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On stationary unilateral problems

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THE OBJECTIVE of the paper is twofold. Firstly, using the methods of nonstandard analysis, we prove that the general form of stationary problems with unilateral (explicit) constraints imposed on displacements, strains and stresses can be derived from a certain class of nonstandard potential unconstrained problems. Secondly, using the theory of maximal monotone operators, we formulate some existence results for problems with unilateral constraints imposed exclusively on: 1) displacements and stresses, 2) strains and stresses, 3) strains.

Praca ma dwojaki cel. Po pierwsze, stosując metody analizy niestandardowej wykazujemy, że ogólną postać zagadnień stacjonarnych z więzami jednostronnymi dla przemieszczeń, odkształceń i naprężeń można otrzymać z pewnej klasy niestandardowych zagadnień potencjalnych bez więzów. Po drugie, korzystając z teorii operatorów maksymalnie monotonicznych, formułujemy pewne twierdzenia o istnieniu dla problemów z więzami jednostronnymi nałożonymi wyłącznie na: 1) przemieszczenia i naprężenia, 2) odkształcenia i naprężenia, 3) odkształcenia.

Работа имеет двоякую цель. Во-первых, применяя методы нестандартного анализа, показываем, что общий вид стационарных задач с односторонними связями для перемещений, деформаций и напряжений можно получить из некоторого класса нестандартных потенциальных задач без связей. Во-вторых, используя теорию максимально монотонных операторов, формулируем некоторые теоремы существования для проблем с односторонними связями, наложенными исключительно на: 1) перемещения и напряжения, 2) деформации и напряжения, 3) деформации.

1. Nonstandard methods in rational mechanics

METHODS of nonstandard analysis (nonstandard methods), introduced by A. ROBINSON [1, 2], and then developed by many authors [3–7], are based on the fact that for every mathematical (full) structure M there exists another structure $^*M$ which is an enlargement of M. The relevant properties of $^*M$ are:

1) every statement k which is meaningful and true for M is also meaningful and true for $^*M$ (statement k has to be expressed in a certain formal language),

2) every entity a of M extends uniquely and naturally to a certain "standard" entity $^*a$ of $^*M$,

3) if $\varrho \in {}^*M$, then $x \in \varrho$ implies $x \in {}^*M$,

4) if a, $a \in M$, is an infinite set, then set $^*(a) := \{^*x | x \in a\}$ (set of all standard entities belonging to $^*a$) is not an element of $^*M$,

5) if a, $a \in M$, is an infinite set, and only in this case, does $^*a$ contains "nonstandard" elements: $^*a \setminus {}^*(a) \neq \phi$.

In what follows we shall always assume that $R \in M$, R being the set of real numbers. From the forementioned properties of $^*M$ it follows that $^*R$ contains standard real numbers (elements of $^*(R)$) as well as nonstandard real numbers (elements of $^*R \setminus {}^*(R)$).

It can be also shown that $^*R$ constitutes a non-Archimedean ordered field and hence it contains numbers which are infinitely small and infinitely large. The existence of such numbers makes it possible to describe situations in mechanics in which objects or phenomena of the "different order" in magnitude ("micro" or "macro" objects) have to be taken into account. The nonstandard description of some problems in mechanics is desirable from a heuristic point of view mainly by the avoidance of passages to a limit at different stages.

Let T be any formalized theory in rational mechanics having as a model a certain mathematical structure M. Structure $^*M$ constitutes what will be called the nonstandard model of T. If A, $A \in M$, stands for (an infinite) set of governing relations of T in M, then $^*A$ is a set of governing relations of T in $^*M$. The main feature of $^*A$ (provided that A is infinite) is that apart from "standard" governing relations it also contains "nonstandard" governing relations. Some relations from the "nonstandard" governing ones can be used in order to describe certain new physical situations. Thus the key point of the procedure is to select from $^*A$ a subset B of nonstandard relations which describes in $^*M$ the new physical situation we are to investigate. We shall confine ourselves to the cases in which for every $\varrho \in B$ we have $\varrho \subset {}^*Z_1 \times \ldots \times {}^*Z_n$ where every Z_i, $i = 1, \ldots, n$, is a certain topological space and $^*Z_i$ is its nonstandard extension. The choice of spaces Z_i (such that $\varrho \subset {}^*Z_1 \times \ldots \times {}^*Z_n$ holds for every $\varrho \in B$) depends on the character of the problem under consideration. Moreover, let $\mu_i(^*z_i)$, $z_i \in Z_i$, stand for a topological monad of a standard point $^*z_i$, $^*z_i \in {}^*Z_i$ (if Z_i is a metric space, then $\mu_i(^*z_i)$ is a set of all points in $^*Z_i$ which are infinitely close to $^*z_i$). To every $\varrho \in B$ we can now assign the relation ${}^{(Z)}\varrho$, ${}^{(Z)}\varrho \in M$, defined by

$$(1.1) \qquad {}^{(Z)}\varrho := \{(z_1, \ldots, z_n) \in Z_1 \times \ldots \times Z_n | \mu_i(^*z_i) \cap \mathrm{dom}_i \varrho \neq \phi, \quad i = 1, \ldots, n\}$$

which is called the shadow of ϱ, $\varrho \in B$, in $Z \equiv Z_1 \times \ldots \times Z_n$, [7, 8]. We shall also assume that every $\varrho \in B$ has in $Z \equiv Z_1 \times \ldots \times Z_n$ a nonempty shadow ${}^{(Z)}\varrho$. This shadow can be interpreted, from the physical point of view, as a certain "macro" description of a situation which in $^*M$ is described by the relation $\varrho \in B$. Putting ${}^{(Z)}B := \{{}^{(Z)}\varrho | \varrho \in B\}$ we shall interpret ${}^{(Z)}\varrho$ as a set of governing relations in M of a certain "new" theory in rational mechanics. The forementioned line of approach is given by the scheme

$$(1.2) \qquad (M \in A) \to ({}^*M \in {}^*A) \to ({}^*A \supset \supset B) \to (M \in {}^{(Z)}B),$$

in which the specification B from $^*A$ and the choice of $Z \equiv Z_1 \times \ldots \times Z_n$ play the most important role, while the set A is assumed to be known. For particulars the reader may consult [8].

2. Nonstandard approach to stationary unilateral problems

2.1. Stationary bilateral problems

Let Ω be the known regular region in R^3 occupied by the body in the reference configuration, $u: \Omega \to R^3$ be the displacement field and $E: \Omega \to R^{(3\times 3)}$ be the strain tensor field ($R^{(3\times 3)}$ stands for the set of all symmetric real 3×3 matrices). Fields u and E will

be treated as elements of certain linear topological spaces V and Y, respectively. By L: $L:V \to Y$ we define a linear mapping, such that $Lu(x)$ stands for a symmetric part of the gradient of u at x, $x \in X$, and by $L^*:Y' \to V'$ we define the conjugate of L, where $(V, \langle , \rangle, V')$ and $(Y, [,], Y')$ are the suitable dual pairings. The external and internal forces will be represented by the linear functionals $f \in V'$ and $S \in Y'$, respectively. In what follows we shall confine ourselves to the stationary problems of the small deformation gradient theory only. Hence the strain field will be related to the displacement field by

$$E = Lu \tag{2.1}$$

and the equilibrium conditions will be assumed in the form

$$L^*S = f. \tag{2.2}$$

In this Subsection we shall confine ourselves to the problems in mechanics in which there are known two differentiable convex potentials $\pi:V \to R$, $\alpha:Y \to R$, such that the following constitutive equations hold:

$$f = -\pi'(u), \quad S = \alpha'(E), \tag{2.3}$$

where π', α' are Gateaux derivatives of π, α, respectively. Every problem under consideration can be stated as follows: for the known potentials α, π, find $(u, E, f, S) \in \in V \times Y \times V' \times Y'$, such that Eqs. (2.1)–(2.3) hold. Let us observe that all the restrictions which a priori are imposed on (u, E, f, S) (and which are called constraints) are determined by the character of the linear spaces V, Y. Thus we deal here with the special kind of constraints which are referred to as the bilateral constraints. Analogously, the problems under consideration can be referred to as stationary (potential) bilateral problems (1).

In the general case, stationary problems in solid mechanics (treated within the small deformation gradient theory) are determined by the form of constitutive relations; in what follows we shall deal with the internal constitutive relations (which will be represented by subsets in $Y \times Y'$) and external constitutive relations (represented by subsets in $V \times V'$).

2.2. Nonstandard approach to the unilateral internal constraints

In many stationary problems in solid mechanics the internal constitutive relations can be postulated in the form, [9].

$$S \in S_0 + \partial\chi_\Gamma(E), \quad E \in E_0 + \partial\chi_\Sigma(S_0), \quad E_0 = \sigma_0'(S_0), \tag{2.4}$$

where Γ and Σ are the known nonempty closed convex subsets of Y and Y', respectively, and $\sigma_0:Y' \to R$ is the known differentiable function; here and in what follows χ stands for the indicator function of subset M in a certain linear topological space X and $\partial\varphi(x)$ is a subdifferential of function $\varphi:X \to \bar{R}$ defined on such a space. If Γ or Σ are not the subspaces of Y or Y', respectively, then the restrictions of the form $E \in \Gamma$, $S_0 \in \Sigma$, which are implied by the relations (2.4), are referred to as unilateral (convex) constraints.

The aim of this Subsection is to prove the following:

(1) It must be stressed that a stationary bilateral problem describes a certain situation in solid mechanics only if the spaces V, Y as well as the potentials α, τ are properly chosen.

PROPOSITION 2.1. Every internal constitutive relation of the form (2.4) coincides with the shadow in $Y\times Y'$ of a certain constitutive relation having the form $(2.3)_2$ with $\alpha\in {}^*A$, A being the set of all constitutive potentials $\alpha: Y\to R$.

The physical sense of Proposition 2.1 is that the constitutive relations with the unilateral internal constraints can be derived, using the nonstandard line of approach (1.2), from the constitutive equations with bilateral internal constraints.

In order to prove Proposition 2.1, we have to assume that there are known:

1) the differentiable convex potential $\sigma_0: Y'\to R$,

2) the closed, convex and nonempty subsets Γ, Σ of Y, Y', respectively.

It follows from the fact that the triple $(\sigma_0, \Gamma, \Sigma)$ uniquely determines the constitutive relation (2.4) as a set of pairs $(E, S)\in Y\times Y'$ satisfying the relation (2.4) for some $(E_0, S_0)\in$ $\in Y\times Y'$. Let $\gamma: {}^*Y\to {}^*R$ and $\sigma_R: {}^*Y'\to {}^*R$ be internal, convex and differentiable functions such that [2]

$$\gamma(E)\in\begin{cases}\{0\} & \text{if} \quad E\in {}^*\Gamma,\\ {}^*R\setminus\mu(R) & \text{if} \quad E\in {}^*Y\setminus\mu(\Gamma),\end{cases}\qquad \sigma_R(S)\in\begin{cases}\{0\} & \text{if} \quad S\in {}^*\Sigma,\\ {}^*R\setminus\mu(R) & \text{if} \quad S\in {}^*Y'\setminus\mu(\Sigma),\end{cases} \tag{2.5}$$

and such that function $(\sigma_0+\sigma_R)^*: {}^*Y\to {}^*R$, which is polar to $(\sigma_0+\sigma_R): {}^*Y'\to {}^*R$, is differentiable [3]. Now define

$$\alpha(E) = [(\sigma_0+\sigma_R)^*+\gamma](E), \quad E\in {}^*Y, \tag{2.6}$$

and

$$S_0 \equiv [(\sigma_0+\sigma_R)^*]'(E), \quad E\in {}^*Y. \tag{2.7}$$

Hence we obtain

$$\alpha'(E) = S_0+\gamma'(E), \quad E = \sigma_0'(S_0)+\sigma_R'(S_0). \tag{2.8}$$

Using the relations (2.8) we see that the (nonstandard) constitutive equation $S = \alpha(E)$, with $\alpha(\cdot)$ given by Eq. (2.6), is equivalent to the system of standard equations

$$S = S_0+S_R, \quad E = E_0+E_R, \quad E_0 = \sigma_0'(S_0) \tag{2.9}$$

and (in general nonstandard) two binary relations in ${}^*Y\times {}^*Y'$:

$$S_R = \gamma'(E), \quad E_R = \sigma_R'(S_0). \tag{2.10}$$

Taking into account the postulated form of $\gamma(\cdot)$ and $\sigma_R(\cdot)$, given by Eq. (2.5), it is easy to conclude that the shadows in $Y\times Y'$ of the relations $(2.10)_1$ and $(2.10)_2$ are given by

$$S_R\in\partial\chi_\Gamma(E), \quad E_R\in\partial\chi_\Sigma(S_0), \tag{2.11}$$

respectively. Now combining Eqs. (2.9) and (2.11) we arrive at the internal constitutive relation (2.4), which ends the proof of Proposition (2.1).

Let us observe that the passage from potential constitutive equation of the form $(2.3)_2$ (with the bilateral constraints only) to the constitutive relations of the form (2.4) (which can involve also unilateral constraints) which was outlined above, is an example of the

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[2] We define $\mu(K)\equiv\bigcup\mu(a)$, $a\in K$, for any subset K of ${}^*X$ where X is an arbitrary topological space K is an arbitrary set of standard points in ${}^*X$, and where $\mu(a)$ is a topological monad of any $a\in K$.

[3] For the existence of the functions φ, σ_R, cf. [19].

nonstandard procedure given on scheme (1.2). The key point of the procedure is to specify a proper form (2.6) of the (nonstandard) constitutive potentials α, $\alpha \in {}^*A$, A being the set of all constitutive potentials which determines the constitutive equations $(2.3)_2$.

2.3. Nonstandard approach to the unilateral external constraints

There exist stationary problems in solid mechanics in which the external constitutive relations (cf. Sect. 2.2) are postulated in the form [9],

$$f \in f_0 - \partial\chi_{\Xi}(u), \quad \mu \in \mu_0 - \partial\chi_{\Delta}(f_0), \quad u_0 = -\gamma_0'(f_0), \tag{2.12}$$

where Ξ and Δ are the known nonempty closed convex subsets in V and V', respectively, and $\delta_0 : V' \to R$ is the known differentiable function. If Ξ or Δ are not the subspaces of V or V', respectively, then the formulas (2.12) imply the restrictions of the form: $u \in \Xi$, $f_0 \in \Delta$, which are called unilateral external (convex) constraints. In this Subsection we are to show that the constitutive relations (2.12) (which can involve unilateral constraints) can be obtained from the constitutive equations $(2.3)_1$ (which can involve bilateral constraints only) by applying the nonstandard line of approach (1.2). To this aid we shall prove the following:

PROPOSITION 2.2. Every external constitutive relation of the form (2.12) coincides with the shadow in $V \times V'$ of a certain constitutive relation having the form $(2.3)_1$ with $\pi \in {}^*P$, P being the set of all constitutive potentials $\pi : V \to R$ of the external forces.

The proof of the Proposition (2.2) is similar to that of Proposition 2.2. Firstly we assume that there is known an arbitrary but fixed triple (δ_0, Ξ, Δ) which uniquely determines the constitutive relation (2.12). Then we introduce the internal, convex and differentiable functions $\xi : {}^*V \to {}^*R$ and $\delta_R : {}^*V' \to {}^*R$, such that

$$\xi(u) \in \begin{cases} \{0\} & \text{if} \quad u \in {}^*\Xi, \\ {}^*R_+ \setminus \mu(R) & \text{if} \quad u \in {}^*V \setminus \mu(\Xi), \end{cases} \qquad \delta_R(f) \in \begin{cases} \{0\} & \text{if} \quad f \in {}^*\Delta, \\ {}^*R \setminus \mu(R) & \text{if} \quad f \in {}^*V' \setminus \mu(\Delta), \end{cases}$$

and that function $(-\delta_0 - \delta_R)^* : {}^*V \to {}^*R$, which is polar to $(-\delta_0 - \delta_R') : {}^*V' \to {}^*R$, is differentiable (4). Now putting

$$\begin{aligned} \pi(u) &\equiv [-(-\delta_0 - \delta_R)^* + \xi](u), \quad u \in {}^*V, \\ f_0 &\equiv [(-\delta_0 - \delta_R)^*]'(u), \quad u \in {}^*V, \end{aligned} \tag{2.13}$$

we can easily conclude that the (nonstandard) constitutive equation $f = -\pi'(u)$ is equivalent to the standard equations

$$f = f_0 + f_R, \quad u = u_0 + u_R, \quad u_0 = -\delta_0'(f_0), \tag{2.14}$$

and, in general, to the nonstandard equations

$$f_R = -\xi'(u), \quad u_R = -\delta_R'(f_0). \tag{2.15}$$

Taking into account the postulated form of $\xi(\cdot)$ and $\delta_R(\cdot)$, we conclude that the relations $(2.15)_{1,2}$ (which are binary relations in ${}^*V \times {}^*V'$) have in $V \times V'$ their shadows in the form

$$f_R \in -\partial\chi_{\Xi}(u), \quad u_R \in -\partial\chi_{\Delta}(f_0). \tag{2.16}$$

(4) For the existence of functions ξ, δ_R, cf. [19].

Combining together Eqs. (2.14) and (2.16) we arrive at the constitutive relation (2.12) which ends the proof of Proposition 2.2. Thus the general conclusion is that using the nonstandard line of approach, discussed in Sect. 1, we can easily pass from the well-known equations $(2.3)_1$ (which determine the potential external forces) to the rather complicated relations (2.12) which can involve unilateral constraints.

2.4. Stationary unilateral problems

Starting from physical considerations, from now on we assume that

$$\sigma_0(S) = \frac{1}{2}[KS, S], \quad \delta_0(f) = \frac{1}{2} < Gf, f >,$$

where $K: Y' \to Y$, $G: V' \to V$ are the known linear operators. Hence $E_0 = KS_0$, $u_0 = -Gf_0$ and the formulas (2.1), (2.2), (2.4) and (2.12) yield the following system of four variational inequalities:

$$\begin{aligned} L^*S - f_0 &\in -\partial\chi_{\Xi}(u), \\ S - S_0 &\in \partial\chi_{\Gamma}(Lu), \\ Lu - KS_0 &\in \partial\chi_{\Sigma}(S_0), \\ u - G(-f_0) &\in -\partial\chi_{\Delta}(f_0). \end{aligned} \tag{2.17}$$

In what follows we shall also confine ourselves to the case of the "dead" external forces, putting $\Delta = \{l\}$, where l is the known element of V'. Then $\partial\chi_{\Delta}(f_0) = V$ and the system (2.17) reduces to

$$\begin{aligned} L^*S - l &\in -\partial\chi_{\Xi}(u), \\ S - S_0 &\in \partial\chi_{\Gamma}(Lu), \\ Lu - KS_0 &\in \partial\chi_{\Sigma}(S_0), \end{aligned} \tag{2.18}$$

and the stationary problem of solid mechanics with (convex) constraints imposed on displacement, strain and stress fields can be stated as follows: for the known linear operator $K: Y' \to Y$ and the known subsets Ξ, Γ, Σ in V, Y, Y', respectively, find $(u, S, S_0) \in \in V \times Y' \times Y'$ such that the relations (2.18) hold. If the constraints (determined by the sets Ξ, Γ, Σ) are unilateral, then we deal with the stationary unilateral problem. In Sect. 3 we are to investigate, from the analytical point of view, three special cases of problems governed by the relations (2.18), namely:

1) Problems with constraints imposed on displacement and stress fields. In this case $\Gamma = Y$ and the relations (2.18) reduce to

$$\begin{aligned} L^*S - l &\in -\partial\chi_{\Xi}(u), \\ Lu - KS &\in \partial\chi_{\Sigma}(S), \end{aligned} \tag{2.19}$$

with (u, S) as the basic unknown.

2) Problems with constraints imposed on strain and stress fields. In this case $\Xi = V$; putting $S_R \equiv S - S_0$ we obtain from the relations (2.18) the following form of governing relations:

$$L^*(S_0 + S_R) - l = 0,$$

$$Lu - KS_0 \in \partial\chi_\Gamma(S_0), \tag{2.20}$$
$$S_R \in \partial\chi\ (Lu),$$

in which (u, S_0, S_R) is the basic unknown.

3) Problems with constraints imposed on strain fields (problems for ideal locking materials). In this case $\Xi = V$, $\Sigma = Y'$ and putting $A \equiv K^{-1}$, $S_R = S - S_0$ we obtain from the relations (2.18) the governing relations

$$L^*(ALu + S_R) - l = 0, \tag{2.21}$$
$$S_R \in \partial\chi_\Gamma(Lu),$$

with (u, S_R) as the basic unknown.

It must be stressed that apart from the constraints mentioned above (i.e. determined by the sets Ξ, Γ, Σ, Δ) which can be unilateral, we can also deal with the bilateral constraints introduced by spaces V and Y. All constraints under consideration are called explicit constraints since the sets Ξ, Γ, Σ and Δ in the formulas (2.17) are known a priori, i.e. they are independent of the basic unknowns.

3. Some stationary problems with constraints for displacements, strains and stresses

3.1. Constraints for displacements and stresses

We start with the mixed problem formulated in Sect 2.4: find the displacement field $u \in \Xi$ and the stress field $S \in \Sigma$ such that the following system of two variational inequalities hold:

$$(P) \qquad \begin{cases} L^*S - l \in -\partial\chi_\Xi(u) \\ Lu - KS \in \partial\chi_\Sigma(S), \end{cases}$$

or, equivalently,

$$\begin{cases} \langle L^*S - l,\ v - u\rangle \geqslant 0, & \forall v \in \Xi, \\ [KS - Lu,\ T - S] \geqslant 0, & \forall T \in \Sigma. \end{cases}$$

It is assumed that V and Y are reflexive Banach spaces and that K is demicontinuous and strong monotone, i.e.

$$[KS - KT,\ S - T] \geqslant c\|S - T\|_Y^2, \quad \forall S, T \in Y', \quad c > 0.$$

The sets $\Xi \subset V$ and $\Sigma \subset Y'$ are assumed to be convex closed and nonempty.

In [10] with the problem (P) have been associated the following two problems:

1) the displacement problem consisting in finding $u \in V$ such Ξ that

$$(P_u) \qquad 0 \in L^*\,(K + \partial\chi_\Sigma)^{-1}Lu + \partial\chi_\Xi(u) - l,$$

and

2) the stress problem consisting in finding $S \in \Sigma$ such that

$$(P_s) \qquad 0 \in KS + \partial\varkappa_\Sigma(S) + \partial\alpha(S),$$

where $\alpha: Y' \to \bar{\mathbb{R}}$ is defined by

$$\alpha(T) \equiv \chi_\Xi^*(-L^*T + l), \quad T \in Y',$$

χ_Ξ^* being the conjugate of χ_Ξ.

The following theorem characterizes the interrelation between solutions of the problems (P), (P_u) and (P_s).

THEOREM 3.1. [10]. *The following conditions are satisfied*:

(i) *If* $(u, S) \in V \times Y'$ *is a solution of* (P) *then* u *is a solution of* (P_u) *and* S *is a solution of* (P_s).

(ii) *If* $u \in V$ *is a solution of* (P_u) *then there exists* $S \in Y'$ *such that* (u, S) *is a solution of* (P).

(iii) *Let us assume in addition that on* Y *Korn's type inequality holds, i.e.*

$$||Lv||_Y \geqslant c||v||_V, \quad v \in V. \tag{3.1}$$

Then, if $S \in Y'$ *is a solution of* (P_s) *then there exists* $u \in V$ *such that* (u, S) *is a solution of* (P).

Theorem 3.1 implies immediately the following:

REMARK 3.1. Suppose that the inequality (3.1) holds. Then the following conditions are equivalent to each other:

(i) (P) has solutions:

(ii) (P_u) has solutions,

(iii) (P_s) has solutions.

From the condition (i) of Theorem 3.1 the necessary condition for the existence of solutions to the problem (P) can be easily derived. It can be expressed in the form

$$\Sigma \cap \operatorname{dom}(\partial\alpha) \neq \emptyset. \tag{3.2}$$

The above relation will be referred to as the compatibility condition for the displacement constraints, the stress constraints and the external forces acting at the body.

Using the known results concerning maximal monotone mappings, [11], and Remark 3.1 we can formulate the following existence theorem for (P):

THEOREM 3.2. *Let us suppose that the inequality* (3.1) *holds and that*

$$\partial\chi_\Sigma + \partial\alpha$$

is a maximal monotone mapping. Then the problem (P) *has at least one solution* $(u, S) \in \in \Xi \times \Sigma$ *with the stress field* S *detremined uniquely.*

From Theorem 3.2 it follows that the existence solution problem of (P) reduces to the investigation conditions under which the sum of maximal monotone mappings $\partial\chi_\Sigma$ and $\partial\alpha$ is again maximal monotone.

3.2. Constraints for strains and stresses

Now we shall consider the class of stationary problems in which both strains and stresses are restricted. As it is shown in Sect. 2.4, a problem of this kind consists in finding $u \in V$, $S_0 \in Y'$ and $S_R \in Y'$ such that the following variational inequalities hold:

$$(Q) \qquad \begin{cases} L^*(S_0 + S_R) - l = 0, \\ Lu - KS_0 \in \partial\chi_\Sigma(S_0), \\ S_R \in \partial\chi_\Gamma(Lu), \end{cases}$$

where $\Gamma \subset Y$ is assumed to be convex closed and such that $\mathrm{dom}(\partial\chi_\Gamma) \cap \mathrm{Im}\, L \neq \emptyset$, (5) or, equivalently,

$$\begin{cases} L^*S_0 - l \in -L^*\partial\chi_\Gamma Lu, \\ Lu - KS_0 \in \partial\chi_\Sigma(S_0). \end{cases}$$

Since, [13],

$$L^*\partial\chi_\Gamma L \subset \partial(\chi_\Gamma \circ L), \tag{3.3}$$

so with problem (Q) can be associated the problem of finding $u \in V$ and $S_0 \in \Sigma$ such that

$$(\bar{Q}) \qquad \begin{cases} L^*S_0 - l \in \partial(\chi_\Gamma \circ L)(u), \\ Lu - KS_0 \in \partial\chi_\Sigma(S_0). \end{cases}$$

The problem $(\bar{Q})$ is not equivalent to (Q) because, in general, the mapping $L^*\partial\chi\, L$ is not maximal monotone.

Note that $\chi_\Gamma \circ L = \chi_{L^{-1}\Gamma}$, where $L^{-1}\Gamma = \{v \in V\colon Lv \in \Gamma\}$ is closed convex and non-empty. Thus all results obtained in Sect. 3.1 hold also for the problem $(\bar{Q})$.

In particular, if

$$L^*\partial\chi_\Gamma L = \partial(\chi_\Gamma \circ L), \tag{3.4}$$

(it is the case in which the mapping $L^*\partial\chi_\Gamma L$ is maximal monotone) then the problems $(\bar{Q})$ and (Q) are equivalent to each other. Thus, in this case, problems with constraints for strains and stresses, due to the equivalence between $(\bar{Q})$ and (Q), can be analysed as problems with constraints for displacements and stresses. Consequently, all results obtained in Sect. 3.1 can be reformulated for problems with constraints for strains and stresses whenever Eq. (3.4) holds. The formulation of the corresponding theorems and propositions is not difficult and therefore will be omitted here.

3.3. Potential case

Now we suppose that K is potential. It means that there exists the function $\gamma\colon Y' \to R$ such that

$$K = \gamma',$$

where $\gamma'\colon Y' \to Y$ is the derivative of γ.

Let us define the function

$$\Phi\colon V \times Y' \to \bar{\mathbf{R}}$$

by means of the formula

$$\Phi(v, T) \equiv -\gamma(T) - \chi_\Sigma(T) + [T, Lv] + \chi_\Xi(v) - \langle f, v\rangle, \qquad v \in V, \quad T \in Y'.$$

PROPOSITION 3.1, [14]. The following conditions are equivalent to each other:

(i) (u, S) is a solution of (P);

(ii) (u, S) is a saddle point of Φ, i.e.

$$\Phi(u, T) \leqslant \Phi(u, S) \leqslant \Phi(v, S), \qquad v \in V, \quad T \in Y'.$$

(5) We use the symbols "dom" and "Im" to denote the effective domain and the range of the corresponding mappings, respectively.

PROPOSITION 3.1 allows us to conclude that if (u, S) is a solution of (P), then

$$\inf_{v\in V}\sup_{S\in Y'}\Phi(v, T) = \sup_{v\in V}\inf_{T\in Y'}\Phi(v, T) = \Phi(u, S).$$

Thus it is natural to consider the following problems:

$$\inf_{v\in V}\sup_{T\in Y'}\Phi(v, T)$$

and

$$\sup_{T\in Y'}\inf_{v\in V}\Phi(v, T),$$

independently. It is easy to verify that

$$\inf_{v\in V}\sup_{T\in Y'}\Phi(v, T) = \inf_{v\in V}\{(\gamma+\chi_\Sigma)^*(Lv)+\chi_\Xi(v)-\langle f, v\rangle\},$$

$(\gamma+\chi_\Sigma)^*: Y\to\bar{\mathbb{R}}$ being the conjugate of $\gamma+\chi_\Sigma$, and

$$\sup_{T\in Y'}\inf_{v\in V}\Phi(v, T) = -\inf_{T\in Y'}\{\gamma(T)+\chi_\Sigma(T)+\alpha(T)\}.$$

Hence we arrive at the following minimization problems for the displacement field u and the stress field S:

$$(\tilde{P}_u)\quad \inf_{v\in V}\{(\gamma+\chi_\Sigma)^*(Lv)+\chi_\Xi(v)-\langle f, v\rangle\}$$

and

$$(\tilde{P}_s)\quad -\inf_{T\in Y'}\{\gamma(T)+\chi_\Sigma(T)+\alpha(T)\},$$

respectively. It is easy to see that the problem (P_u) is the generalization of the well-known minimum potential energy principle and the problem $(\tilde{P}_s)$ is the generalization of the minimum complementary energy principle. To this aid it suffices to put $\Xi = V$ and $\Sigma = Y'$ in $(\tilde{P}_u)$ and $(\tilde{P}_s)$, respectively.

It is important to know whether the existence of solutions to the minimization problems $(\tilde{P}_u)$ and $(\tilde{P}_s)$ implies the existence of solutions to the problem (P). Results in this direction can be summarized in the following propositions, [14]:

PROPOSITION 3.2. Suppose that $u\in V$ is a solution of $(\tilde{P}_u)$. Then there exists $S\in Y'$ such that (u, S) is a solution of (P).

PROPOSITION 3.3. Let the inequality (3.1) be satisfied and let $\partial\chi_\Sigma+\partial\alpha$ be maximal monotone. Then, if $S\in Y'$ is a solution of $(\tilde{P}_s)$, then there exists $u\in V$ such that (u, S) is a solution of (P).

3.4. Variational problems for ideal locking materials

This section is devoted to the study of the system of variational inequalities encountered in problems in which constraints are imposed on strain field only. We consider such constraints which lead to, so-called, locking materials introduced by PRAGER in [16–17]. The existence solution problem will be investigated.

Assume that

$$V = \mathring{H}^1(\Omega)^3,$$
$$Y = Y' = L^2(\Omega)^{(3\times 3)},$$

where $\mathring{H}^1(\Omega)^3$ is the space of all vector functions square integrable in Ω together with their first partial derivatives and vanishing on a given part $\partial_1\Omega$ of the boundary $\partial\Omega$, $L^2(\Omega)^{(3\times 3)}$ is the space of all symetric tensor functions square integrable in Ω.

Here we are to deal with problems in which constraints are imposed on the strain field only. It is the case in which the set of all admissible stress fields coincides with the space $L^2(\Omega)^{(3\times 3)}$, i.e.

$$\Sigma = L^2(\Omega)^{(3\times 3)}.$$

To determine the set of all admissible strain fields let us denote by Δ a closed convex nonempty subset of $R^{(3\times 3)}$ such that $0 \in \Delta$ and $\operatorname{int}\Delta \neq \phi$, $R^{(3\times 3)}$ being the space of all symmetric real valued 3×3 matrices. Let $\Gamma \subset L^2(\Omega)^{(3\times 3)}$ be given by

$$\Gamma = \{E \in L^2(\Omega)^{(3\times 3)} : E(x) \in \Delta \quad \text{for} \quad \text{a.e.} \quad x \in \Omega\}. \tag{3.5}$$

By admissible strain fields we shall mean such elements of $L^2(\Omega)^{(3\times 3)}$ which have the form Lv for some $v \in \mathring{H}^1(\Omega)^3$ and $Lv \in \Gamma$.

If the body is subjected to the "dead" load $l \in (\mathring{H}^1(\Omega)^3)'$ only, then the corresponding system of variational inequalities reduces to the form

$$(Q_1) \qquad \begin{cases} L^*(ALu + S_R) - l = 0, \\ S_R \in \partial\chi_\Gamma(Lu), \end{cases}$$

or, equivalently,

$$L^*ALu - l \in -L^*\partial\chi_\Gamma(Lu),$$

where A is the inverse of K, i.e. $A = K^{-1}$. In the sequel it is assumed that A is demicontinuous and strong monotone.

It is convenient to reformulate (Q_1) to the equivalent form

$$(Q_2) \qquad \begin{cases} \langle L^*ALu - l, v\rangle = -(S_R, Lv), & v \in \mathring{H}^1(\Omega)^3, \\ \langle L^*ALu - l, u\rangle \leqslant -(S_R, E), & E \in \Gamma, \\ Lu \in \Gamma, \end{cases}$$

where $\langle .,. \rangle_{H^1}$ and $(.,.)_{L^2}$ stand for the pairing over $(\mathring{H}^1(\Omega)^3)' \times \mathring{H}^1(\Omega)^3$ and the inner product in $L^2(\Omega)^{(3\times 3)}$, respectively.

Since on $\mathring{H}^1(\Omega)^3$ Korn's inequality holds, A being demicontinuous and strong monotone, so by the known results for variational inequalities, [11, 13], the solution to the problem (Q_1) exists for any $l \in (\mathring{H}^1(\Omega)^3)'$ provided that $L^*\partial\chi_\Gamma L$ is maximal monotone, or, equivalently, $L^*\partial\chi_\Gamma L = \partial(\chi_\Gamma \circ L)$. Unfortunately, in the problem under consideration the forementioned equality does not hold. Thus, in general, the problem (Q_1) has no solution if $\mathring{H}^1(\Omega)^3$ and $L^2(\Omega)^{(3\times 3)}$ are taken as the displacement space and the stress space, respectively.

The goal of further investigation is to find such spaces in which the constraints under consideration can be realized. To this end we shall consider, for any $\varepsilon > 0$, the approximation problem (Q_ε) which consists in finding $u \in \mathring{H}^1(\Omega)^3$ and $S \in L^2(\Omega)^{(3\times 3)}$ such that

$$(Q_\varepsilon) \qquad \begin{cases} L^*(ALu + S_R) - l = 0, \\ S_R \in \partial\chi_{\Gamma_\varepsilon}(Lu), \end{cases}$$

where

$$\Gamma_\varepsilon = \{E \in L^2(\Omega)^{(3\times 3)} \ \omega * E(x) \in \Delta \quad \text{for} \quad x \in \Omega)\},$$

$$\omega_\varepsilon * E(x) \stackrel{\text{df}}{=} \int_\Omega E(y)\omega_\varepsilon(x-y)dy, \quad x \in \Omega.$$

In the above relations $\omega_\varepsilon: R^3 \to R$ stands for the function having the following properties:

$$\omega_\varepsilon \geqslant 0, \quad \omega_\varepsilon(x) = \omega_\varepsilon(-x), \quad x \in R^3, \quad \omega_\varepsilon \text{ is continuous},$$

$$\int_{R^3} \omega_\varepsilon(x)dx = 1, \quad \operatorname{supp}\omega_\varepsilon = \{x \,||x|| \leqslant \varepsilon\}.$$

Note that replacing in (Q_1) the set Γ by the approximation set Γ_ε we arrive at (Q_ε). Since, as it is shown in [15],

$$\Gamma_\varepsilon \supset \Gamma, \quad \varepsilon > 0,$$

$$\bigcap_{\varepsilon>0} \Gamma_\varepsilon = \Gamma,$$

the family $\{\Gamma_\varepsilon\}_{\varepsilon>0}$ is a proper approximation of the set Γ.

Now we shall assume the following hypothesis concerning the set Γ:

(*H*) There exist an element $u_0 \in \mathring{H}^1(\Omega)^3$ and the closed convex nonempty subset $\tilde{\Delta}$, $\tilde{\Delta} \subset \Delta$, with dist $(\tilde{\Delta}, \partial\Delta) = \delta > 0$ such that

$$Lu_0(x) \in \tilde{\Delta} \quad \text{for a.e.} \quad x \in \Omega.$$

Under the Hypothesis (*H*), $Lu_0 \in \operatorname{int}\Gamma_\varepsilon$, $\varepsilon > 0$ [15]. Thus, as it is known, [13], the following equality has to hold:

$$L^*\partial\chi_{\Gamma_\varepsilon}L = \partial(\chi_{\Gamma_\varepsilon}L).$$

So the mapping $L^*\partial\chi_{\Gamma_\varepsilon}L$ is maximal monotone. Thus, under the above assumption, for any $\varepsilon > 0$ the problem (Q_ε) has a solution.

Let us denote by $(u_\varepsilon, S_{R\varepsilon}) \in \mathring{H}^1(\Omega)^3 \times L^2(\Omega)^{(3\times 3)}$ a solution to (Q_ε). It implies that the following relations hold:

$$\begin{cases} L^*(ALu_\varepsilon + S_{R\varepsilon}) - l = 0, \\ S_{R\varepsilon} \in \partial\chi_{\Gamma_\varepsilon}(Lu_\varepsilon). \end{cases}$$

Denoting by $S_\varepsilon \equiv ALu_\varepsilon + S_{R\varepsilon}$, the above relations can be expressed as

$$\text{(3.6)} \qquad \begin{cases} (ALu_\varepsilon - S_\varepsilon, \ E - Lu_\varepsilon)_{L^2} \geqslant 0, \quad E \in \Gamma_\varepsilon, \\ L^*S_\varepsilon - l = 0 \\ Lu_\varepsilon \in \Gamma_\varepsilon. \end{cases}$$

PROPOSITION 3.4. There exists a positive constant C, not depending on ε, such that

$$||u_\varepsilon||_{H^1} \leqslant C.$$

Proof. Putting $E = Lu_0 \in \Gamma_\varepsilon$ in the set (3.6) we arrive at the inequality

$$(ALu_0, Lu_0 - Lu_\varepsilon)_{L^2} - \langle f, u_0 - u_\varepsilon\rangle_{H^1} \geqslant 0.$$

Hence, using the strong monotonicity of A and Korn's inequality, we obtain

$$(ALu_0, Lu_0 - Lu_\varepsilon)_{L^2} - \langle f, u_0 - u_\varepsilon\rangle_{H^1} \geqslant (ALu_0 - ALu_\varepsilon, Lu_0 - Lu_\varepsilon)_{L^2} \geqslant c||u_0 - u_\varepsilon||^2_{\mathring{H}^1}.$$

From the above inequalities we easily obtain the boundedness of (u_ε).

PROPOSITION 3.5. Suppose, in addition, that A maps bounded sets into bounded sets. Then there exists a positive constant C, not depending on ε, such that

$$\begin{aligned} &||S_{R\varepsilon}||_{L^1} \leqslant C, \\ &||L^*S_{R\varepsilon}||_{(H^1)'} \leqslant C, \end{aligned} \tag{3.7}$$

where $||\cdot||_{L^1}$ is the norm in $L^1(\Omega)^{(3\times 3)}$, given by

$$||T||_{L^1} = \int_\Omega \operatorname{tr}[T\operatorname{sgn} T]dv, \qquad T \in L^2(\Omega)^{(3\times 3)}.$$

P r o o f. Under Hypothesis (H) we have

$$E_{0\varepsilon} \equiv Lu_0 + \frac{\delta}{3}\operatorname{sgn} S_\varepsilon \in \Gamma.$$

So $E_{0\varepsilon} \in \Gamma_\varepsilon$, [15], and consequently putting $E = E_{0\varepsilon}$ in the set (3.6) we obtain easily the following inequality:

$$||S_\varepsilon||_{L^1} \leqslant \frac{3}{\delta}\langle L^*ALu_\varepsilon - l, u_0 - u_\varepsilon\rangle_{H^1} + (ALu_\varepsilon, \operatorname{sgn} S_\varepsilon)_{L^2}.$$

Hence, by virtue of Proposition 3.4 and by the hypotheses for A, we obtain the boundedness of S_ε in $L^1(\Omega)^{(3\times 3)}$. Since $S_{R\varepsilon} = S_\varepsilon - ALu_\varepsilon$ and ALu_ε is bounded in $L^2(\Omega)^{(3\times 3)}$, so $S_{R\varepsilon}$ has to be bounded in $L^1(\Omega)^{(3\times 3)}$, too ($\Omega$ is bounded). The boundedness of $L^*S_{R\varepsilon}$ in $(H^1(\Omega)^3)'$ follows immediately from the equality $L^*S_{R\varepsilon} = l - L^*\, ALu_\varepsilon$ and from Proposition 3.4. This ends the proof of Proposition 3.5.

The boundedness of u_ε in $\mathring{H}^1(\Omega)^3$ allows us to choose an infinite subsequence ε_n, $\varepsilon_n \to 0$, such that u_{ε_n} converges weakly to some $u \in \mathring{H}^1(\Omega)^3$. It implies that Lu_{ε_n} converges weakly to Lu in $L^2(\Omega)^{(3\times 3)}$ and since $Lu_{\varepsilon_n} \in \Gamma_{\varepsilon_n}$, we obtain, [15]:

$$Lu \in \Gamma. \tag{3.8}$$

By the boundedness of ALu_{ε_n} we can extract a subsequence, again denoted by ALu_{ε_n}, converging weakly to some $W \in L^2(\Omega)^{(3\times 3)}$

$$ALu_{\varepsilon_n} \to W \quad \text{weakly in} \quad L^2(\Omega)^{(3\times 3)}. \tag{3.9}$$

Putting $E = Lu \in \Gamma \subset \Gamma_{\varepsilon_n}$ in the set (3.6), we obtain

$$(ALu_{\varepsilon_n}, Lu)_{L^2} - \langle f, u - u_{\varepsilon_n}\rangle_{H^1} \geqslant \langle ALu_{\varepsilon_n}, Lu_{\varepsilon_n}\rangle_{L^2}.$$

Hence

$$\limsup (ALu_{\varepsilon_n}, Lu_{\varepsilon_n})_{L^2} \leqslant (W, Lu)_{L^2}.$$

Thus, by the known argument for maximal monotone mapping, [11] we conclude that

$$W = ALu,$$

$$(ALu_{\varepsilon_n}, Lu_{\varepsilon_n})_{L^2} \to (ALu, Lu)_{L^2} \quad \text{as} \quad \varepsilon_n \to 0.$$

To investigate the stress problem let us introduce the following space (see also [18]):

$$M(\Omega) = \{T \in \bar{M}(\Omega)^{(3\times 3)} : L^*T \in (\mathring{H}^1(\Omega)^3)'\},$$

where $\bar{M}(\Omega)$ is the space of bounded measures on $\bar{\Omega}$ (the dual of $C(\bar{\Omega})$).

From Proposition (3.5) it follows that the sequence $S_{R\varepsilon_n}$ is bounded in $\overline{M}(\Omega)^{(3\times 3)}$. Since $\overline{M}(\Omega)^{(3\times 3)}$ is a dual space, so we can extract a subsequence of $S_{R\varepsilon_n}$, again denoted by $S_{R\varepsilon_n}$, converging weak (star) to some element $S_R \in \overline{M}(\Omega)^{(3\times 3)}$.

$$S_{R\varepsilon_n} \to S_R \quad \text{weak (star) in} \quad \overline{M}(\Omega)^{(3\times 3)}, \tag{3.10}$$

or, equivalently,

$$\int_{\Omega} \operatorname{tr}\,[S_{R\varepsilon_n} E]dv \to \int_{\bar{\Omega}} \operatorname{tr}[S_R E], \qquad \forall E \in C(\bar{\Omega})^{(3\times 3)}.$$

Using the boundedness of $L^*S_{R\varepsilon_n}$ in $\mathring{H}^1(\Omega)^3$ (Proposition 3.5) and taking into account the relation (3.10), we obtain the following estimation:

$$\left|\int_{\bar{\Omega}} \operatorname{tr}[S_R Lv]\right| \leqslant c\|v\|_{H^1}, \qquad \forall v \in \mathring{C}^1(\bar{\Omega})^3,$$

which implies

$$L^*S_R \in \left(\mathring{H}^1(\Omega)^3\right)'.$$

Thus $S_R \in M(\Omega)$.

By virtue of (Q_2), for any $\varepsilon_n > 0$ we have

$$\begin{cases} \langle L^*ALu_{\varepsilon_n} - f, v\rangle_{H^1} = -(S_{R\varepsilon_n}, Lv)_{L^2}, & \forall v \in \mathring{H}^1(\Omega)^3, \\ \langle L^*ALu_{\varepsilon_n} - f, u_{\varepsilon_n}\rangle_{H^1} \leqslant -(S_{R\varepsilon_n}, E)_{L^2}, & \forall E \in \Gamma, \\ Lu_{\varepsilon_n} \in \Gamma_{\varepsilon_n}. \end{cases} \tag{3.11}$$

Now, taking into account Eqs. (3.7)–(3.10) we can pass to the limit with $\varepsilon_n \to 0$ in the set (3.11). It yields

$$\begin{cases} \langle L^*ALu - f, v\rangle_{H^1} = -\int_{\bar{\Omega}} \operatorname{tr}[LvS_R], & \forall v \in \mathring{H}^1(\Omega)^3, \\ \langle L^*ALu - f, u\rangle_{H^1} \leqslant -\int_{\bar{\Omega}} \operatorname{tr}[ES_R], & \forall E \in \Gamma \cap C(\Omega)^{(\bar{3}\times 3)}, \\ Lu \in \Gamma. \end{cases} \tag{3.12}$$

The obtained results can be summarized in the following

THEOREM 3.3. *Suppose that A is demicontinuous strong monotone and maps bounded sets into bounded sets. Let Hypothesis (H) holds. Then for any $l \in \left(\mathring{H}^1(\Omega)^3\right)'$ there exists $u \in \mathring{H}^1(\Omega)^3$ and $S_R \in M(\Omega)$ such that the set* (3.12) *is satisfied.*

Theorem 3.3 allows us to conclude that the constraints for strains determined by the set Γ, Γ given by Eq. (3.5), can be realized if $H^1(\Omega)^3$ and $M(\Omega)$ are taken as the displacement space and the stress space, respectively. The solution of the problem under consideration is understood in the sense given by the set (3.12).

REMARK 3.2. We claim that if A is a linear continuous potential operator, then in fact, u_{ε_n} converges strongly to u in $\mathring{H}^1(\Omega)^3$. To prove this fact let us note that under the above assumptions u_{ε_n} is a solution to the following minimization problem:

$$\inf_{v \in W_{\varepsilon_n}} \{\|v\|_A^2 - \langle l, v\rangle\}, \tag{3.13}$$

where

$$||v||_A^2 \stackrel{\text{df}}{=} \int_\Omega \text{tr}[ALvLv]dv$$

is the norm in $H^1(\Omega)^3$ which, due to the strong monotonicity of A and Korn's inequality, is equivalent to the usual norm $||\cdot||_{H^1}$,

$$W_{\varepsilon_n} \stackrel{\text{df}}{=} \{v \in \mathring{H}(\Omega)^3 \;\; Lv \in \Gamma_{\varepsilon_n}\}.$$

Denoting by

$$W \stackrel{\text{df}}{=} \{v \in \mathring{H}^1(\Omega)^3 \;\; Lv \in \Gamma\}$$

and using the fact that $\Gamma_{\varepsilon_n} \supset \Gamma$, we obtain

$$W_{\varepsilon_n} \supset W.$$

Since $u_{\varepsilon_n} \in W_{\varepsilon_n}$ and $u \in W$, so by virtue of the problem (3.13) we get

$$||u_{\varepsilon_n}||_A^2 - \langle l, u_{\varepsilon_n}\rangle_{H^1} \leqslant ||u||_A^2 - \langle l, u\rangle_{H^1}. \tag{3.14}$$

From the lower semicontinuity of the mapping

$$v \mapsto ||v||_A^2 - \langle l, v\rangle_{H^1}, \qquad v \in H^1(\Omega)^3,$$

we deduce

$$\liminf\,(||u_{\varepsilon_n}||_A^2 - \langle l, u_{\varepsilon_n}\rangle_{H^1}) \geqslant ||u||_A^2 - \langle l, u\rangle_{H^1}.$$

The above inequality and the relation (3.14) imply

$$\lim ||u_{\varepsilon_n}||_A^2 = ||u||_A^2,$$

which is equivalent to

$$\lim ||u_{\varepsilon_n}||_{H^1}^2 = ||u||_{H^1}^2.$$

This last equality together with the weak convergence of u_{ε_n} to u leads to the strong convergence of u_{ε_n} to u in the space $\mathring{H}^1(\Omega)^3$ since it is a Hilbert space. This ends the proof of Remark 3.2.

REMARK 3.3 It is easy to observe that if $0 \in \text{int}\Delta$, then Hypothesis (H) is satisfied immediately. In this case as u_0 we can put 0.

REMARK 3.4. Let us denote by

$$V_d = \{v \in H^1(\Omega)^3 \;\; v|_{\partial_1\Omega} = u^d\},$$

$u^d \in H^{\frac{1}{2}}(\partial_1\Omega)^3$ being a given function, the set of all admissible displacement fields. By admissible strain fields we shall mean such elements of the form $Lv \in L^2(\Omega)^{(3\times 3)}$ that $v \in V_d$. Assume the following hypothesis:

(H^d) There exists $u_0 \in H^1(\Omega)^3$, $u_0|_{\partial_1\Omega} = u^d$ such that

$$Lu_0(x) \in \tilde{\Delta} \quad \text{for a.e.} \quad x \in \Omega,$$

$\tilde{\Delta}$ is the same as in (H).

The governing relations for problems with constraints defined above can be written in the form of the following system:

$$(Q_1^d) \qquad \begin{cases} L^*(ALu + S_R) - l = 0, \\ S_R \in \partial\chi_{\tilde{\Gamma}}(Lu), \end{cases}$$

where $A(\cdot) = A(\cdot) + ALu_0$, $\tilde{\Gamma} = \Gamma - Lu_0$, in which the basic unknowns are $u \in \mathring{H}^1(\Omega)^3$

and $S_R \in L^2(\Omega)^{(3\times 3)}$. The above system has the same structure as that given by (Q_1). Thus all results obtained for (Q_1) can be easily reformulated for (Q_1^d), and therefore will be omitted.

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