575.

ON A SPECIAL QUARTIC TRANSFORMATION OF AN ELLIPTIC FUNCTION.

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It is remarked by Jacobi that a transformation of the order $n'n''$ may lead to a modular equation

$$\frac{\Delta'}{\Delta}=\frac{n'}{n''}\frac{K'}{K},$$

and in particular when $n'=n''$, or the order is square, then the equation may be $\frac{\Delta'}{\Delta}=\frac{K'}{K}$; viz. that instead of a transformation we may have a multiplication. A quartic transformation of the kind in question may be obtained as follows: writing

$$X=(a,\ b,\ c,\ d,\ e\between x,\ 1)^4=a\,(x-\alpha)\,(x-\beta)\,(x-\gamma)\,(x-\delta),$$

H the Hessian, Φ the cubi-covariant, I and J the two invariants, then there is a well known quartic transformation

$$z=\frac{2H}{X},$$

leading to

$$\frac{dz}{\sqrt{(Z)}}=\frac{2\sqrt{(-2)}\,dx}{\sqrt{(X)}},$$

where $Z=z^3-Iz+2J$. In fact we have

$$Z=\frac{2}{X^3}(4H^3-IH^2X+JX^3),\quad =\frac{-2}{X^3}\Phi^2,$$

that is,

$$\sqrt{(Z)}=\frac{\sqrt{(-2)}\,\Phi}{X^2}\sqrt{(X)},$$

so that, by Jacobi's general principle, it at once appears that we have a transformation of the form in question.

Now we may establish a linear transformation

$$z = \frac{py+q}{y-\delta},$$

such that to the roots z_1, z_2, z_3 of the equation $z^3 - Iz + 2J = 0$ correspond the values α, β, γ of y; and this being so, we have between y, z the relation

$$\frac{dz}{\sqrt{(Z)}} = \frac{\sqrt{(-2)}\,dy}{\sqrt{(Y)}},$$

where $Y = a(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)$, $= (a, b, c, d, e \between y, 1)^4$; that is, we have

$$\frac{py+q}{y-\delta} = \frac{2H}{X},$$

such that

$$\frac{dy}{\sqrt{(Y)}} = \frac{2dx}{\sqrt{(X)}},$$

which is a quartic transformation giving a duplication of the integral. The foundation of the theorem is that we can determine p, q in such wise that the functions

$$\frac{p\alpha+q}{\alpha-\delta},\quad \frac{p\beta+q}{\beta-\delta},\quad \frac{p\gamma+q}{\gamma-\delta}$$

shall be the roots z_1, z_2, z_3 of the equation $z^3 - Iz + 2J = 0$. For writing

$$A = (\beta-\gamma)(\alpha-\delta),$$
$$B = (\gamma-\alpha)(\beta-\delta),$$
$$C = (\alpha-\beta)(\gamma-\delta),$$

and observing the equations

$$I = \frac{a^2}{24}(A^2+B^2+C^2),\quad = -\frac{a^2}{12}(BC+CA+AB),$$

(since $A+B+C=0$) and

$$2J = -\frac{a^3}{216}(B-C)(C-A)(A-B),$$

the equation in z is

$$\{z - \tfrac{1}{6}a(B-C)\}\{z - \tfrac{1}{6}a(C-A)\}\{z - \tfrac{1}{6}a(A-B)\},$$

and the equations for the determination of p, q thus are

$$p\alpha + q = \tfrac{1}{6}a(\alpha-\delta)(B-C),\ = \tfrac{1}{6}a(\alpha-\delta)\{2(\alpha\delta+\beta\gamma) - (\alpha+\delta)(\beta+\gamma)\},$$
$$p\beta + q = \tfrac{1}{6}a(\beta-\delta)(C-A),\ = \tfrac{1}{6}a(\beta-\delta)\{2(\beta\delta+\gamma\alpha) - (\beta+\delta)(\gamma+\alpha)\},$$
$$p\gamma + q = \tfrac{1}{6}a(\gamma-\delta)(A-B),\ = \tfrac{1}{6}a(\gamma-\delta)\{2(\gamma\delta+\alpha\beta) - (\gamma+\delta)(\alpha+\beta)\},$$

giving

$$p=\tfrac{1}{6}a\,\{-3\delta^2+2\delta(\alpha+\beta+\gamma)-\beta\gamma-\gamma\alpha-\alpha\beta\},$$
$$q=\tfrac{1}{6}a\,\{\delta^2(\alpha+\beta+\gamma)-2\delta(\beta\gamma+\gamma\alpha+\alpha\beta)+3\alpha\beta\gamma\},$$

or, as these may also be written

$$p=\tfrac{1}{6}a\,\{(\beta-\delta)(\gamma-\delta)+(\gamma-\delta)(\alpha-\delta)+(\alpha-\delta)(\beta-\delta)\},$$
$$q=\tfrac{1}{6}a\,\{\alpha(\beta-\delta)(\gamma-\delta)+\beta(\gamma-\delta)(\alpha-\delta)+\gamma(\alpha-\delta)(\beta-\delta)\};$$

and observe also

$$p\delta+q=\tfrac{1}{2}a(\alpha-\delta)(\beta-\delta)(\gamma-\delta).$$

Taking X in the standard form $=(1-x^2)(1-k^2x^2)$, and writing

$$\gamma=-1,\quad \delta=1,\quad \alpha=+\frac{1}{k},\quad \beta=-\frac{1}{k},$$

we have

$$z=\frac{py+q}{y-1}=\frac{-\frac{1}{6}\{2k^2(1+k^2)(1+k^2x^4)+(1-10k^2+k^4)x^2\}}{(-x^2)(1-k^2x^2)},$$

$$A=-1+\frac{2}{k}-\frac{1}{k^2},$$

$$B=\ \ 1+\frac{2}{k}+\frac{1}{k^2},$$

$$C=\ \ -\frac{4}{k};$$

$$z_1=\ \ \tfrac{1}{6}(1+6k+k^2),$$
$$z_2=\ \ \tfrac{1}{6}(1-6k+k^2),$$
$$z_3=-\tfrac{1}{3}(1+k^2);$$

$$Z=z^3-\tfrac{1}{12}(1+14k^2+k^4)z+\tfrac{1}{108}(1+k^2)(1-34k^2+k^4)$$
$$=(z-z_1)(z-z_2)(z-z_3),$$
$$p=\tfrac{1}{6}(1-5k^2),\quad q=\tfrac{1}{6}(5-k^2),\quad p+q=1-k^2;$$

giving as they should do

$$z_1=\frac{\frac{p}{k}+q}{\frac{1}{k}-1},\quad z_2=\frac{-\frac{p}{k}+q}{-\frac{1}{k}-1},\quad z_3=\frac{-p+q}{-2}.$$

Write for shortness

$$-\tfrac{1}{6}\{2k^2(1+k^2)(1+x^4)+(1-10k^2+k^4)x^2\}=Q,$$

so that

$$\frac{py+q}{y-1}=\frac{Q}{X},$$

then

$$\frac{Q}{X} - z_1 = \frac{p+q}{k-1} \cdot \frac{ky-1}{y-1},$$

$$\frac{Q}{X} - z_2 = \frac{p+q}{k+1} \cdot \frac{ky+1}{y+1},$$

$$\frac{Q}{X} - z_3 = \frac{p+q}{2} \cdot \frac{y+1}{y-1}.$$

The last of these is

$$\frac{1-k^2}{2} \frac{y+1}{y-1} = \frac{\frac{1}{2}(1-k^2)^2 x^2}{(1-x^2)(1-k^2x^2)},$$

that is,

$$\frac{y+1}{y-1} = \frac{(1-k^2) x^2}{(1+x^2)(1-k^2x^2)},$$

from which the foregoing equation

$$\frac{dy}{\sqrt{(Y)}} = \frac{2dx}{\sqrt{(X)}}$$

may be at once verified.