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A geometric description of distortional plastic hardening of deviatoric materials

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A CERTAIN geometric concept is introduced for the description of subsequent yield surfaces which generalizes well-known models of plastic hardening and includes the possibility of describing nonaffine, distortional transformation of yield surfaces. This phenomenon has been revealed in many experiments but is not properly reflected by the existing hardening theories. The description is formulated for the class of pressure - insensitive materials (deviatoric materials) and is performed in appropriate five-dimensional deviatoric vector stress spaces. General and simplified cases are discussed in detail. For a certain simplified case the convexity is proved in two-dimensional stress space, whereas the general case may exhibit concavities.

W pracy przedstawiono pewną nową koncepcję opisu kolejnych powierzchni plastyczności, uogólniającą znane modele wzmocnienia plastycznego i uwzględniającą możliwość opisu nieafinicznej, dystorsyjnej transformacji powierzchni plastyczności — zjawiska obserwowanego w wielu doświadczeniach i nie odzwierciedlonego poprawnie przez istniejące teorie wzmocnienia. Opis sformułowany jest dla klasy materiałów o własnościach niezależnych od ciśnienia hydrostatycznego (materiałów dewiatorowych) i prowadzony jest w odpowiednich pięciowymiarowych dewiatorowych przestrzeniach wektorowych naprężenia. Przeprowadzono szczegółową dyskusję ogólnego i uproszczonych przypadków proponowanej powierzchni plastyczności. Dla pewnego uproszczonego przypadku wykazano wypukłość powierzchni w dwuwymiarowej przestrzeni naprężenia, natomiast w przypadku ogólnym rozważane powierzchnie mogą wykazywać wklęsłości.

В работе представлена некоторая новая концепция описания последовательных поверхностей пластичности, обобщающая известные модели пластического упрочнения и учитывающая возможность описания неаффинного, дисторсного преобразования поверхностей пластичности — явления наблюдаемого в многих экспериментах и не отображенного правильно существующими теориями упрочнения. Описание сформулировано для класса материалов со свойствами независящими от гидростатического давления (девиаторных материалов) и ведется в соответствующих пятимерных девиаторных векторных пространствах напряжения. Проведено подробное обсуждение общих и упрощенных случаев предлагаемой поверхности пластичности. Для некоторого упрощенного случая показана выпуклость поверхности в двумерном пространстве напряжения, в общем случае же рассматриваемые поверхности могут обладать вогнутостями.

1. Introductory remarks

A DESCRIPTION of plastic hardening consists usually of the following typical elements: equations of subsequent yield surfaces (neutral surfaces) described by internal state variables, constitutive equations and evolution equations for those state variables. In the present paper we discuss the first of the above elements. In many approaches this step is also regarded as sufficient to construct constitutive equations (e.g. if we assume the validity of the normality rule for the vector of plastic strain rates); the relevant evolution equations will be discussed separately.

The classical hardening rules describe the proportional expansion or rigid translation of the initial yield surface (isotropic and kinematic hardening, respectively). These elements are not sufficient for more adequate description. Hence two further elements of transformation of the initial yield surface have been introduced: affine deformation (e.g. an ellipse is transformed into another ellipse, with a changed ratio of semi-axes) and rotation. The above four elements describe the most general transformation of a quadratic function — as used in the Huber-Mises-Hencky (HMH) yield condition — into another, anisotropic quadratic function.

The relevant equation

$$N_{ijkl}(s_{ij}-a_{ij})(s_{kl}-a_{kl})-1 = 0, \tag{1.1}$$

mentioned by F. EDELMAN and D. C. DRUCKER [1], was investigated in detail by A. BALTOV and A. SAWCZUK [2], V. L. DANILOV [3], M. TANAKA and Y. MIYAGAWA [4]. It contains 20 parameters: 15 independent components of the fourth-order tensor N_{ijkl} describing expansion, affine deformation and rotation of the yield surface, and five independent components of the deviator a_{ij}, describing translation; s_{ij} denote here the deviatoric stress components.

However, even Eq. (1.1) cannot describe the fifth, very important element of transformation of the initial yield surface, observed in most experimental investigations, namely nonaffine deformation of that surface, called distortion. The equation of such subsequent yield surfaces must exceed the quadratic terms used in Eq. (1.1). Certain proposals of that type are due to J. F. WILLIAMS and N. L. SVENSSON [5] (fourth-degree terms), E. SHIRATORI, K. IKEGAMI, F. YOSHIDA, K. KANEKO and S. KOIKE [6, 7] (also fourth-degree terms), H. P. SHRIVASTAVA, Z. MRÓZ and R. N. DUBEY [8] (third-degree terms introduced via the third deviatoric stress invariant), M. ORTIZ and E. P. POPOV [9] (trigonometric polynomials).

The above proposals are mostly of algebraic character. Such an approach has some advantages: for example, a relatively easy discussion of invariance of the equations proposed. However, it also has some disadvantages: poor visualization, difficult analysis of convexity, unexpected corner points, etc. Conversely, the proposal given in the present paper is mainly of geometric character, so its pictorial aspect will be easier. However, the equations of subsequent yield surface and their analysis will also be given.

2. Basic assumptions

We consider subsequent transformations of the initial yield surface in the form of a hypersphere in an appropriate five-dimensional space. A. A. ILYUSHIN [10, 11] introduced such an auxiliary, deviatoric vector space, defining the components σ_i, $i = 1, 2, \ldots 5$, of the stress vector $\boldsymbol{\sigma}$ as follows:

$$\tag{2.1} \begin{aligned} \sigma_1 &= \frac{3}{2} s_{xx}, \\ \sigma_2 &= \frac{\sqrt{3}}{2}(s_{yy}-s_{zz}), \end{aligned}$$

$$\sigma_3 = s_{xy}\sqrt{3}, \quad \sigma_4 = s_{yz}\sqrt{3}, \quad \sigma_5 = s_{zx}\sqrt{3}.$$

Indeed, the HMH yield surface is then described by the equation

$$|\boldsymbol{\sigma}| = \sigma_0, \tag{2.2}$$

and hence it is represented by a hypersphere; σ_0 denotes the yield-point stress in uniaxial tension.

Ilyushin's space (2.1) is clearly connected with the HMH initial yield condition: other yield conditions are not represented here by a fully symmetric surface, i.e. by a hypersphere. However, Ilyushin's space may be generalized as to cover a broader class of initial yield conditions, in other words a broader class of materials.

Such generalizations were proposed by the authors in [12]: in an appropriately modified Ilyushin's space any yield condition for isotropic bodies depending on the second and on the third deviatoric stress invariant may be represented by a hypersphere, and also homogeneous quadratic yield conditions for anisotropic bodies (Mises, Hill). The relevant materials are called here "deviatoric materials" since their properties can be described in a five-dimensional deviatoric space. In what follows we refer our considerations to any auxiliary five-dimensional stress space in which the initial yield condition is represented by a hypersphere.

We make use of Ilyushin's postulate of isotropy. It states that plastic hardening depends on intrinsic geometry of the trajectory in the five-dimensional space, but does not depend on the initial direction of that trajectory. Of course, this postulate requires the initial yield surface to be a hypersphere and this requirement is satisfied. The postulate of isotropy was verified experimentally by many investigators, mostly in the Soviet Union (V. S. LENSKY [13], L. S. ANDREEV [14]) and recently in Japan (Y. OHASHI *et al.* [15], E. SHIRATORI, K. IKEGAMI, K. KANEKO [6, 7, 16]). Usually it is accepted that it holds with a reasonable accuracy which may further be improved by introducing a modified space.

The description of plastic hardening proposed in the present paper is of a purely phenomenological character; however, certain parameters, regarded as internal state variables, may be interpreted physically, e.g. as residual microstresses. We confine our considerations to small strains and quasi-static loading.

Yield surfaces may be defined variously and their shape may depend essentially on the definition adopted, R. M. HAYTHORNTHWAITE [17]. We do not discuss this problem here; the construction proposed may be used, in principle, for any particular definition. It may also serve for particular surfaces if we employ a multisurface description of plastic hardening (Z. MRÓZ [18,19], E. SHIRATORI, K. IKEGAMI and F. YOSHIDA [20]).

3. The geometric procedure proposed

The procedure will be first shown in a two-dimensional case, i.e. in a two-dimensional subspace of a five-dimensional space. It is well known that an ellipse may be obtained from two concentric circles by a projecting procedure where the pole of projecting radii coincides with the centres of both circles (Fig. 1a). Now, a much more general curve may

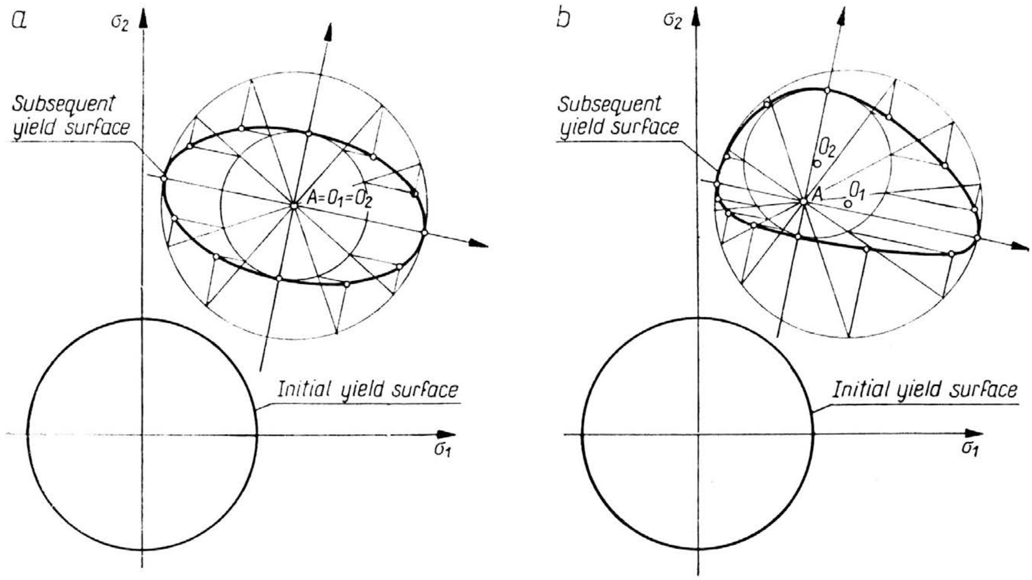

FIG. 1. Proposed description of subsequent yield surfaces (b) as a generalization of quadratic surfaces (a).

be obtained by similar projecting if we distinguish the centres of the circles O_1 and O_2 and the pole A (Fig. 1b). Indeed, such a curve resembles a subsequent yield curve obtained from experiments for a general curvilinear trajectory.

In the general five-dimensional case we introduce five hyperspheres with various radii $R_{(i)}$ and various centres $O_{(i)}$, a pole A and a system of mutually perpendicular projecting directions. The pole must be located within all the hyperspheres. Then the current radius intersects all the hyperspheres and the projecting procedure means simply taking one coordinate (in the rotated system of projecting directions) from each hypersphere.

The simplest analytical description will be obtained in a moving system of coordinates $\hat{\sigma}_i$, translated and rotated with respect to the original system σ_i. The directions of $\hat{\sigma}_i$ coincide with the projecting directions. The versors of $\hat{\sigma}_i$ will be denoted by $\hat{\mathbf{w}}_i$; they are related to the versors $\mathbf{w}_i$ of the original system σ_i by the orthogonal transformation formulae

$$\hat{\mathbf{w}}_i = Q_{ij}\mathbf{w}_j, \quad i, j = 1, 2, \dots 5, \tag{3.1}$$

where $\mathbf{Q} = (Q_{ij})$ is an orthogonal tensor ($\mathbf{Q}^{-1} = \mathbf{Q}^T$, $\det\mathbf{Q} = \pm 1$) and where the summation convention holds. Now we define in the moving system of coordinates the vector of "active stress" $\hat{\boldsymbol{\sigma}} = \hat{\sigma}_i\hat{\mathbf{w}}_i$, related to the stress vector $\boldsymbol{\sigma}$ by the formula

$$\hat{\sigma}_i = Q_{ij}(\sigma_j - a_j), \tag{3.2}$$

where the vector $\mathbf{a} = a_i\mathbf{w}_i$ describes the translation of the centre of the moving coordinate system (Fig. 2), and may be interpreted as a vector of residual microstresses.

The position of the centres of hyperspheres $O_{(i)}$ will be defined in the moving coordinate system by five vectors $\mathbf{d}_{(i)} = d_{(i)j}\hat{\mathbf{w}}_j$. These vectors are responsible for nonelliptic distortion of the yield surface, hence the notation $\mathbf{d}$. Here a bracketed index denotes

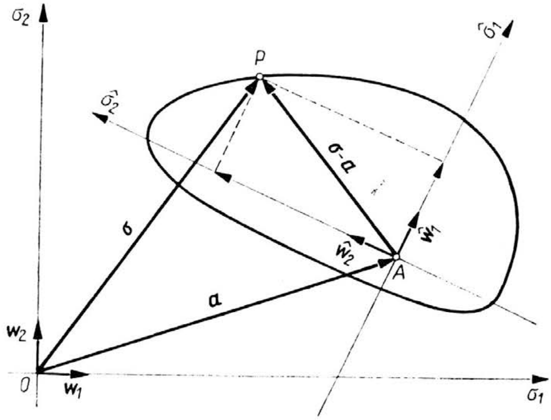

FIG. 2. Basic notations.

a label (number of individual hyperspheres) and is not subject to summation. In what follows, an underlined index will not participate in summation, either.

Instead of the Cartesian system $\hat{\sigma}_i$, we introduce a spherical system of coordinates where the position of a point will be determined by the radius ϱ and four angular coordinates β_i, $i = 1, 2, 3, 4$.

For a three-dimensional space this system is shown in Fig. 3. The versor **t** of the direction ϱ has the following Cartesian coordinates:

$$
\begin{aligned}
t_1 &= \cos\beta_1 \cos\beta_2 \cos\beta_3 \cos\beta_4,\\
t_2 &= \sin\beta_1 \cos\beta_2 \cos\beta_3 \cos\beta_4,\\
t_3 &= \sin\beta_2 \cos\beta_3 \cos\beta_4,\\
t_4 &= \sin\beta_3 \cos\beta_4,\\
t_5 &= \sin\beta_4,
\end{aligned}
\tag{3.3}
$$

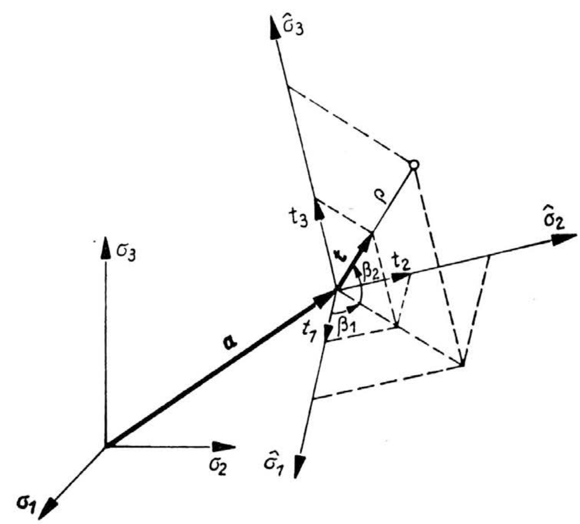

FIG. 3. Spherical system of coordinates in the moving reference frame.

where, obviously,

$$t_i t_i = 1, \quad i = 1, 2, \dots 5. \tag{3.4}$$

Using these coordinates we may write

$$\hat{\sigma}_i = \varrho t_i. \tag{3.5}$$

Parametric equations of the subsequent yield surface investigated may be obtained by deriving the equations of nonconcentric hyperspheres in the spherical system introduced and by taking one corresponding coordinate from each hypersphere:

$$\varrho_{(i)} = \varrho_{(i)}(\beta_1, \beta_2, \beta_3, \beta_4) = \varrho_{(i)}(t_1 \dots t_5). \tag{3.6}$$

Making use of Eq. (3.5), we derive the parametrical equations of those hyperspheres:

$$\hat{\sigma}_{(i)j} = \varrho_{(i)} t_j, \quad i, j = 1, 2, \dots 5, \tag{3.7}$$

and now $\hat{\sigma}_i$ of the subsequent yield surface equals $\hat{\sigma}_{(i)i}$ of the hypersphere, hence

$$\hat{\sigma}_i = \varrho_{(i)} t_i. \tag{3.8}$$

Finally, making use of Eq. (3.2), we write in the original system of coordinates the equation of the subsequent yield surface as follows:

$$\sigma_j = Q_{ij}^{-1} \hat{\sigma}_i + a_j = Q_{ij}^{-1} \varrho_{(i)} t_i + a_j. \tag{3.9}$$

4. Parametric equations for the general case

Equations of nonconcentric hyperspheres may be written in the form (Fig. 4)

$$(\hat{\sigma}_j - d_{(i)j})(\hat{\sigma}_j - d_{(i)j}) = R_{(i)}^2, \tag{4.1}$$

where the summation convention (over j) holds.

Substituting here Eq. (3.7), we obtain for each hypersphere an equation quadratic with respect to the current radius $\varrho_{(i)}$:

$$\varrho_{(i)}^2 - 2\varrho_{(i)} d_{(i)j} t_j + d_{(i)j} d_{(i)j} - R_{(i)}^2 = 0. \tag{4.2}$$

This equation has two solutions: we should select the solution corresponding to the sense of the vector $\mathbf{t}$:

$$\varrho_{(i)} = d_{(i)j} t_j + [(d_{(i)j} t_j)^2 - d_{(i)j} d_{(i)j} + R_{(i)}^2]^{1/2}. \tag{4.3}$$

Finally, substituting Eq. (4.3) into Eq. (3.9), we obtain

$$\sigma_j = Q_{ij}^{-1} \{d_{(i)k} t_k + [(d_{(i)k} t_k)^2 - d_{(i)k} d_{(i)k} + R_{(i)}^2]^{1/2}\} t_i + a_j, \tag{4.4}$$

where the summation goes over i (without brackets) and over k; $i, j, k = 1, 2, \dots 5$. The parameters $R_{(i)}$ and $\mathbf{d}_{(i)}$ must satisfy the conditions

$$R_{(i)}^2 - d_{(i)k} d_{(i)k} \geqslant 0, \quad i = 1, 2, \dots 5, \tag{4.5}$$

ensuring the location of the pole within all the hyperspheres.

Equations (4.4) describe the surface under investigation in parametrical form. Indeed, substituting the coordinates (3.3), we express five coordinates σ_j in terms of four parameters β_i. The number of parameters defining the surface is here rather high: one ortho-

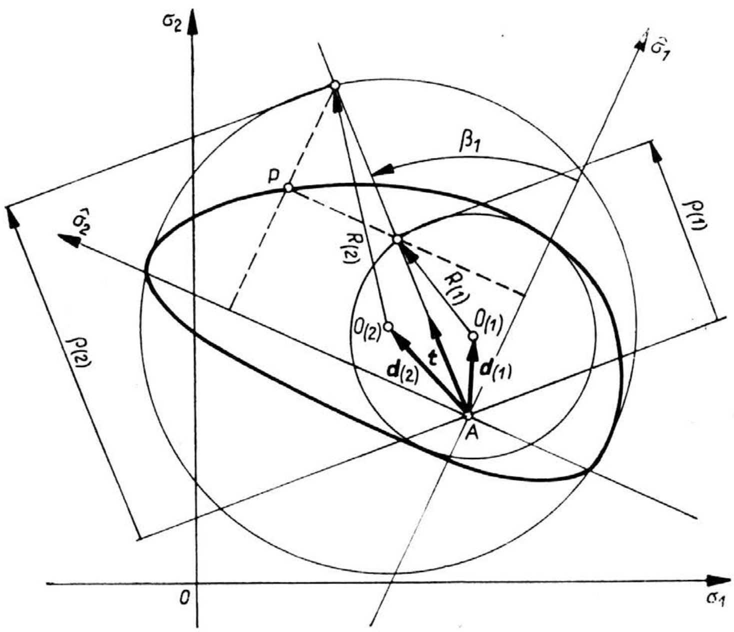

FIG. 4. General case of the proposed yield surface.

gonal second-order tensor **Q** has 10 independent coordinates (since 25 coordinates must satisfy 15 orthogonality relations), six vectors $\mathbf{d}_{(i)}$ and $\mathbf{a}$ have 30 coordinates and five scalars $R_{(i)}$ make the total sum equal to 45. In a n-dimensional space this number equals $3n(n+1)/2$. Hence the number of scalar parameters is approximately twice higher than that describing the general quadratic function (1.1), namely 45:20 in the five-dimensional case, and 9:5 in the two-dimensional case.

In view of this fact we are looking for a reduction of the number of parameters, possibly without visible loss of accuracy of description.

5. Parametric equations for the simplified case

A careful analysis of the surfaces (4.4) shows that, indeed, a substantial reduction of the number of parameters is possible without greater loss of accuracy.

Almost all the features of the distorted surfaces are retained if we assume that the centre of each hypersphere lies on the corresponding axis (Fig. 5), this means

$$d_{(i)j} = 0 \quad \text{for} \quad j \neq i. \tag{5.1}$$

The only nonzero coordinate $d_{(i)i}$ will be denoted briefly by d_i, and the bracket of $R_{(i)}$ will also be omitted. Then in the five-dimensional case 20 parameters (coordinates $d_{(i)j}$) vanish and only 25 remain; this number is only slightly larger than the number 20 parameters in Eq. (1.1). One can conclude that the whole gain of describing adequately non-elliptic distortion of yield surfaces is here due to the 5 additional parameters d_i. Finally, the general parametric equations (4.4) are replaced by

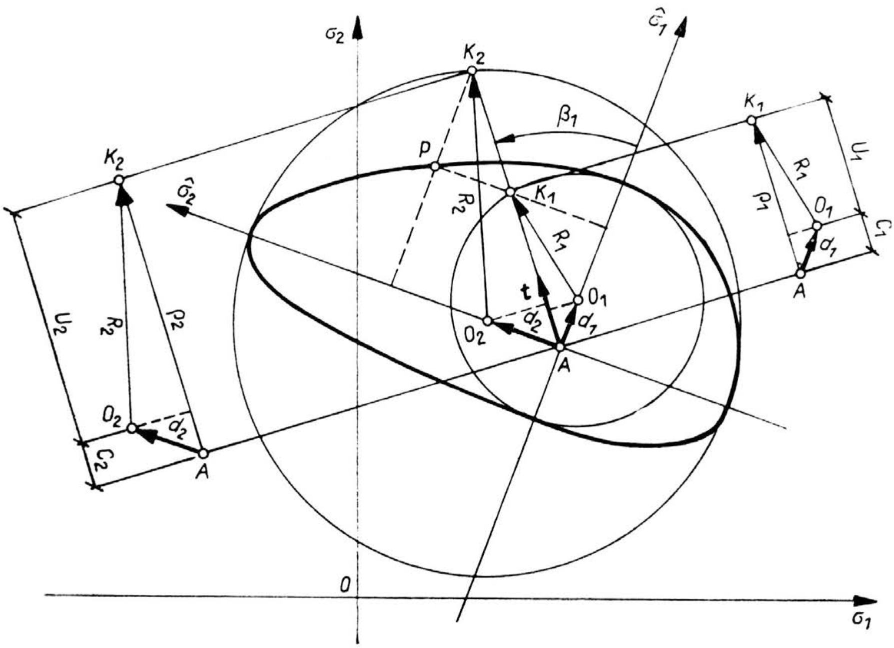

FIG. 5. Simplified case of the proposed yield surface.

$$\sigma_j = Q_{ij}^{-1}[d_{\underline{i}} t_{\underline{i}} + (d_{\underline{i}}^2 t_{\underline{i}}^2 - d_{\underline{i}}^2 + R_{\underline{i}}^2)^{1/2}] t_i + a_j, \tag{5.2}$$

$i, j = 1, 2, \ldots 5$, summation over i, no summation over $\underline{i}$. The inequalities (4.5) are simplified to

$$R_i - |d_i| \geqslant 0. \tag{5.3}$$

In a n-dimensional space the number of parameters is here equal to $n(n+5)/2$. Hence in a two-dimensional case we have 7 parameters versus 5 required in this case by Eq. (1.1).

6. Implicit equation for the simplified case

The simplifying assumption (5.1) results not only in a substantial reduction of the number of parameters, but also in a much simpler form of Eqs. (5.2) and the possibility of deriving an implicit equation in this case. Indeed, the parameters t_i may easily be eliminated since only one of these parameters appears in each equation for the coordinate $\hat{\sigma}_i$:

$$\hat{\sigma}_i = [d_{\underline{i}} t_{\underline{i}} + (d_{\underline{i}}^2 t_{\underline{i}}^2 - d_{\underline{i}}^2 + R_{\underline{i}}^2)^{1/2}] t_i. \tag{6.1}$$

The solution of Eq. (6.1) with respect to t_i looks as follows:

$$t_{\underline{i}}^2 = \frac{\hat{\sigma}_{\underline{i}}^2}{R_{\underline{i}}^2 + 2d_{\underline{i}} \hat{\sigma}_{\underline{i}} - d_{\underline{i}}^2} \tag{6.2}$$

and substitution of Eq. (6.2) into the relation (3.4) results in the following implicit equation of the surfaces under consideration:

$$\sum_{\underline{\mathrm{i}}=1}^{5} \frac{\hat{\sigma}_{\underline{\mathrm{i}}}^2}{R_{\underline{\mathrm{i}}}^2+2d_{\underline{\mathrm{i}}}\hat{\sigma}_{\underline{\mathrm{i}}}-d_{\underline{\mathrm{i}}}^2} = 1. \tag{6.3}$$

The above equation may be written in a more convenient form by using matrix notation. To this aim let us define the following functional diagonal matrix $\mathbf{D} = \mathrm{diag}(D_{\underline{\mathrm{ii}}})$ where $D_{\underline{\mathrm{ii}}}$ depend on stresses:

$$D_{\underline{\mathrm{ii}}} = R_{\underline{\mathrm{i}}}^2+2d_{\underline{\mathrm{i}}}\hat{\sigma}_{\underline{\mathrm{i}}}-d_{\underline{\mathrm{i}}}^2. \tag{6.4}$$

Using this matrix we may write Eq. (6.3) in the form

$$\hat{\boldsymbol{\sigma}}^T\mathbf{D}^{-1}\hat{\boldsymbol{\sigma}}-1 = 0. \tag{6.5}$$

Now we may return to the original system of coordinates, using Eq. (3.2):

$$(\boldsymbol{\sigma}-\mathbf{a})^T\mathbf{Q}^T\mathbf{D}^{-1}\mathbf{Q}(\boldsymbol{\sigma}-\mathbf{a})-1 = 0. \tag{6.6}$$

In indicial notation the equation

$$C_{ij}(\sigma_i-a_i)(\sigma_j-a_j)-1 = 0 \tag{6.7}$$

resembles Eq. (1.1) but rather generalizes it since the tensor C_{ij} depends here on $\boldsymbol{\sigma}$:

$$C_{ij} = Q_{ki}Q_{kj}D_{\underline{\mathrm{kk}}}^{-1}, \tag{6.8}$$

$\underline{\mathbf{k}} = k$, but no summation over $\underline{\mathbf{k}}$,

$$D_{\underline{\mathrm{kk}}} = R_{\underline{\mathrm{k}}}^2+2d_{\underline{\mathrm{k}}}Q_{\underline{\mathrm{k}}j}(\sigma_j-a_j)-d_{\underline{\mathrm{k}}}^2. \tag{6.9}$$

7. Surfaces corresponding to simple loading

Simple loading is characterized by a proportional increase of the components of the stress deviator (and not necessarily of the stress tensor); the trajectory in auxiliary Ilyushin or generalized spaces is a straight line with radial direction and this fact combined with the postulate of isotropy results in symmetry requirements (rotational symmetry of the yield surface). Suppose that the direction $\hat{\sigma}_1$ coincides with the direction of the loading trajectory $\boldsymbol{\xi}$; then the hyperspheres (2)÷(5) cannot be distinguished from each other (Fig. 6). The first projecting direction coincides also with $\boldsymbol{\xi}$,

$$Q_{1j} = \xi_j, \quad j = 1, 2, \dots 5, \tag{7.1}$$

whereas other projecting directions are orthogonal but arbitrary (only in a two-dimensional space they are determined uniquely). Also translation takes place in the direction ξ,

$$a_i = |\mathbf{a}|\,\xi_i, \quad i = 1, 2, \dots 5. \tag{7.2}$$

In the general case of the proposed description two coordinates $d_{(i)j}$ may be different from zero, namely $d_{(1)1}$ and $d_{(j)1}$, equal to each other for any j, $j = 2, 3, 4, 5$. Then the number of scalar parameters is equal to 5: $|\mathbf{a}|$, R_1, R_j, $d_{(1)1}$, $d_{(j)1}$, $j = 2$ (Fig. 6a). For the simplified description we have $d_{(j)1} = 0$, and hence the number of independent scalar parameters is reduced to 4 (Fig. 6b).

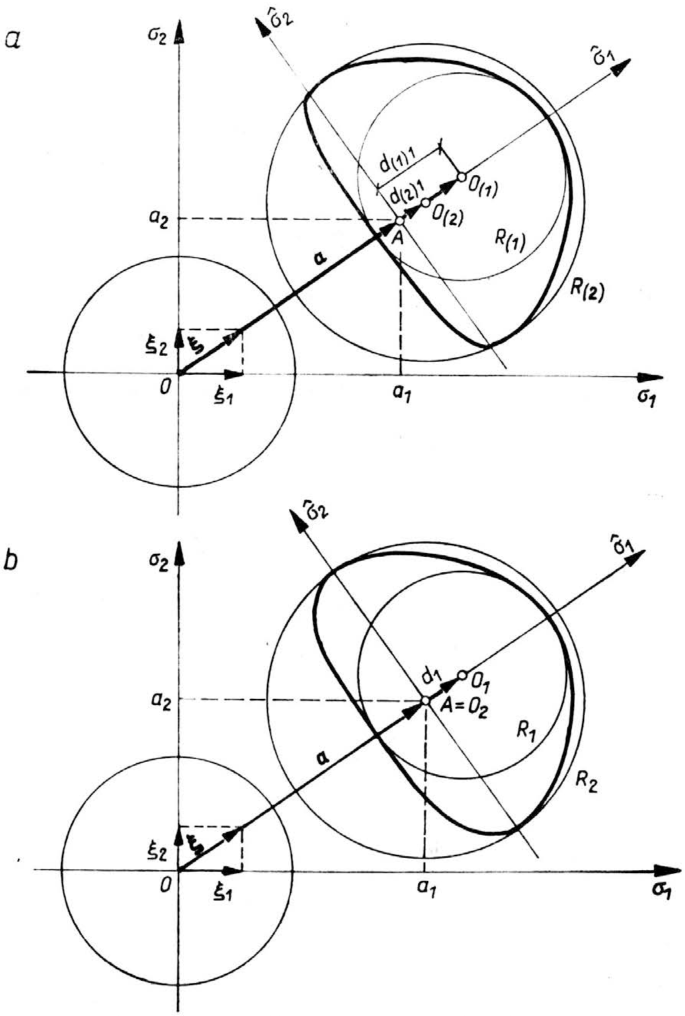

FIG. 6. Subsequent yield surfaces corresponding to simple loading: (a) general case, (b) simplified case.

8. Investigation of convexity of subsequent yield surfaces in the two-dimensional case

The analysis of convexity of an assumed model of yield surface is important in view of the basic axioms and postulates of the theory of plasticity and their consequences, e.g. regarding the uniqueness of solution of boundary value problems.

As to the present description we shall investigate the convexity of the proposed yield surface only for the two-dimensional case of the vectorial stress space. Although not general, this analysis gives a hint of the possible behaviour of the yield surface for more general stress states.

For the simplified case of the yield surface proposed, the parametric equations (6.1) written in a moving coordinate system $\hat{\sigma}_1\ \hat{\sigma}_2$ in the plane $\sigma_1\sigma_2$ have the form

$$\hat{\sigma}_1 = [d_1 t_1 + (d_1^2 t_1^2 - d_1^2 + R_1^2)^{\frac{1}{2}}] t_1, \tag{8.1}$$

$$\sigma_2 = [d_2 t_2 + (d_2^2 t_2^2 - d_2^2 + R_2^2)^{\frac{1}{2}}] t_2 .$$

For this case $t_1 = \cos\beta_1$, $t_2 = \sin\beta_1$ (Fig. 5), while the inequalities (5.3) are here

$$R_1 - |d_1| \geqslant 0, \qquad R_2 - |d_2| \geqslant 0. \tag{8.2}$$

With the abbreviated notation shown in Fig. 5 Eqs. (8.1) may be written in the following way:

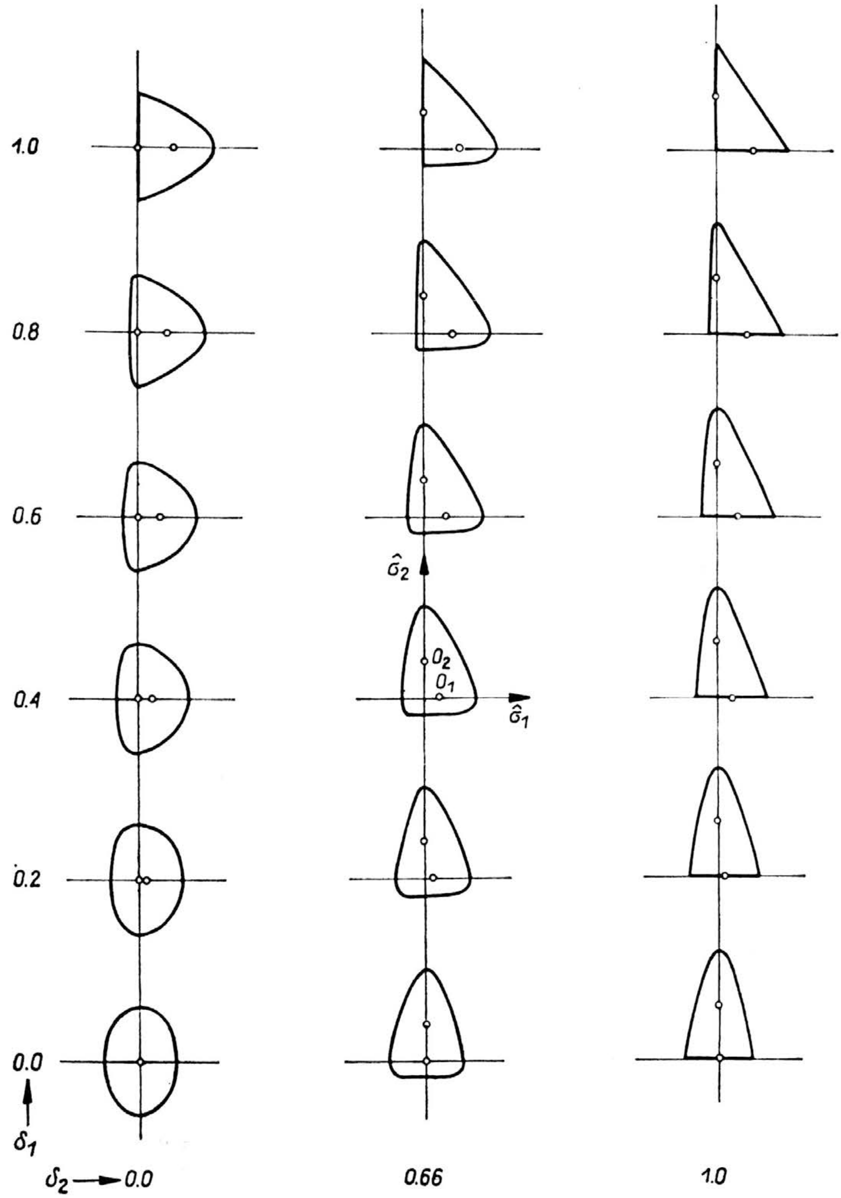

FIG. 7. Dependence of the simplified yield surfaces on the parameters $\delta_1 = d_1/R_1$, $\delta_2 = d_2/R_2$ ($R_2/R_1 = 1.5$).

$$\begin{aligned}\sigma_1 &= \varrho_1(\beta_1)\cos\beta_1 = [c_1(\beta_1)+u_1(\beta_1)]\cos\beta_1,\\ \sigma_2 &= \varrho_2(\beta_1)\sin\beta_1 = [c_2(\beta_1)+u_2(\beta_1)]\sin\beta_1,\end{aligned} \tag{8.3}$$

where c_1, c_2, u_1, u_2 denote corresponding terms in the square brackets in (8.1).

The region enclosed by the curve (8.1) is convex if the curvature k of this curve is non-negative, that is if

$$k = (\hat\sigma_1'\hat\sigma_2''-\hat\sigma_1''\hat\sigma_2')[(\hat\sigma_1')^2+(\hat\sigma_2')^2]^{-\frac{3}{2}} \geqslant 0, \tag{8.4}$$

where the primes denote the derivatives of $\hat\sigma_1$ and $\hat\sigma_2$ with respect to the angular parameter β_1. Calculating these derivatives, we arrive after some transformations at the following expression for the curvature k:

$$\begin{aligned}k = \varrho_1^2\varrho_2^2[(R_1^2\sin\beta_1-2d_2u_2\cos^2\beta_1)u_1^2\sin\beta_1\\ +(R_1^2\cos\beta_1-2d_1u_1\sin^2\beta_1)u_2^2\cos\beta_1]\,(\varrho_1^4u_2^2\sin^2\beta_1+\varrho_2^4u_1^2\cos^2\beta_1)^{-\frac{3}{2}},\end{aligned} \tag{8.5}$$

which can be shown to be non-negative since after some manipulations the first square bracket in Eq. (8.5) may be written as a sum of non-negative terms:

$$(R_2^2u_1d_1\sin^2\beta_1-R_1^2u_2^2\cos\beta_1)^2+(R_1^2u_2d_2\cos^2\beta_1-R_2^2u_1^2\sin\beta_1)^2 \geqslant 0, \tag{8.6}$$

and the second one is evidently non-negative.

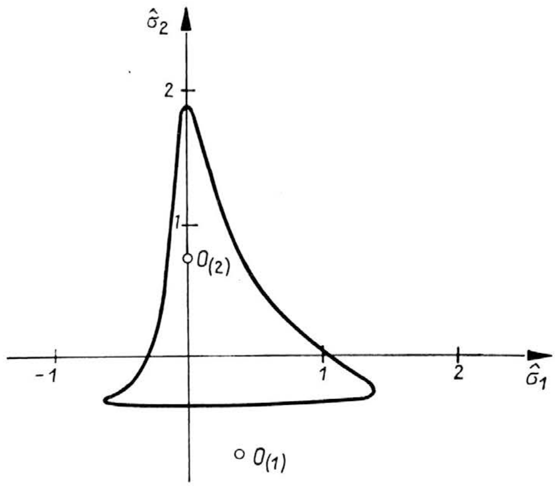

FIG. 8. An example of a concave surface in the general case.

This means that the curve (8.1) corresponding to the simplified case of the proposed yield surface is convex for the whole range (8.2) of the parameters R_i, d_i for which it is defined.

For the general case of the yield surface proposed (Eqs. (4.4)) a similar proof of convexity cannot be performed. On the contrary, investigation of this type of surfaces has revealed the possibility of occurrence of concave surfaces for certain combinations of values of distortional parameters $\mathbf{d}_{(i)}$ and radii $R_{(i)}$. Hence the generality of Eqs. (4.4) may be regarded even as too great if we confine our considerations to convex yield surfaces.

Figure 7 shows typical subsequent yield surfaces described by Eq. (6.1). On the other hand, Fig. 8 shows an example of a nonconvex surface obtained in the general case (4.4) for

$$R_{(1)} = 1, \quad R_{(2)} = 1.135, \quad d_{(1)1} = 0.375, \quad d_{(1)2} = -0.75, \quad d_{(2)1} = 0, \quad d_{(2)2} = 0.75.$$

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