mathbf{w}}(\boldsymbol{\xi})$ correspond at each $\boldsymbol{\xi}$ to the vectors $\mathbf{m}$, $\mathbf{n}$ which defined a triangular strain cycle considered in the Subsect. 4.3. The constant μ is chosen sufficiently small so that at every point of B the perturbed strain path $\tilde{\mathbf{q}}$, defined by Eqs. (5.2) and (6.1), corresponds to one and the same constitutive cone as the fundamental path $\tilde{\mathbf{q}}^0$. Therefore, we may use the formula (4.10) to calculate ΔW. On account of the above regularity assumptions in $\hat{B}\times[\theta_1, \theta_2]$ and in terms of the nominal stress $\mathbf{s}$ and deformation gradient $\mathbf{F}$ as a specific pair of $\mathbf{p}$, $\mathbf{q}$-variables, this formula at every $\boldsymbol{\xi}\in B$ can be rewritten as

$$(6.2)\qquad (w-w_0)|_\theta = \frac{1}{3}\mu\vartheta^2(\hat{C}^0_{ijkl}-\hat{C}^0_{klij})\hat{v}^0_{j,i}(u_{l,k}+w_{l,k})+\frac{1}{2}\mu^2\vartheta^2(\hat{C}^0_{ijkl}-\hat{C}^0_{klij})u_{l,k}w_{j,i} + \vartheta^2\big(O(\hat{R})+O(\hat{\theta})\big)+O\big((\theta-\bar{\theta})^3\big)\quad \text{for}\quad \theta\geqslant\bar{\bar{\theta}},$$

where $\hat{C}^0_{ijkl}$ and $\hat{v}^0_{j,i}$ are evaluated at $(\hat{\boldsymbol{\xi}}, \theta_1)$ for the fundamental path.

Integration of Eq. (6.2) over B yields

$$(6.3)\qquad \Delta W|_\theta = \frac{1}{2}\mu^2\vartheta^2(\hat{C}^0_{ijkl}-\hat{C}^0_{klij})\int_B u_{l,k}w_{j,i}\,d\xi + |B|\vartheta^2\big(O(\hat{R})+O(\hat{\theta})\big)+|B|\,O\big((\theta-\bar{\theta})^3\big)\quad \text{for}\quad \theta\geqslant\bar{\bar{\theta}},$$

since the integral of the first term in Eq. (6.2) vanishes by the divergence theorem.

To proceed further, we need the corollary of the following lemma:

LEMMA 2. Let Ω be a bounded domain in R^3, and let A_{ijkl} $(i, j, k, l = 1, 2, 3)$ be constants (³) satisfying the conditions

$$(6.4)\qquad A_{ijkl} = -A_{klij},$$

$$(6.5)\qquad A_{ijkl} = A_{jilk}.$$

Suppose that

$$(6.6)\qquad \int_\Omega A_{ijkl}u_{j,i}w_{l,k}\,d\xi = 0$$

for all real-valued functions $\tilde{u}_j$, $\tilde{w}_l$ twice continuously differentiable on Ω and vanishing with their derivatives on $\partial\Omega$.

Then $A_{ijkl} = 0$.

Proof. By usual arguments of the calculus of variations, we obtain that for a fixed $\tilde{u}_j$ the necessary condition for Eq. (6.6) is

$$(6.7)\qquad A_{ijkl}u_{j,ik} = 0 \quad \text{in } \Omega.$$

In turn, the condition (6.7) holds for every $u_{j,ik} = u_{j,ki}$ if and only if

$$(6.8)\qquad A_{ijkl} = -A_{kjil}.$$

By combining Eqs. (6.4), (6.5) and (6.8), we obtain

$$A_{ijkl} = -A_{kjil} = -A_{jkli} = A_{lkji} = A_{klij} = -A_{ijkl}.$$

Hence, $A_{ijkl} = 0$.

COROLLARY. If A_{ijkl} satisfy Eqs. (6.4) and (6.5) and are not identically zero, then the integral in Eq. (6.6) is nonzero for some $\tilde{u}_j$, $\tilde{w}_l$ and, consequently, can take any value.

(³) The lemma can be easily generalized for A_{ijkl} varying (piecewise) smoothly in Ω.

Define now

$$\hat{A}_{ijkl} = \hat{L}^0_{ijkl} - \hat{L}^0_{klij}, \tag{6.9}$$

where $\hat{L}^0_{ijkl}$ are the moduli associated with the Green measure of strain, evaluated at $(\hat{\boldsymbol{\xi}}, \theta_1)$ for the actual constitutive cone. From the relation (2.4) it is seen that $\hat{A}^0_{ijkl}$ satisfy the conditions (6.4) and (6.5). Hence there exist smooth vector fields $\tilde{\mathbf{u}}^1$, $\tilde{\mathbf{w}}^1$ vanishing outside the unit ball $B^1 = \{\boldsymbol{\xi}: |\boldsymbol{\xi}| < 1\}$ and such that

$$\hat{A}_{ijkl} \int_{B^1} u^1_{j,i} w^1_{l,k} d\xi = -\hat{c}|B^1| < 0. \tag{6.10}$$

For any ball B, we define the fields $\tilde{\mathbf{u}}$, $\tilde{\mathbf{w}}$ by

$$\begin{aligned} \tilde{u}_j(\boldsymbol{\xi}) &= R\hat{F}^{-1}_{rj}\tilde{u}^1_r\left(\frac{\boldsymbol{\xi}-\bar{\boldsymbol{\xi}}}{R}\right), \\ \tilde{w}_l(\boldsymbol{\xi}) &= R\hat{F}^{-1}_{sl}\tilde{w}^1_s\left(\frac{\boldsymbol{\xi}-\bar{\boldsymbol{\xi}}}{R}\right), \end{aligned} \tag{6.11}$$

where $\hat{\mathbf{F}}^{-1}$ is the inverse of the deformation gradient $\hat{\mathbf{F}}$ at $(\hat{\boldsymbol{\xi}}, \theta_1)$. Note that $\tilde{\mathbf{u}}$, $\tilde{\mathbf{w}}$ vanish on ∂B and have uniformly bounded gradients, as required above. By combining Eqs. (2.9), (6.9), (6.10) and (6.11), we obtain

$$\begin{aligned} (\hat{C}^0_{ijkl} - \hat{C}^0_{klij}) \int_B u_{j,i} w_{l,k} d\xi &= \hat{F}_{jp}\hat{F}_{lq}\hat{A}_{ipkq}\hat{F}^{-1}_{rj}\hat{F}^{-1}_{sl} \int_B \tilde{u}^1_{r,i}\left(\frac{\boldsymbol{\xi}-\bar{\boldsymbol{\xi}}}{R}\right)\tilde{u}^1_{s,k}\left(\frac{\boldsymbol{\xi}-\bar{\boldsymbol{\xi}}}{R}\right) d\xi \\ &= \hat{A}_{irks} \int_{B^1} u^1_{r,i}(\boldsymbol{\xi})\tilde{u}^1_{s,k}(\boldsymbol{\xi}) R^3 d\xi = -R^3|B^1|\hat{c} = -|B|\hat{c}. \end{aligned} \tag{6.12}$$

By using Eq. (6.12), we reduce Eq. (6.3) to

$$\frac{1}{|B|}\Delta W|_\theta = -\frac{1}{2}\mu^2\hat{c}\vartheta^2 + \vartheta^2\big(O(\hat{R}) + O(\hat{\theta})\big) + O\big((\theta-\bar{\theta})^3\big) \quad \text{for} \quad \theta \geqslant \bar{\bar{\theta}}. \tag{6.13}$$

Finally, we take $\hat{R}$ and $\hat{\theta}$ sufficiently small, so that the second term in Eq. (6.13) is negligible (for all balls) when compared with the first term. Then the positive constants $c_2 \approx \frac{1}{2}\mu^2\hat{c}$ and c_3 can be found such that the required estimate $(5.4)_2$ with $r = 2$ is obtained. Note that $\Delta W|_{\bar{\bar{\theta}}}$ *is negative for* ϑ *sufficiently small.*

For $\theta \in [\bar{\theta}, \bar{\bar{\theta}}]$, the work difference $\Delta W|_\theta$ need not be negative. To obtain its estimate, we integrate over B the formula (4.3) written in terms of $\mathbf{s}$, $\mathbf{F}$ — measures for the two processes and subtract the results. This yields

$$\Delta W|_\theta = \int_B \bar{s}^0_{ij}(F_{ji} - F^0_{ji})|_\theta d\xi + |B|\, O\big((\theta-\bar{\theta})^2\big). \tag{6.14}$$

On account of the equilibrium condition $s^0_{ij,i} = 0$ which is necessary for stability [18], the integral in Eq. (6.14) vanishes by the divergence theorem since $x_j = x^0_j$ on ∂B. Hence we have the estimate $(5.4)_1$ with $r = 2$.

To summarize: for $c^0_{\alpha\beta} \neq c^0_{\beta\alpha}$ at a regular point $(\hat{\boldsymbol{\xi}}, \theta_1)$ we have shown that, for $\hat{R}$ and $\hat{\theta}$ sufficiently small, for every ball $B \subset \hat{B}$ and every time interval $[\bar{\theta}, \bar{\bar{\theta}}] \subset [\theta_1, \theta_2]$ the superposition of the constrained cycle of deformation defined by Eqs. (6.1) and (6.11) on the fundamental process $\boldsymbol{\chi}^0$ results in the work difference satisfying the estimates (5.4) with $r = 2$. By applying the Lemma 1, we obtain that the process $\boldsymbol{\chi}^0$ is unstable in the time interval $[\theta_1, \theta_2]$. The point $\hat{\xi}$ and instants θ_1 and θ_2 are arbitrary. Hence we have proved the following statement:

For stability of a deformation process in the energy sense it is necessary that at every regular point the moduli relating the actual rates of work-conjugate measures of stress and strain have the principal symmetry property.

Comparison with the result obtained in the Subsect. 4.2 shows the relation between the stability of a process and the second-order path-independence of deformation work.

7. Stability of equilibrium of elastic-plastic bodies

Stability of equilibrium is defined here by the Definition 1 as the stability of the degenerate deformation process in which all quantities do not vary in time. In the case of dead loading on S_T, the necessary condition for stability of equilibrium is of the form [17]:

$$\int_V \dot{s}_{ij} v_{j,i} d\xi \geqslant 0 \tag{7.1}$$

for all smooth velocity fields $\tilde{\mathbf{v}}$ vanishing on S_u. Those velocity fields represent possible modes of departure from the considered equilibrium configuration. The criterion (7.1) states that the internal work done on any direct deformation path leading to a neighbouring configuration, defined as a path in which the displacement field is increased proportionally to $\tilde{\mathbf{v}}$ with accuracy to first order, is not less than the corresponding work done by dead loads, to second order in the additional displacements. A stability criterion of this type has been widely used in the literature. Following HILL [3, 4], it has been frequently taken in a slightly stronger form, viz.

$$\int_V \dot{s}_{ij} v_{j,i} d\xi > 0 \tag{7.2}$$

for all smooth velocity fields $\tilde{\mathbf{v}}$ which vanish on S_u and are not identically zero. Hill proposed the criterion (7.2) as a sufficient condition for stability of equilibrium in a dynamic sense, however, under certain assumptions which are not easy to be verified [4, 13].

Below we show that a body of an elastic-plastic material with a smooth yield surface and a non-normality flow rule, being at least partly stressed to the yield point, cannot be in a stable equilibrium in the present energy sense. Hence, at least for such solids, the inequality (7.2) is not sufficient for stability of equilibrium in the present sense. Instability is proved by showing the existence of arbitrarily small cycles of constrained deformation in which energy is extracted from the material, and then by appealing again to the Lemma 1. This demonstrates why the criterion (7.2) which excludes a spontaneous departure from equilibrium along a direct deformation path, is not able without further assumptions to assure stability of equilibrium for more complex paths taken into account.

7.1. Constitutive relations for elastic-plastic solids

In this section attention is focused on elastic-plastic solids with a smooth yield surface bounding an elastic domain in strain space. Within the elastic domain, the stress is a function of strain only. The constitutive rate equations for a material element stressed to the yield point are taken in the form (2.1), with the restriction that at a given $\mathbf{H}$ the $\dot{\mathbf{e}}$-space is divided by the hyperplane of equation $\lambda_{ij}\dot{e}_{ij} = 0$ into two half-spaces, playing the role of constitutive cones, in which the moduli L_{ijkl} take different constant values, viz.

$$(7.3) \qquad L_{ijkl} = \begin{cases} L^p_{ijkl} & \text{if} \quad \lambda_{ij}\dot{e}_{ij} > 0, \\ L^e_{ijkl} & \text{if} \quad \lambda_{ij}\dot{e}_{ij} < 0, \end{cases}$$

where $\boldsymbol{\lambda}$ is the unique normal to the yield surface in strain space, directed outward from the elastic domain (cf. [7]). The elastic moduli L^p_{ijkl} correspond to further plastic loading while the elastic moduli L^e_{ijkl} correspond to elastic unloading. The moduli L^p_{ijkl} and L^e_{ijkl} vary continuously during plastic loading which causes displacements of the yield surface such that the point representing the current strain remains on it, while there is no movement of the yield surface during elastic unloading or straining within the elastic domain. We assume that stress-rate depends continuously on strain-rate; then we must have

$$(7.4) \qquad L^p_{ijkl} - L^e_{ijkl} = -\mu_{ij}\lambda_{kl}.$$

If $L^p_{ijkl} = L^p_{klij}$ and $L^e_{ijkl} = L^e_{klij}$, then $\boldsymbol{\lambda}$ is proportional to $\boldsymbol{\mu}$. The classical normality flow rule is obtained [7,11] when the proportionality factor, g say, is taken positive, so that

$$(7.5) \qquad L^p_{ijkl} = L^p_{klij} = L^e_{ijkl} - \frac{1}{g}\lambda_{ij}\lambda_{kl}, \qquad g > 0$$

The form of the relationship (7.5) and the normality flow rule are measure-invariant [8, 10, 11].

7.2. Symmetry of moduli and stability of equilibrium

We assume here Eqs. (7.3) and (7.4) but not Eq. (7.5); we shall show that Eq. (7.5) is implied by the postulate of stability of equilibrium. Consider a body of an elastic-plastic material being in equilibrium under arbitrary boundary conditions. The body may be partly in elastic state (the elastic zone) and is partly stressed to the yield point (the plastic zone).

Consider first the elastic zone. The moduli L_{ijkl} do not depend there on the strain rate and there is only one constitutive cone coinciding with the whole $\dot{\mathbf{e}}$-space. Therefore, the moduli L_{ijkl} are well defined also for zero rate of strain. The statement from Sect. 6 can thus be directly applied to the degenerate deformation process of zero velocities in the elastic zone. Hence, at every regular point in the elastic zone (or in elastic solids), the incremental moduli associated with work conjugate variables must have the principal symmetry property for the equilibrium to be stable in the energy sense.

In the plastic zone the moduli are not well defined by the zero rate of strain, so that we cannot apply directly the statement from Sect. 6 to show their principal symmetry. However, for nonsymmetric moduli we shall prove the existence of cycles of constrained deformation of the properties specified in the Lemma 1. Of course, the work done in the fundamental process is now zero, so that ΔW is simply the work done in a cycle of constrained deformation.

Suppose first that at a regular point $\hat{\boldsymbol{\xi}}$ of the plastic zone the elastic moduli are not symmetric. A regular point is now defined as a point $\hat{\boldsymbol{\xi}} \in V$, in the neighbourhood of which the moduli L^{p}_{ijkl} and L^{e}_{ijkl} are smooth functions of place. In a sufficiently small neighbourhood $\hat{B}$ of $\hat{\boldsymbol{\xi}}$, $\hat{B} = \{\boldsymbol{\xi}: |\boldsymbol{\xi}-\hat{\boldsymbol{\xi}}| < \hat{R}\} \subset V$, the variations of L^{e}_{ijkl} or L^{p}_{ijkl} are of order $O(\hat{R})$.

For any ball B contained in $\hat{B}$ and for any time interval $[\bar{\theta}, \bar{\bar{\theta}}]$ we define on $B \times [\bar{\theta}, \bar{\bar{\theta}}]$ a cycle of constrained deformation consisting of a constant number K of subsequent, identical subcycles; the first of the subcycles is described again by Eq. (6.1) with $\vartheta = \frac{1}{K}(\bar{\bar{\theta}}-\bar{\theta})$, and the other analogically. The fields $\tilde{\mathbf{u}}$, $\tilde{\mathbf{w}}$ are defined by the relations (6.11) as previously, with the only difference that $\hat{L}^{0}_{ijkl}$ in Eq. (6.9) denote now the elastic moduli at $\hat{\boldsymbol{\xi}}$, associated with the Green measure of strain. For sufficiently small amplitude of the subcycles, i.e. for sufficiently small $\mu\vartheta$, plastic deformations can take place in the first subcycle only. The work done over each of the elastic subcycles (i.e. except the first subcycle) is the same and can be expressed by a formula analogous to Eq. (6.13), viz.

$$\Delta W|_{\theta+\vartheta} - \Delta W|_{\theta} = -\mu^{2}\hat{c}|B|\vartheta^{2} + |B|\vartheta^{2}O(\hat{R}) + |B|\,O(\vartheta^{3}), \quad \text{for} \quad \theta-\bar{\theta} = \vartheta, 2\vartheta, \ldots, (K-1)\vartheta. \tag{7.6}$$

We take $\hat{R}$ sufficiently small and K sufficiently large, so that the second and the third term in Eq. (7.6) become negligible in comparison with the first term; the work done over an elastic subcycle is then negative. From Eq. (6.14) it is seen that the work done up to any θ in the first subcycle is, at most, of order $|B|\vartheta^{2}$. Therefore, there is a constant c_1 such that the estimate $(5.4)_1$ is true with $r = 2$, since the maximum of $\Delta W|_{\theta}$ must appear in the first two subcycles on account of negativeness of Eq. (7.6). Evidently, the second required estimate in the relation (5.4) with $c_3 = 0$, and thus with any $c_3 > 0$, follows for K sufficiently large.

Application of the Lemma 1 shows that the equilbrium is unstable in any finite time interval. Since the point $\hat{\boldsymbol{\xi}}$ is arbitrary, we arrive at the following result:

For stability of equilibrium of an elastic or elastic-plastic body it is necessary that the elastic moduli associated with work-conjugate measures are symmetric at every regular point of the body.

This result is also valid if the yield surface is not smooth since the presence of corners on the yield surface does not affect the above proof.

Suppose now that the elastic moduli are symmetric, but at a regular point $\hat{\boldsymbol{\xi}}$ of the plastic zone the plastic moduli do not satisfy Eq. (7.5), that is, the tensors $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ appearing in Eq. (7.4) are not proportional with a positive factor. For any ball $B \subset \hat{B}$, where $\hat{B}$ is a neighbourhood of $\hat{\boldsymbol{\xi}}$ as above, and any interval $[\bar{\theta}, \bar{\bar{\theta}}]$ we construct a cycle of constrained deformation, again described by Eq. (6.1) but now with $\vartheta = \bar{\bar{\theta}}-\bar{\theta}$ and $\tilde{\mathbf{u}} = -\tilde{\mathbf{w}}$. It is easy to see that for sufficiently small $\mu\vartheta$ at a typical point of B a simple loading-unloading cycle of strain takes place, followed or preceded by the opposite, purely elastic cycle. By applying the formula (4.7) to each of the smooth segments of the cycles and using the constitutive equations (2.5) to express the stress increments on each segment, we obtain that the work density is expressed by

$$w|_{\bar{\theta}} = \frac{1}{2}\mu^2\vartheta^2(C^e_{ijkl}-C^p_{ijkl})u_{j,i}u_{l,k}+O(\vartheta^3), \tag{7.7}$$

where C^e_{ijkl} and C^p_{ijkl} correspond to L^e_{ijkl} and L^p_{ijkl}, respectively. By writing Eq. (7.7) in terms of the moduli $\hat{C}^e_{ijkl}$ and $\hat{C}^p_{ijkl}$ evaluated at $\hat{\boldsymbol{\xi}}$, and integrating over B, we obtain

$$\Delta W|_{\bar{\theta}} = \frac{1}{2}\mu^2\vartheta^2(\hat{C}^e_{ijkl}-\hat{C}^p_{klij})\int_B u_{j,i}u_{l,k}\,d\xi+|B|\vartheta^2 O(\hat{R})+|B|\,O(\vartheta^3). \tag{7.8}$$

As previously, we define the field $\tilde{\mathbf{u}}$ for any ball with help of a field $\tilde{\mathbf{u}}^1$ defined on the unit ball B^1 (cf. Sect. 6). From the known theorems of the calculus of variations [1, 14], it follows that the quantity

$$A_{ijkl}\int_{B^1} u^1_{j,i}u^1_{l,k}\,d\xi \tag{7.9}$$

takes nonnegative values for all smooth fiel $\tilde{\ }^1$ vanishing over ∂B_1 if and only if

$$A_{ijkl}b_j a_i b_l a_k \geqslant 0 \tag{7.10}$$

for all vectors $\mathbf{a}$, $\mathbf{b}$. Take $A_{ijkl} = \hat{L}^e_{ijkl}-\hat{L}^p_{ijkl} = \hat{\mu}_{ij}\hat{\lambda}_{kl}$ (cf. Eq. (7.4)) where $\hat{L}^e_{ijkl}$ and $\hat{L}^p_{ijkl}$ are the elastic and plastic moduli at $\hat{\boldsymbol{\xi}}$, associated with the Green measure of strain. Since we have excluded the case $\hat{\boldsymbol{\lambda}} = g\hat{\boldsymbol{\mu}}$ with $g > 0$, it can be shown that a pair $(\mathbf{a}, \mathbf{b})$ can be found such that $\hat{\mu}_{ij}a_ib_j$ and $\hat{\lambda}_{ij}a_ib_j$ are of opposite signs. Hence, the inequality (7.10) does not hold and a smooth field $\tilde{\mathbf{u}}^1$ vanishing over ∂B_1 exists such that

$$(\hat{L}^e_{ijkl}-\hat{L}^p_{ijkl})\int_{B^1} u^1_{j,i}u^1_{l,k}\,d\xi = -|B^1|\hat{c} < 0. \tag{7.11}$$

For that $\tilde{\mathbf{u}}^1$, we define the field $\tilde{\mathbf{u}}$ for any ball B by Eq. $(6.11)_1$. By combining Eq. (7.11), (2.8) and $(6.11)_1$, we obtain (cf. Eq. (6.12))

$$(\hat{C}^e_{ijkl}-\hat{C}^p_{ijkl})\int_B u_{j,i}u_{l,k}\,d\xi = -|B|\hat{c}. \tag{7.12}$$

For $\hat{R}$ and ϑ sufficiently small, Eqs. (7.8) and (7.12) yield the estimation of the work done over the cycle of constrained deformation in the form

$$\frac{1}{|B|}\Delta W|_{\bar{\theta}} < -c_2\vartheta^2 \tag{7.13}$$

with $c_2 \approx \frac{1}{2}\mu^2\hat{c}$. Obviously, this gives the estimation $(5.4)_2$ with $r = 2$ and $c_3 = 0$ since $W|_{\theta>\bar{\theta}} = W|_{\bar{\theta}}$. The first estimation in the relation (5.4) for $\theta < \bar{\theta}$ follows from Eq. (6.14).

By applying the Lemma 1 and the above statement on symmetry of elastic moduli, we arrive at the following result:

For stability of equilibrium of an elastic-plastic solid with a smooth yield surface it is necessary that the plastic moduli satisfying Eq. (7.4) *are of the form* (7.5) *at every regular point in the plastic zone.*

Shortly, we may say that the postulate of stability of equilibrium excludes a non-normality flow rule.

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