581.

ON A THEOREM IN ELLIPTIC MOTION.

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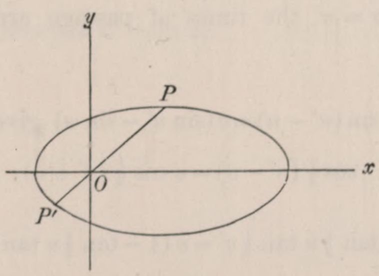
LET a body move through apocentre between two opposite points of its orbit, say from the point P, eccentric anomaly u, to the point P', eccentric anomaly u', where



u, u' are each positive, $u < \pi$, $u' > \pi$. Taking the origin at the focus, and the axis of x in the direction through apocentre, then—

Coordinates of P are $x = a(-\cos u + e)$, $y = a\sqrt{1-e^2}\sin u$,

„ P' „ $x = a(-\cos u' + e)$, $y = a\sqrt{1-e^2}\sin u'$;

whence, expressing that the points P, P' are in a line with the focus,

$$\sin u'(-\cos u + e) - \sin u(-\cos u' + e) = 0,$$

that is,

$$\sin(u' - u) = e(\sin u' - \sin u),$$

which is negative, viz. $u' - u$ is $> \pi$.

The time of passage from P to P' is

$$nt = (u' - e\sin u') - (u - e\sin u),$$
$$= u' - u - e(\sin u' - \sin u),$$
$$= u' - u - \sin(u' - u),$$

which, $u' - u$ being greater than π and $-\sin(u' - u)$ positive, is greater than π; viz. the time of passage is greater than one-half the periodic time. Of course, if P and P' are at pericentre and apocentre, the time of passage is equal one-half the periodic time.

The time of passage from P' to P through the pericentre is

$$nt = 2\pi - (u' - u) + \sin(u' - u),$$

which is

$$= 2\pi - (u' - u) - \sin\{2\pi - (u' - u)\},$$

where $2\pi - (u' - u)$, $= \alpha$ suppose, is an angle $< \pi$. Writing, then

$$nt = \alpha - \sin\alpha,$$

and comparing with the known expression for the time in the case of a body falling directly towards the centre of force, we see that the time of passage from P' to P through the pericentre, is equal to the time of falling directly towards the same centre of force from rest at the distance $2a$ to the distance $a(1 + \cos\alpha)$, where, as above $\alpha = 2\pi - (u' - u)$, $u' - u$ being the difference of the eccentric anomalies at the two opposite points P, P'. If $\alpha = \pi$, the times of passage are each $= \frac{\pi}{n}$, that is, one-half the periodic time.

The foregoing equation $\sin(u' - u) = e(\sin u' - \sin u)$ gives obviously

$$\cos\tfrac{1}{2}(u' - u) = e\cos\tfrac{1}{2}(u' + u);$$

that is,

$$1 + \tan\tfrac{1}{2}u\tan\tfrac{1}{2}u' = e(1 - \tan\tfrac{1}{2}u\tan\tfrac{1}{2}u'),$$

or,

$$-\tan\tfrac{1}{2}u\tan\tfrac{1}{2}u' = \frac{1-e}{1+e};$$

(in the figure $\tan\frac{1}{2}u$ is positive, $\tan\frac{1}{2}u'$ negative); and we thence obtain further

$$\sin\tfrac{1}{2}(u' - u) = \cos\tfrac{1}{2}u'\cos\tfrac{1}{2}u(\tan\tfrac{1}{2}u' - \tan\tfrac{1}{2}u),$$
$$\sin\tfrac{1}{2}(u' + u) = \cos\tfrac{1}{2}u'\cos\tfrac{1}{2}u(\tan\tfrac{1}{2}u' + \tan\tfrac{1}{2}u),$$
$$\cos\tfrac{1}{2}(u' - u) = \cos\tfrac{1}{2}u'\cos\tfrac{1}{2}u\,.\,\frac{2e}{1+e},$$
$$\cos\tfrac{1}{2}(u' + u) = \cos\tfrac{1}{2}u'\cos\tfrac{1}{2}u\,.\,\frac{2}{1+e};$$

and thence also

$$\cos u + \cos u' = 2 \cos \tfrac{1}{2} (u' + u) \cos \tfrac{1}{2} (u' - u),$$

$$= \cos^2 \tfrac{1}{2} u' \cos^2 \tfrac{1}{2} u \,.\, \frac{8e}{(1+e)^2}.$$

But we have

$$1 + \cos (u' - u) = 2 \cos^2 \tfrac{1}{2} (u' - u) = \cos^2 \tfrac{1}{2} u' \cos^2 \tfrac{1}{2} u \,.\, \frac{8e^2}{(1+e)^2},$$

or, comparing with the last equation,

$$1 + \cos (u' - u) = e (\cos u + \cos u'),$$

or, what is the same thing,

$$1 - \cos (u' - u) = (1 - e \cos u') + (1 - e \cos u);$$

and in like manner,

$$1 + \cos (u' + u) = 2 \cos^2 \tfrac{1}{2} (u' + u) = \cos^2 \tfrac{1}{2} u' \,.\, \cos^2 \tfrac{1}{2} u \frac{8}{(1+e)^2};$$

or, comparing with the same equation,

$$1 + \cos (u' + u) = \frac{1}{e} (\cos u + \cos u'):$$

which are formulæ corresponding with the original equation

$$\sin (u' - u) = e (\sin u' - \sin u).$$