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Isothermal and adiabatic simple waves in a thin-walled tube

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THE ANALYSIS is presented of isothermal and adiabatic simple waves propagating along the semi-infinite, thin-walled tube and produced by normal pressures and torques applied to its end. Several conclusions are drawn concerning the stress trajectories. A numerical example is given in which the stress trajectories are calculated for two temperature ranges, and the isothermal and adiabatic wave velocities are compared. It is shown that in the case of finite deformations the differences between isothermal and adiabatic slow wave velocities may be considerably great.

Przedstawiono analizę izotermicznych i adiabatycznych fal prostych w półnieskończonej cienkosciennej rurze, wywołanych ciśnieniem normalnym i momentem skręcającym przyłożonym do brzegu. Wyciągnięto szereg wniosków dotyczących trajektorii naprężeń. Przedstawiono przykład numeryczny, w którym podano trajektorie naprężeń dla dwóch zakresów temperatury oraz porównano prędkości fal izotermicznych i adiabatycznych. Pokazano, że w zakresie dużych deformacji różnica między wolnymi falami izotermicznymi i adiabatycznymi może być bardzo duża.

Представлен анализ изотермических и адиабатических простых волн в полубесконечной тонкостенной трубе, вызванных нормальным давлением и скручивающим моментом, приложенным к границе. Сделан ряд выводов, касающихся траекторий напряжений. Представлен численный пример, в котором приведены траектории напряжений для двух интервалов температуры, а также сравнены скорости изотермических и адиабатических волн. Показано, что в области больших деформаций разница между свободными изотермическими и адиабатическими волнами может быть очень большой.

Notations

$\mathbf{A}\cdot\mathbf{B}$ A_iB_i or $A_{ij}B_{ij}$,
$\mathbf{AB}$ $A_{ij}B_j$ or $A_{ijkl}B_{kl}$,
$\mathrm{tr}\,\mathbf{A}$ A_{kk},
$\mathbf{1}$ unit tensor,
$\bar{\mathbf{A}}$ $\mathbf{A}-\frac{1}{3}(\mathrm{tr}\,\mathbf{A})\mathbf{1}$,
$\overset{T}{\mathbf{A}}$ transpose of a sensor.

1. Introduction

IN THE PRESENT literature numerous solutions may be found concerning the problems of propagation of waves in complex states of stress (for one space variable and two-parameter loadings). The solutions are usually presented for two types of geometric objects:

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a semi-infinite space and a thin-walled tube. An extensive review on the literature concerning such problems may be found in the papers by N. CRISTESCU [4], W. K. NOWACKI [17], W. HERRMAN [10], E. H. LEE [12].

Three types of waves have been determined analytically in elastic-plastic materials: the acceleration waves, simple waves and unloading (or loading) waves. In the case of acceleration waves, practically all experiments show that all predictions based on such models are false [2]. On the other hand, simple wave processes expected to take place in these models are in a better agreement with the experiments performed on metallic rods and tubes. The available experimental results concerning the unloading waves do not allow for drawing the suitable conclusions.

Simple waves propagating in a thin-walled, semi-infinite tube made of elastic-plastic materials and produced by normal pressure and torques applied at the end of the tube, were subject to the analysis of several authors, to start with the known paper by CLIFTON [1]. The material considered in the paper was isotropic and characterized by isotropic hardening. Later on, LIPKIN and CLIFTON [14] considered a similar problem in the case of materials with kinematic hardening. GOEL and MALVERN [8] proposed a method taking into account simultaneous effects of isotropic and kinematic hardening. In [21] TING derived the solution for all possible combinations of stepwise variable loading of the boundary at the time instant $t = 0$. He also obtained an explicit solution for the case of materials with linear hardening [23]. HAN-CHIN and HUAN-CHI LIN [9] solved the analogous problem under the assumption of endochronic plasticity proposed by Valanis.

FUKUOKA [5, 7], who treated the waves as moving singular surfaces, studied the cases in which the states of material in front and behind the wave were the same or different. Adiabatic simple waves travelling along a thin-walled tube were investigated by RANIECKI [19], PODOLAK and RANIECKI [18]. The effect of energy dissipation on the velocity and profile of simple wave was also analyzed.

In all the papers mentioned above, small deformations were assumed and two wave velocities were established: fast wave velocity C_f and slow wave velocity C_s, satisfying the inequality

$$C_s \leqslant C_2 \leqslant C_f \leqslant C_1 .$$

Here $C_1 = \sqrt{E/\varrho}$ denotes the velocity of longitudinal elastic wave, and $C_2 = \sqrt{\mu/\varrho}$ — the velocity of transversal elastic waves. The results were given earlier by CRAGGS [3] for an infinite space. The case of $C_f = C_s = C_2$ was considered by TING [22].

LIPKIN and CLIFTON [13, 14] established the existence of fast and slow simple waves. Some experimental data were given in paper [6]; experimental and theoretical results were compared by TING [22].

This paper will present the discussion on the propagation velocity of adiabatic simple waves of second order in a thin-walled tube, on the basis of the constitutive equations derived in [20, 16], valid in the range of finite deformations of isotropic metals. Elastic distortions and voluminal plastic deformations will be disregarded.

2. Fundamental equations

It was shown in [16] that the set of fundamental equations in cylindrical coordinates (r, φ, z) consists of the following relations:

(i) Continuity equations

$$\beta\left(\frac{\partial V_r}{\partial_r}+\frac{V_r}{r}+\frac{1}{r}\frac{\partial V^{\varphi}}{\partial\varphi}+\frac{\partial V_z}{\partial z}\right)=\dot{\beta}, \tag{2.1}$$

in which $\beta = \varrho_0/\varrho$, ϱ_0 and ϱ are the respective densities measured in the reference configurations and actual configurations, according to the theory of elastic-plastic materials introduced by MANDEL [15]; V_r, V_φ, V_z are the Eulerian velocity components in coordinates r, φ, z. The material time derivative of an arbitrary physical parameter is calculated from the formula

$$\dot{A}=\left(\frac{\partial}{\partial t}+V_r\frac{\partial}{\partial r}+\frac{V^{\varphi}}{r}\frac{\partial}{\partial\varphi}+V_z\frac{\partial}{\partial z}\right)A. \tag{2.2}$$

(ii) Equations of motion

$$\begin{gathered}
\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\varphi}}{\partial\varphi}+\frac{\partial\sigma_{rz}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\varphi\varphi}}{r}=\varrho\left(\dot{V}_v-\frac{V_{\varphi}^2}{r}\right),\\
\frac{\partial\sigma_{\varphi r}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\varphi\varphi}}{\partial\varphi}+\frac{\partial\sigma_{\varphi z}}{\partial z}+2\frac{\sigma_{r\varphi}}{r}=\frac{\varrho}{r}\frac{d}{dt}(rV_{\varphi}),\\
\frac{\partial\sigma_{zr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{z\varphi}}{\partial\varphi}+\frac{\partial\sigma_{zz}}{\partial z}+\frac{\sigma_{zr}}{r}=\varrho\dot{V},
\end{gathered} \tag{2.3}$$

where σ_{rr}, $\sigma_{\varphi\varphi}\ldots$ are the physical components of Cauchy's stress tensor σ.

(iii) Constitutive equations

$$\dot{\boldsymbol{\tau}}-(\boldsymbol{\omega}+\mathscr{C})\boldsymbol{\tau}+\boldsymbol{\tau}(\boldsymbol{\omega}+\mathscr{C})=\mathbf{LD}-\frac{j}{H}(\overline{\mathbf{m}}\cdot\overline{\mathbf{D}})\overline{\mathbf{m}}. \tag{2.4}$$

Here $\boldsymbol{\tau}=\beta\boldsymbol{\sigma}$ — Kirchhoff's stress tensor, $\mathbf{D}=\frac{1}{2}(\mathrm{grad}\,\mathbf{v}+\mathrm{grad}^T\mathbf{v})$ — strain rate tensor, $\boldsymbol{\omega}=\frac{1}{2}(\mathrm{grad}\mathbf{v}-\mathrm{grad}^T\mathbf{v})$ — spin tensor. Matrix $\mathscr{C}$ has the components

$$\mathscr{C}=\begin{pmatrix} 0 & \frac{V_{\varphi}}{r} & 0\\ -\frac{V_{\varphi}}{r} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{2.5}$$

$L_{ijkl}=\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\left(K^r-\frac{2\mu}{3}\right)\delta_{ij}\delta_{kl}$, $K^r=\beta(K-p)$, $p=-\frac{\sigma_{kk}}{3}$ — mean pressure, K — bulk modulus, μ — Lamé constant.

The Mises–Huber condition is assumed in the form

$$f = \bar{\boldsymbol{\tau}} \cdot \bar{\boldsymbol{\tau}} - 2k^2(\alpha, \theta) = 0, \tag{2.6}$$

where $\dot{\alpha} = \dot{W}^p$ is the power of plastic deformation, and θ temperature.

Function f is normed in the following manner:

$$f = \frac{\bar{\boldsymbol{\tau}} \cdot \bar{\boldsymbol{\tau}}}{\sqrt{8}\,k} - \frac{k}{\sqrt{2}} = 0, \tag{2.7}$$

so that

$$\frac{\partial f}{\partial \bar{\boldsymbol{\tau}}} = \overline{\mathbf{m}}, \quad \mathrm{tr}\,\overline{\mathbf{m}} = 0, \quad \overline{\mathbf{m}} \cdot \overline{\mathbf{m}} = 1, \quad \text{for} \quad f = 0.$$

The hardening function H in isothermal processes has the form

$$H^i = \frac{1}{2\mu}\left(\frac{h}{2\mu} + 1\right) \quad \text{where} \quad h = -\frac{\partial f}{\partial \alpha}(\bar{\boldsymbol{\tau}} \cdot \overline{\mathbf{m}}) \tag{2.8}$$

and in adiabatic processes — the form

$$H^a = \frac{1}{2\mu}\left(\frac{h}{2\mu} + 1 + q_d m_\theta\right), \tag{2.9}$$

where q_d is the thermal coefficient of energy dissipation

$$q_d = \frac{1-\pi}{\varrho_0 C_e}\,\overline{m} \cdot \bar{\boldsymbol{\tau}}, \quad \pi = \varrho_0 \frac{d\varphi(\alpha)}{d\alpha} \tag{2.10}$$

φ — specific stored energy which may be measured experimentally. In most metals π assumes the values from 0.02 to 0.1. C_e — specific heat at constant deformation, $m_\theta = -\frac{1}{2\mu}\frac{\partial f}{\partial \theta}$ — thermal softening coefficient.

Eqs. (2.4) describe both the isothermal and adiabatic processes; the difference between them consists only on the fact that $H^i \neq H^a$, provided certain minor coupling effects are disregarded (heat of elastic deformation and thermal expansion produced by the dissipation energy).

$$j = \begin{cases} 1 & \text{if} \quad f = 0 \quad \text{and} \quad \overline{\mathbf{m}} \cdot \overline{\mathbf{D}} \geqslant 0 \\ 0 & \text{if} \quad f < 0 \quad \text{or} \quad f = 0 \quad \text{and} \quad \overline{\mathbf{m}} \cdot \overline{\mathbf{D}} < 0. \end{cases} \tag{2.11}$$

3. Formulation of the problem

Let us consider a thin-walled, semi-infinite tube made of isotropic elastic-plastic material with isotropic hardening (Fig. 1). Axis z coincides with the axis of the tube, its mean radius being denoted by $\bar{r}$. The tube is loaded at its end by normal pressure and a torque; it is assumed to be thin enough to secure a uniform distribution of stresses along the axis. In view of symmetry of the problem, the stress tensor contains only three non-vanishing physical components

$$\sigma_{z\varphi} = \sigma_{\varphi z} = \tau, \quad \sigma_{zz} = \sigma,$$

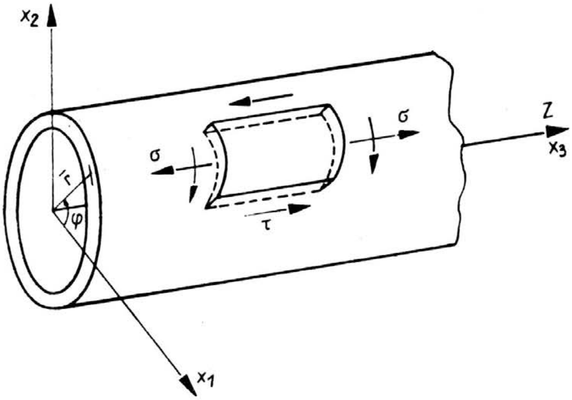


FIG. 1.

σ, τ being functions of z and t. From the assumptions it follows that V_r, V_φ, V_z, β are independent of r, φ and are functions of z and t. On disregarding the radial inertia forces, elimination of V_r from the fundamental equations yields:

$$\dot{\beta} = \beta - \left(-\frac{1}{2\mu}\beta\tau\frac{\partial V_\varphi}{\partial z} + \frac{\partial V_z}{\partial z}\right),$$

$$\frac{\partial \tau}{\partial z} = \varrho\dot{V}\varphi,$$

(3.1)
$$\frac{\partial \sigma}{\partial z} = \varrho\dot{V}_z,$$

$$(\beta\dot{\tau}) - \frac{1}{2}\beta\sigma\frac{\partial V_\varphi}{\partial z} = \mu\frac{\partial V_\varphi}{\partial z} - \frac{j\beta\tau}{6Hk^2}\left[-\beta\sigma\left(-\frac{1}{2\mu}\beta\tau\frac{\partial V_\varphi}{\partial z}\right)\right.$$
$$\left. + 2\beta\sigma\frac{\partial V_z}{\partial z} + 3\beta\tau\frac{\partial V_\varphi}{\gamma z}\right],$$

$$(\beta\dot{\sigma}) + \beta\tau\frac{\partial V_\varphi}{\partial z} = 2\mu\frac{\partial V_z}{\partial z} + \frac{K^r - 2\mu/3}{3K^r}(\beta\dot{\sigma}) - \frac{j\beta\sigma}{9Hk^2}\left[-\beta\sigma\left(-\frac{1}{2\mu}\beta\tau\frac{\partial V_\varphi}{\partial z}\right)\right.$$
$$\left. + 2\beta\sigma\frac{V\partial_z}{\partial z} + 3\beta\tau\frac{\partial V_\varphi}{\partial z}\right],$$

where $\dot{A} = \left(\frac{\partial}{\partial t} + V_z\frac{\partial}{\partial z}\right)A$. The remaining equations are identically satisfied (approximately).

The system (3.1) contains the following unknowns: β, V_φ, V_z, σ, τ, being functions of z and t. It was shown in [16] that in the case of small deformations the system of equations is reduced to the set given by CLIFTON [1], (in isothermal processes), or to the set derived by RANIECKI [19] (in adiabatic processes). In these papers two particular cases were also discussed: pure tension and pure torsion.

4. Simple waves in thin-walled tubes

Simple waves represent a solution depending on z and t through a certain function $\eta(z, t)$:

$$\tau = \tau(\eta), \quad \sigma = \sigma(\eta), \quad V_z = V_z(\eta), \quad V_\varphi = V_\varphi(\eta), \quad \beta = \beta(\eta) \tag{4.1}$$

and, hence, they are constant at surfaces η = const, moving in the medium. It was shown in [16] that simple waves propagate at the same velocity as the transversal waves.

Let us consider the system (3.1) assuming the material to be in the plastic state ($j = 1$). We have then

$$\begin{gathered}
\frac{d\beta}{d\eta}\dot{\eta} = \beta\left(-\frac{1}{2\mu}\beta\tau\frac{dV_\varphi}{d\eta}\eta_{,z} + \frac{dV_z}{d\eta}\eta_{,z}\right),\\
\frac{d\tau}{d\eta}\eta_{,z} = \frac{\varrho_0}{\beta}\frac{dV_\varphi}{d\eta}\dot{\eta},\\
\frac{d\sigma}{d\eta}\eta_{,z} = \frac{\varrho_0}{\beta}\frac{dV_z}{d\eta}\dot{\eta},\\
\left(\beta\frac{d\tau}{d\eta} + \tau\frac{d\beta}{d\eta}\right)\bar{\eta} - \frac{1}{2}\beta\sigma\frac{dV_\varphi}{d\eta}\eta_{,z} = \mu\frac{dV_\varphi}{d\eta}\eta_{,z}\\
- \frac{\beta\tau}{6Hk^2}\left[\beta\tau\left(\frac{\beta\sigma}{2\mu} + 3\right)\frac{dV_\varphi}{d\eta}\eta_{,z} + 2\beta\sigma\frac{dV_z}{d\eta}\eta_{,z}\right],\\
\bar{\alpha}\left(\beta\frac{d\sigma}{d\eta} + \sigma\frac{d\beta}{d\eta}\right)\dot{\eta} + \beta\tau\frac{dV_\varphi}{d\eta}\eta_{,z} = 2\mu\frac{dV_z}{d\eta}\eta_{,z}\\
- \frac{\beta\sigma}{8Hk^2}\left[\beta\tau\left(\frac{\beta\sigma}{2\mu} + 3\right)\frac{dV_\varphi}{d\eta}\eta_{,z} + 2\beta\sigma\frac{dV_z}{d\eta}\eta_{,z}\right]
\end{gathered} \tag{4.2}$$

with the notation

$$\bar{\alpha} = \frac{2(3K^r + \mu)}{9K^r}.$$

Let us multiply Eq. $(4.2)_{2,3}$ by $\dot{\eta}$, and Eq. $(4.2)_{4,5}$ by η_z; substitute the first two equations into Eqs. $(4.2)_4$, and take into account $(4.2)_1$. The system (4.2) is then reduced to the set of two following equations

$$\begin{gathered}
\tau\beta\left(1 + \frac{\beta\sigma}{3Hk^2}\right)V_z^* + \left[\varrho_0\Omega^2 - \frac{\beta^2\tau^2}{2\mu} - \frac{\beta\sigma}{z} - \mu + \frac{\beta^2\tau^2}{6Hk^2}\left(\frac{\beta\sigma}{2\mu} + 3\right)\right]V_\varphi^* = 0,\\
\left(\varrho_0\bar{\alpha}\Omega^2 + \bar{\alpha}\sigma\beta - 2\mu + \frac{2\beta^2\sigma^2}{9Hk^2}\right)V_z^* + \left[-\frac{\bar{\alpha}\beta^2\sigma\tau}{2\mu} + \beta\tau + \frac{\beta^2\sigma\tau}{9Hk^2}\left(\frac{\beta\sigma}{2\mu} + 3\right)\right]V_\varphi^* = 0,
\end{gathered} \tag{4.3}$$

where, in this case, $\Omega = W - v_z$ is the local velocity, and W is the motion velocity of the wave, $V_z^* = \dfrac{dv_z}{d\eta}$, $V_\varphi^* = \dfrac{dv_\varphi}{d\eta}$.

Equation (4.3) makes it possible to determine the velocity of simple waves; the condition of existence of the solution requires the principal determinant to vanish. It may be verified that, in the case of small deformations, Eqs. (4.3) are of the form given by RANIECKI in [19].

In view of considerable complexity of the problem let us now assume that the mean pressure is not very high. This assumption makes it possible to use the ordinary values of elastic coefficients taken from the tables (instead of K^r), what means that the elastic deformations are small. Due to the theory presented here, the voluminal components of plastic deformations have been disregarded so that the material may be assumed to be perfectly incompressible: $\varrho = \varrho_0$, $\beta = 1$. Under such assumptions the systems (3.1) and (4.3) take the following form

$$\frac{\partial \tau}{\partial z} = \varrho \dot{v}, \qquad \frac{\partial \sigma}{\partial z} = \varrho \dot{v}_z,$$

$$(4.4) \qquad \dot{\tau} - \frac{1}{2}\sigma \frac{\partial v_\varphi}{\partial z} = \mu \frac{\partial v_\varphi}{\partial z} - \frac{j\tau}{2Hk^2}\left(\sigma \frac{\partial v_z}{\partial z} + \tau \frac{\partial v_\varphi}{\partial z}\right),$$

$$\dot{\sigma} + \tau \frac{\partial v_\varphi}{\partial z} = 2\mu \frac{\partial v_z}{\partial z} + \frac{K - 2\mu/3}{3K}\dot{\sigma} - \frac{j\sigma}{3Hk^2}\left(\sigma \frac{\partial v_z}{\partial z} + \tau \frac{\partial v_\varphi}{\partial z}\right).$$

Here ϱ, μ, K are the material constants,

$$\dot{\tau} = \frac{\partial \tau}{\partial t} + v_z \frac{\partial \tau}{\partial z}, \qquad \dot{\sigma} = \frac{\partial \sigma}{\partial t} + v_z \frac{\partial \sigma}{\partial z},$$

τ, σ, v_φ, v_z are functions of z and t, and

$$(4.5) \qquad \begin{aligned} &\frac{\tau\sigma}{2Hk^2} v_z^* + \left(\varrho \Omega^2 - \mu - \frac{\sigma}{2} + \frac{\tau^2}{2Hk^2}\right) v_\varphi^* = 0, \\ &\left(\varrho \Omega^2 - E + \frac{E\sigma^2}{6\mu Hk^2}\right) v_z^* + \frac{E\tau}{2\mu}\left(1 + \frac{\sigma}{3Hk^2}\right) v_\varphi^* = 0. \end{aligned}$$

In the case of elastic deformations $H \to \infty$ we obtain the elastic wave velocities

$$\varrho \Omega^2 = \mu + \frac{\sigma}{2} = \mu\left(1 + \frac{\sigma}{2\mu}\right) \approx \mu, \qquad \varrho \Omega^2 = E, \qquad (E \text{ — Young's modulus}).$$

Velocities of simple waves may be found by equating the determinant of Eqs. (4.5) to zero,

$$(4.6) \qquad (X - E)\left(X - \mu - \frac{\sigma}{2}\right) + X\left(\frac{3\mu\tau^2 + E\sigma^2}{6\mu Hk^2}\right) - \frac{6E\mu\tau^2 + 2\left(\mu + \frac{\sigma}{2}\right)E\sigma^2 + 3E\sigma\tau^2}{12\mu Hk^2} = 0,$$

where $\varrho \Omega^2 = X$.

The above equation may be written in the form

$$(4.7) \qquad \begin{aligned} &Y_1 = Y_2, \\ &Y_1 = (X - E)\left(X - \mu - \frac{\sigma}{2}\right), \\ &Y_2 = A(X_0 - X), \end{aligned}$$

where

(4.8)
$$A = \frac{\tau^2+\sigma^2}{2Hk^2} > 0,$$

(4.9)
$$X_0 = \frac{3k^2\left(\mu+\frac{\sigma}{2}\right)}{\tau^2+\sigma^2}.$$

In the last two expressions the relation $E = 3\mu$ (incompressible material) is taken into account.

It may be shown that for stresses much smaller than the elasticity modulus μ, the inequality holds

(4.10)
$$\frac{\sigma}{2}+\mu \leqslant X_0 \leqslant E = 3\mu.$$

The geometric interpretation of Eq. (4.6) is shown in Fig. 2. It is seen that the simple waves may be propagated at two possible velocities:

$$\varrho\Omega_f^2 = X_f \quad \text{(fast waves)},$$
$$\varrho\Omega_s^2 = X_s \quad \text{(slow waves)},$$

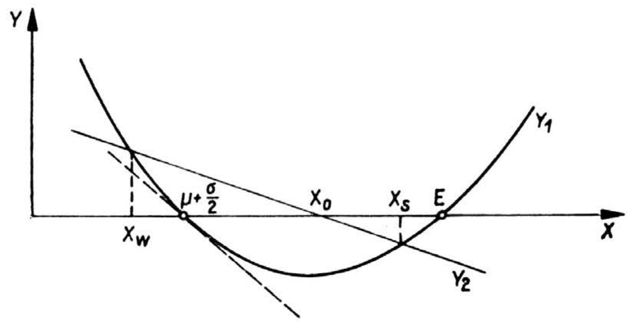


FIG. 2.

which satisfy the inequality

(4.11)
$$\Omega_s \leqslant \Omega_2^e \leqslant \Omega_f \leqslant \Omega_1^e.$$

Here $\varrho\Omega_1^{e^2} = E$ and $\varrho\Omega_2^{e^2} = \mu+\frac{\sigma}{2}$. In Fig. 2 are shown the graphs of functions Y_1, Y_2 satisfying the conditions (4.7), (4.8), (4.9). Point X_0 lies in the interval $\left[\mu+\frac{\sigma}{z}, E\right]$.

The equation of simple waves velocities (4.6) may be solved in certain particular cases (for instance, in the cases of pure shear and torsion), [16].

Analysis of Eq. (4.6) and relations (4.7)–(4.9) leads to the following conclusions:

a) Velocity of fast waves is equal to that of the longitudinal elastic waves if and only if $\sigma = 0$. The sufficient condition holds true if the ratio of the stress to the Lamé constants μ is much less than unity, while the necessary condition is always true.

b) The velocity of fast waves is equal to that of the transversal elastic waves if and only if $\tau = 0$ and $|\sigma| \geqslant \hat{\sigma}$, the value of $\hat{\sigma}$ being found from the equation

$$2\mu H\Big|_{\substack{\tau=0\\ \sigma=\hat{\sigma}}} = \frac{E}{E-\left(\mu+\frac{\sigma}{2}\right)}. \tag{4.12}$$

c) The velocity of slow waves is equal to that of the transversal elastic waves if and only if $\tau = 0$ and $|\sigma| < \hat{\sigma}$.

The proof of (a) is based on the analysis of the condition $X_0 = E$, and the proofs of (b) and (c) — on the relation $X_0 = \mu + \frac{\sigma}{z}$ and on the comparison of slopes of the line $Y_2 = 0$ with the slopes of the line tangent to curve $Y_1 = 0$ at point $\mu + \frac{\sigma}{2}$.

5. Stress trajectories

Let us consider the possible stress trajectories in the plane $\sigma - \tau$ connected with simple waves; multiply Eq. $(4.5)_1$ by σ, Eq. $(4.5)_2$ by τ and sum the results up. After simple transformations we obtain

$$\frac{d\sigma}{d\tau} = \frac{\left(\varrho\Omega^2 - \mu - \frac{\sigma}{2} - \frac{3\tau^2}{2\sigma}\right)\sigma}{(\varrho\Omega^2 - E)\tau}, \tag{5.1}$$

where, for an incompressible material, $E = 3\mu$. Equation (5.1) differs from the equation of the stress trajectory at small deformations by the last two terms in parentheses of the numerator. The velocity of simple waves $\varrho\Omega^2 = X$ is obtained by solving the Eq. (4.6),

$$X_{1,2} = \frac{4\mu + \frac{\sigma}{2} - \frac{\sigma^2+\tau^2}{2Hk^2} \pm \sqrt{\Delta}}{2}, \tag{5.2}$$

where

$$\Delta = \left(4\mu + \frac{\sigma}{2} - \frac{\sigma^2+\tau^2}{2Hk^2}\right)^2 - 12\left(\mu+\frac{\sigma}{2}\right)\left(\mu - \frac{1}{2H}\right), \tag{5.3}$$

$$X_1 = \varrho\Omega_f^2, \quad X_2 = \varrho\Omega_s^2.$$

It is easily seen from Eq. (5.1) that in the region $\sigma > 0$, $\tau > 0$, the angle of inclination of the stress trajectory of fast waves is negative; in the case of slow waves, this angle is positive. In contrast to the case of small deformation, orthogonality of the stress trajectories cannot be established here.

6. Numerical example

As an example, let us consider simple waves propagating in a thin-walled tube made of pure zirconium. This metal was tested by KEELER [11]. The corresponding stress-strain curves were obtained for pure, annealed zirconium tested at temperatures from $-195°C$

to 370°C and velocities ranging from 0.005 cm/min to 5 cm/min. The following function describes the relation between the actual stress σ, logarithmic plastic strain ε^p and temperature θ°C = θ°K − 273 (in tension):

$$\sigma = C_1(1-a\theta)(\varepsilon P+b_1)^x. \tag{6.1}$$

Relation (6.1) describes fairly well the results obtained by Keeler in the range of temperatures from −195°C to 25°C and from 200°C to 300°C. The following values of constants C_1, a, b_1, x are obtained:

(i) In the range from −195°C to 25°C:

$$b = 0.01, \qquad x = 0.25,$$
$$a = 2.8\cdot 10^{-3}\ 1/°\mathrm{C}, \qquad C_1 = 4.76\cdot 10^8 \mathrm{N/m^2}.$$

(ii) In the range from 200°C to 370°C:

$$b = 0.01, \qquad x = 0.25,$$
$$a = 8.37\cdot 10^{4}\ 1/°\mathrm{C}, \qquad C_1 = 3.95\cdot 10^8 \mathrm{N/m^2}.$$

Moreover, we obtain for zirconium

$$\varrho = 6440\ \mathrm{kg/m^3}, \qquad \mu = 3266{,}73\cdot 10^7 \mathrm{N/m^2},$$
$$C_e = 284.84\ \mathrm{J/kg°C}.$$

In Keeler's experiments the temperature was controlled and kept constant, so that the relation between the hardening parameter α, stress σ and temperature θ may be evaluated from the equation

$$\alpha = \int_{\sigma_Y}^{\sigma} \frac{\partial \varepsilon^e(\sigma_1, \theta)}{\partial \sigma_1}\sigma_1 d\sigma_1, \tag{6.2}$$

where $\sigma_Y = C_1(1-a\theta)b_1^x$ is the initial yield limit at $\varepsilon^p = 0$ and $\varepsilon^p = \varepsilon^p(\sigma, \theta)$ is the solution of Eq. (6.1) with respect to ε^p. Let us integrate Eq. (6.2); we obtain

$$\alpha = \frac{1}{(x+1)^x \sqrt{C_1(1-a\theta)}}\left(\sigma^{\frac{x+1}{x}} - \sigma_Y^{\frac{x+1}{x}}\right). \tag{6.3}$$

On the other hand, during the process of loading the temperature equation assumes the form

$$\dot{\theta} = \frac{q_d}{2\mu H}(\overline{\mathbf{m}}\cdot\overline{\mathbf{D}}), \qquad q_d = \frac{1-\pi}{\varrho_0 c_e}(\overline{\mathbf{m}}\cdot\overline{\boldsymbol{\tau}}).$$

On comparing this result with the equation of evolution we obtain

$$\dot{\theta} = \frac{1-\pi(\alpha)}{\varrho_0 c_e}\dot{\alpha}. \tag{6.4}$$

Assuming that $\pi = 0.1$ we arrive at the result

$$\theta - \theta_0 = \frac{1-\pi}{\varrho_0 c_e}\alpha. \tag{6.5}$$

Hence, in the interval of temperatures from $-195°C$ to $25°C$

$$\theta+195 = \frac{0.9}{\varrho_0 c_e}\alpha \tag{6.6}$$

and for temperatures from the interval (200°C, 370°C) $\theta_0 = 200°C$ and

$$\theta-200 = \frac{0.9}{\varrho_0 c_e}\alpha. \tag{6.7}$$

The set of Eqs. (6.3), (6.6) or (6.7) may be solved for σ and θ; substitution of $k\sqrt{3}$ for σ yields the function $k = k(\alpha, \theta)$. In the adiabatic process, the hardening function may thus be calculated from the formula

$$H^a = \frac{1}{2\mu}\left(\frac{h}{2\mu}+1+q_a m_\theta\right) = \frac{1}{2\mu}\left(\frac{k}{\mu}\frac{\partial k}{\partial \alpha}+1+\frac{k}{\mu}\frac{1-\pi}{\varrho_0 c_e}\frac{\partial k}{\partial \theta}\right). \tag{6.8}$$

In the isothermal process it is assumed that $\theta = \theta_0$ and $\sigma = k\sqrt{3}$. From Eq. (6.3) it follows that $k = k(\alpha)$, so that the hardening function H^i is of the form

$$H^i = \frac{1}{2\mu}\left(\frac{k}{\mu}\frac{\partial k}{\partial \alpha}+1\right). \tag{6.9}$$

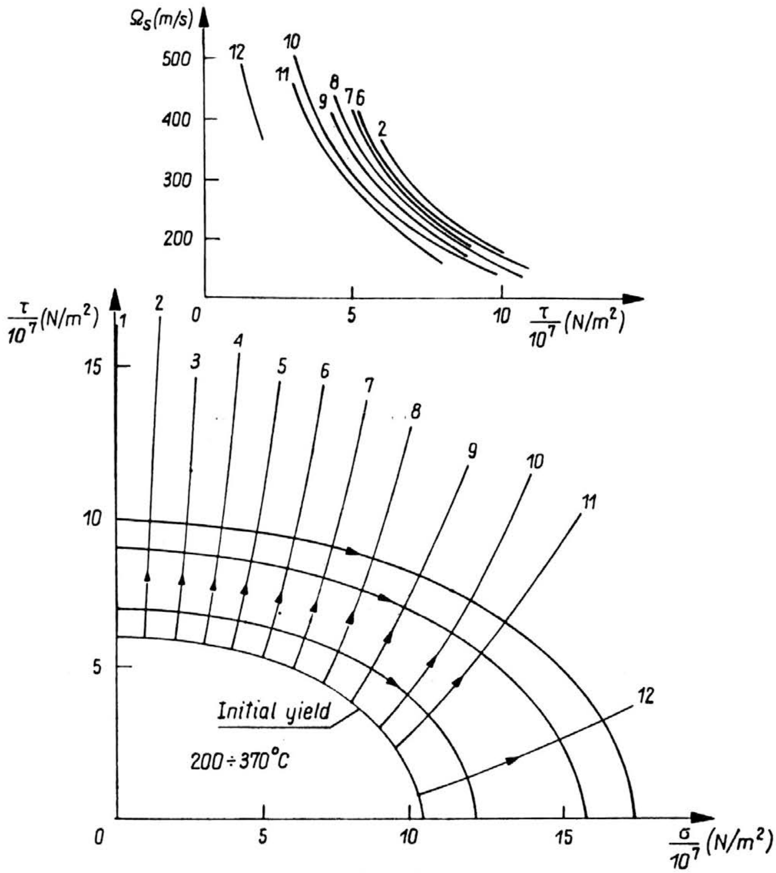


FIG. 3.

The results of calculations made according to the Runge–Kutta method are shown in graphical form. In Figs. 3 and 4 are plotted the stress trajectories and the velocities of slow waves as functions of the stress component τ for the isothermal process at $\theta_0 = 200°C$ and $\theta_0 = -195°C$; similar diagrams for an adiabatic process and temperatures ranging from 200°C to 370°C and from $-195°C$ to 25°C are shown in Figs. 5 and 6. The ratios of velocities of slow waves taking place in both the processes and in two temperature ranges are demonstrated in Figs. 7 and 8.

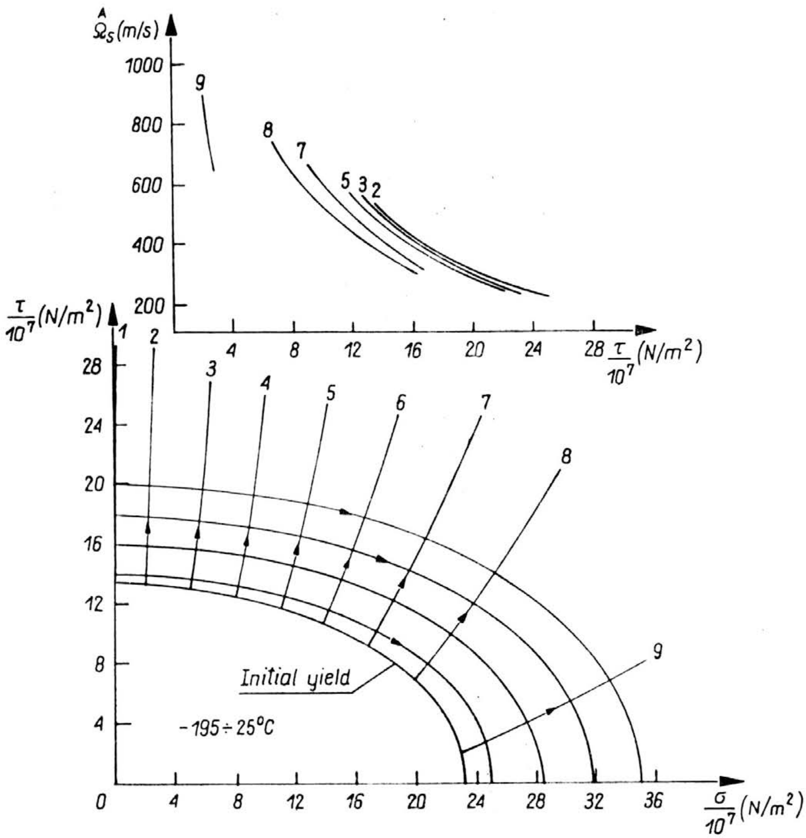


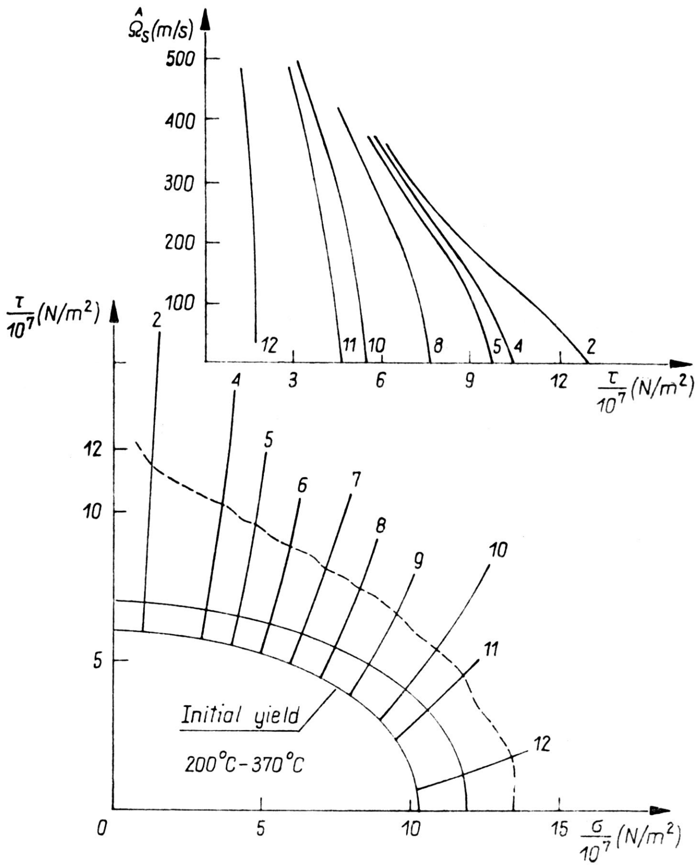
FIG. 4.

Analysis of these processes makes it possible to draw the following conclusions.

(a) No substantial differences may be observed between the stress trajectories in the adiabatic and isothermal processes. Also the velocities of fast waves in both the processes are almost the same.

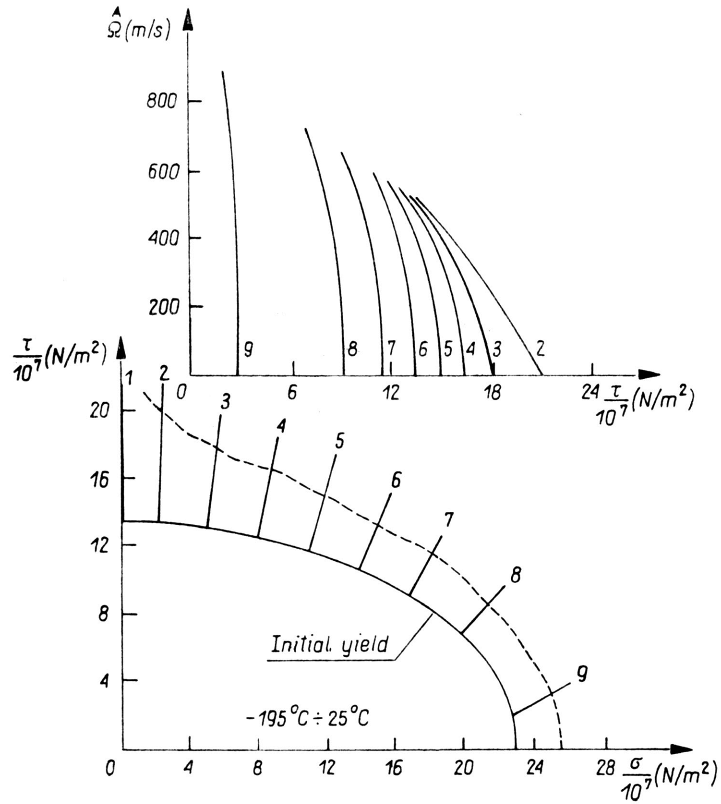
(b) From the previous analysis it followed that the stress trajectories of slow and fast waves were not orthogonal. This fact is not observed from the graphs.

(c) The velocities of fast waves decrease in the directions indicated in Figs. 3 and 4 from the value of the longitudinal elastic wave velocity (at the τ-axis) to the value of the transversal elastic wave velocity (at the σ-axis). In the present case, the latter velocity



$\hat{\Omega}_s$(m/s)
500
400
300
200
100
12 11 10 8 5 4 2
3 6 9 12
$\frac{\tau}{10^7}$(N/m^2)
$\frac{\tau}{10^7}$(N/m^2)
2 4 5 6 7 8 9 10 11 12
12
10
5
Initial yield
200°C–370°C
0 5 10 15
$\frac{\sigma}{10^7}$(N/m^2)

FIG. 5.



$\hat{\Omega}$(m/s)
800
600
400
200
9 8 7 6 5 4 3 2
0 6 12 18 24
$\frac{\tau}{10^7}$(N/m^2)
$\frac{\tau}{10^7}$(N/m^2)
1 2 3 4 5 6 7 8 9
20
16
12
8
4
Initial yield
−195°C ÷ 25°C
0 4 8 12 16 20 24 28
$\frac{\sigma}{10^7}$(N/m^2)

FIG. 6.

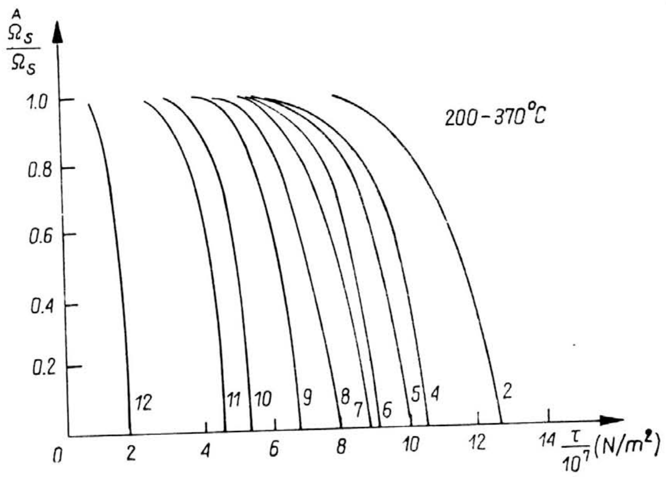


FIG. 7.

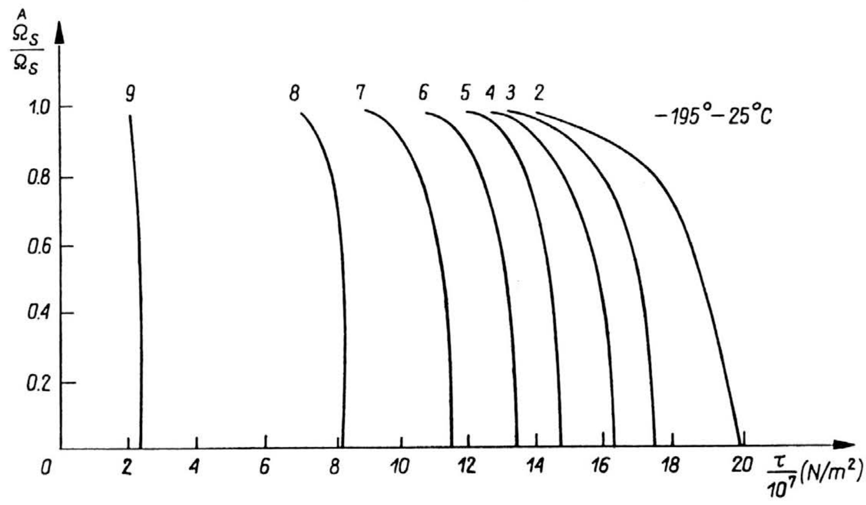


FIG. 8.

assumes the value of $\sqrt{\left(\mu+\frac{\sigma}{2}\right)\Big/\varrho}$ instead of $\sqrt{\mu/\varrho}$, like in the case of small deformations. This difference, however, is also small.

(d) In the case considered here, the root of Eq. (4.12) does not exist, and condition $\tau = 0$, $|\sigma| \geqslant \hat{\sigma}$ (cf. Eq. (4.12)) is fulfilled for each $|\sigma| \geqslant \sigma_Y$, σ_Y denoting the initial yield stress at tension.

(e) The velocity of slow waves decreases in the direction shown in Figs. 3 and 4.

(f) A considerable difference is observed between the velocities of slow waves in the adiabatic and isothermal processes. The slow wave velocity in adiabatic processes tends rapidly to zero at relatively small stresses, what means that the energy dissipation at finite deformations exerts a considerable influence on the process. It is seen from Eqs. (5.2), (5.3) that $\Omega_s = 0$ if

$$\mu = \frac{1}{2H^a}$$

and so from Eq. (2.9) it follows that the term $q_\alpha m_\theta$ is then equal to $-h/2\,\mu$.

Dashed lines in Figs. 5, 6 denote the loci of points at which the velocities of adiabatic slow waves are zero.

Similar conclusions were found to be true in the case of small deformations in [19] (except for (b) and (f)). The method presented here cannot be used in the cases of arbitrary boundary conditions; it may be applied only in the case of constant stresses at the boundary of the tube. The method of solution of the initial-boundary-value problem was discussed by CLIFTON [1].

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